

On the periodic logistic equation

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Abstract

We show that the p -periodic logistic equation $x_{n+1} = \mu_{n \bmod p} x_n (1 - x_n)$ has cycles (periodic solutions) of minimal periods $1, p, 2p, 3p, \dots$. Then we extend Singer's theorem to periodic difference equations, and use it to show the p -periodic logistic equation has at most p stable cycles. Also, we present computational methods investigating the stable cycles in case $p = 2$ and 3 .

Key words: Logistic map; Nonautonomous; Periodic solutions; Singer's theorem; Attractors

1 Introduction

Since Robert May published his famous work "Simple models with very complicated dynamics" [15], the logistic difference equation

$$x_{n+1} = \mu x_n (1 - x_n), \mu \in (0, 4), x_n \in [0, 1], n \in \mathbb{N} := \{0, 1, \dots\}$$

became of particular interest. The simplicity of this equation and complexity of its dynamics make it powerful in illustrating many fundamental notions in discrete dynamical systems. See for instance Devaney [2], Elaydi [3,4], Martelli [14], Peitgen et al. [16], Gleick [8], and Hao and Zheng [10].

In this paper, we assume a periodically fluctuating environment, and thus we focus our attention on the p -periodic version of the logistic equation, i.e.

$$x_{n+1} = \mu_{n \bmod p} x_n (1 - x_n), \mu_0, \mu_1, \dots, \mu_{p-1} \in (0, 4), x_n \in [0, 1] \forall n \in \mathbb{N}, \quad (1)$$

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where p is assumed to be minimal. In the past two decades or so, some attention has been given to this equation, see for instance Grinfeld et al. [9], Jia [13], and Kot and Schaffer [12]. However, as in the autonomous case, this equation is rich enough to be worth further investigation. Thanks to recent developments in the theory of periodic difference equations, AlSharawi et al. [1], Elaydi and Sacker [5,6], Henson [11], Selgrade and Roberds [17], Franke and Selgrade [7], and computer simulations, we are able to give more details about the dynamics of the p -periodic logistic equation in (1). We show the existence of cycles of minimal periods $1, p, 2p, \dots$. Then we extend Singers's theorem to periodic difference equations and show equation (1) has at most p stable cycles (Section 2). In Sections 3 and 4, we use MAPLE and MATLAB to explore the stable cycles in the 2 and 3 periodic cases.

2 Existence and stability of cycles

AlSharawi et al. [1] gave an extension of the well known Sharkovsky's theorem to periodic difference equations. For the reader's convenience, we state it here, but first we need the p -Sharkovsky's ordering [1]. Given positive integers $p > 1$ and $r \geq 1$, denote the least common multiple of p and r by $lcm(p, r)$, define $\mathcal{A}_{p,r} = \{n : lcm(n, p) = rp\}$, then the p -Sharkovsky's ordering is given by

$$\begin{aligned} &\mathcal{A}_{p,3} \prec \mathcal{A}_{p,5} \prec \mathcal{A}_{p,7} \prec \dots \\ &\mathcal{A}_{p,2 \cdot 3} \prec \mathcal{A}_{p,2 \cdot 5} \prec \mathcal{A}_{p,2 \cdot 7} \prec \dots \\ &\vdots \\ &\mathcal{A}_{p,2^n \cdot 3} \prec \mathcal{A}_{p,2^n \cdot 5} \prec \mathcal{A}_{p,2^n \cdot 7} \prec \dots \\ &\vdots \\ &\dots \prec \mathcal{A}_{p,2^n} \prec \dots \prec \mathcal{A}_{p,2^2} \prec \mathcal{A}_{p,2} \prec \mathcal{A}_{p,1}. \end{aligned}$$

Theorem 1 (Sharkovsky's theorem for periodic difference equations)

Let $f_i : X \rightarrow X$, $i = 0, 1, \dots, p-1$ be continuous maps on a closed interval X . Suppose that the p -periodic difference equation $x_{n+1} = f_{n \bmod p}(x_n)$ has a geometric r -cycle, and let $\ell := \frac{lcm(p,r)}{p}$. Then each set $\mathcal{A}_{p,q}$, such that $\mathcal{A}_{p,\ell} \prec \mathcal{A}_{p,q}$, contains a period of a geometric cycle.

To have an equilibrium point $x^* \in [0, 1]$, the maps $f_0(x) = \mu_0 x(1-x), \dots, f_{p-1}(x) = \mu_{p-1} x(1-x)$ need to intersect at $x = x^*$, which implies $x^* = 0$ is the only equilibrium point. Li [13] proved the p -periodic logistic equation in (1) has a p -cycle when $\prod_{i=0}^{p-1} \mu_i > 1$. By Theorem 1, the existence of a p -cycle does not assure the existence of any other cycles. Using a graphical approach, Grinfeld et al. [9] have shown the existence of a stable 2^n -cycle in

the 2-periodic case. In fact, depending on the bifurcation theory of Henson [11], we get a stronger result. Let us state a special version of Henson's theory that suits our need.

Theorem 2 *Suppose $F(\mu, x)$ is nonlinear in x , one to one in μ , C^2 in both μ and x , and the autonomous equation $x_{n+1} = F(\mu, x_n)$ has a cycle $\{x_0, x_1, \dots, x_{q-1}\}$ of minimal period q . Then for sufficiently small ϵ , there exist $\mu_0, \dots, \mu_{p-1} \in [\mu - \epsilon, \mu + \epsilon]$ such that the p -periodic difference equation $x_{n+1} = f_{n \bmod p}(x_n) = F(\mu_{n \bmod p}, x_n)$ has $\gcd(p, q)$ distinct cycles, each of which is of minimal period $\text{lcm}(p, q)$. Furthermore, if $|F_x(\mu, x_0)F_x(\mu, x_1)\dots F_x(\mu, x_{p-1})| \neq 1$ then either all of the $\gcd(p, q)$ cycles are asymptotically stable, or they are all unstable.*

PROOF. See Henson [11].

Now, we have the following result.

Theorem 3 *The only possible cycles of the p -periodic logistic equation (1) are cycles of minimal periods 1 or mp , $m \in \mathbb{Z}^+$. Furthermore, for each $m \in \mathbb{Z}^+$, there exists $\mu_0, \mu_1, \dots, \mu_{p-1} \in (0, 4)$ such that equation (1) has cycles of minimal periods rp for all $m \prec r$ in the 1-Sharkovsky's ordering.*

PROOF.

Denote the greatest common divisor between p and r ($\gcd(p, r)$) by s . Then we proceed by eliminating two possibilities. First, when r divides p , $r \neq p, 1$. Suppose $\bar{x}_0 \rightarrow \bar{x}_1 \rightarrow \dots \rightarrow \bar{x}_{r-1}$ is an r -cycle of equation (1). Then it is necessary for the p equations

$$\bar{x}_{j+1 \bmod r} = \mu_j \bar{x}_{j \bmod r} (1 - \bar{x}_{j \bmod r}), \quad 0 \leq j \leq p-1$$

to be satisfied, but having $\mu_i \neq \mu_j$ for some $i, j \in \{0, 1, \dots, p-1\}$ produces a contradiction. Second, when r is neither a divisor nor a multiple of p . Again suppose there is an r -cycle as before, then we need the $\frac{kp}{s}$ equations

$$\bar{x}_{j+1 \bmod r} = \mu_{j \bmod p} \bar{x}_{j \bmod r} (1 - \bar{x}_{j \bmod r}), \quad 0 \leq j \leq \frac{rp}{s}.$$

to be satisfied. Observe that $\frac{rp}{s} > \max\{k, p\}$. If $s = 1$ then $\mu_0 = \mu_1 = \dots = \mu_{p-1}$, which contradicts the fact that (1) is nonautonomous. If $s > 1$ then the $\frac{rp}{s}$ equations reduces our p -periodic equation to a p^* -periodic one, for some positive integer $p^* < p$, and this contradicts the minimality of the period p . To this end we verified the nonexistence of cycles of minimal periods different from $1, mp$. Next, given $m \in \mathbb{Z}^+$, and let $r \in \mathcal{A}_{p,m}$, there exists $\mu \in (0, 4)$ such

that the autonomous equation $x_{n+1} = \mu x_n(1 - x_n)$ has a cycle of minimal period r . By Henson's Theorem 2, we perturb μ to get $\mu_0, \mu_1, \dots, \mu_{p-1} \in (0, 4)$ so that equation (1) has a cycle of minimal period $\text{lcm}(r, p) = mp$. Finally, invoke Theorem 1 to complete the proof.

Remark 4 *From Theorems 2 and 3, the cascade of cycles associated with the p -periodic logistic equation along a line sufficiently close and parallel to (t, t, t) $0 < t < 4$ is*

$$3p \prec 5p \prec \dots \prec 2^n p \prec \dots \prec 4p \prec 2p \prec p.$$

Singer's theorem [18] is a useful tool in finding an upper bound for the number of stable cycles in autonomous difference equations $x_{n+1} = f(x_n)$. For our convenience, we write Singers theorem, then we extend it to periodic difference equations in the form $x_{n+1} = f_n(x_n)$. A stability definition in the autonomous case can be found in many undergraduate texts [3,4]; however, for the periodic nonautonomous case we adopt the following definition [1].

Definition 5 *Let $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ be a geometric r -cycle of $x_{n+1} = f_{n \bmod p}(x_n)$ in an interval X . Then*

- (i) c_r is uniformly stable (US) if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n_0 = 0, 1, \dots, p-1$, and $x \in X$,

$$|x - \bar{x}_{n_0 \bmod r}| < \delta \quad \text{implies} \quad |\Phi_n(f_{n_0})x - \Phi_n(f_{n_0})\bar{x}_{n_0 \bmod r}| < \varepsilon$$

for all $n \in \mathbb{Z}^+$, where $\Phi_n(f_{n_0}) = f_{(n_0+n-1) \bmod p} \circ \dots \circ f_{(n_0+1) \bmod p} \circ f_{n_0}$.

- (ii) c_r is uniformly attracting (UA) if there exists $\eta > 0$ such that for any $n_0 = 0, 1, \dots, p-1$, and $x \in X$,

$$|x - \bar{x}_{n_0 \bmod r}| < \eta \quad \text{implies} \quad \lim_{n \rightarrow \infty} \Phi_{ns}(f_{n_0})x = \bar{x}_{n_0 \bmod r},$$

where $s = \text{lcm}(r, p)$.

- (iii) c_r is uniformly asymptotically stable (UAS) if it is both uniformly stable and uniformly attracting.
- (iv) c_r is globally asymptotically stable (GAS) if it is UAS and $\eta = \infty$.

For a better understanding of stability, uniform stability, and a stability criteria, we refer the reader to [1] and [3].

Definition 6 *Let $f : X \rightarrow X$ be differentiable on the closed interval X . Then the Schwarzian derivative of f is defined by*

$$S(f(x)) := \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2.$$

Theorem 7 (Singer's theorem) *Let $f : X \rightarrow X$ be defined on the closed interval X such that the Schwarzian derivative is negative for all $x \in X$. If f has m critical points in X , then the difference equation $x_{n+1} = f(x_n)$ has at most $2 + m$ attracting cycles.*

PROOF. See Singer [18], or Elaydi [4], page 63.

Theorem 8 (An extension of Singer's theorem) *Let $f_i : X \rightarrow X$ be defined on the closed interval X such that $S(f_i(x)) < 0$ for all $x \in X$ and $\forall i = 0, 1, \dots, p-1$. If f_i has m_i critical points in X , then the p -periodic difference equation $x_{n+1} = f_{n \bmod p}(x_n)$ has at most $2 + \sum_{i=0}^{p-1} m_i$ uniformly attracting cycles.*

PROOF. Define $g(x) = f_{p-1} \circ f_{p-2} \circ \dots \circ f_0(x)$, then we divide the proof into two steps

Step One: $x_{n+1} = g(x_n)$ has at most $2 + \sum_{i=0}^{p-1} m_i$ stable cycles. Since $S(f_i) < 0$ for each $i = 0, 1, \dots, p-1$, then it can be shown by mathematical induction that $S(g) < 0$. Now, in Singers theorem, the number of stable cycles of $g(x)$ is built on the number of critical points. Thus let us concentrate on the critical points of $g(x)$. Denote the critical points of f_i by $c_{i,j}$, $1 \leq j \leq m_i$. Then

$$g'(x) = f'_{p-1}(f_{p-2} \circ \dots \circ f_0(x)) \cdot f'_{p-2}(f_{p-3} \circ \dots \circ f_0(x)) \cdot \dots \cdot f'_0(x)$$

implies

$$\begin{aligned} x &= c_{0,j}, \quad j = 1, \dots, m_0 \\ f_0(x) &= c_{1,j}, \quad j = 1, \dots, m_1 \\ f_1(f_0(x)) &= c_{2,j}, \quad j = 1, \dots, m_2 \\ &\vdots \\ f_{p-2}(\dots(f_1(f_0(x)))) &= c_{p-1,j}, \quad j = 1, \dots, m_{p-1}. \end{aligned}$$

Clearly, the first line of equations in this system has m_0 solutions, and contribute to the m_0 orbits

$$\mathcal{O}^+(c_{0,j}) = \{c_{0,j}, g(c_{0,j}), g^2(c_{0,j}), \dots\}, \quad 1 \leq j \leq m_0.$$

It is difficult to determine how many solutions the second line of equations may provide; however, they contribute to the m_1 orbits

$$\mathcal{O}^+(y_{1,j}) = \{y_{1,j}, g(y_{1,j}), g^2(y_{1,j}), \dots\}, \quad 1 \leq j \leq m_1, \text{ and } y_{1,j} = f_{p-1} \circ \dots \circ f_1(c_{1,j}).$$

Similarly, for the rest of the above equations. Thus, instead of determining the number of critical points of g , which is difficult, we count the number of

orbits associated with the critical points. Thus we have $2 + \sum_{i=0}^{p-1} m_i$ orbits associated with the critical points, and this is the maximum number of stable cycles of g .

Step Two: $g(x)$ has an attracting r -cycle if and only if $x_{n+1} = f_{n \bmod p}(x_n)$ has a uniformly attracting q -cycle for some $q \in \mathcal{A}_{p,r}$. Suppose $g(x)$ has the attracting r -cycle, $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}$, we set these points on the fiber \mathcal{F}_0 (see [1], [5] and [6] for more details about the structure of the fibers). Then we set the r distinct points $f_0(\bar{x}_i)$, $0 \leq i < r$ on the fiber \mathcal{F}_1 , the r points $f_1 \circ f_0(\bar{x}_i)$, $0 \leq i < r$ on the fiber \mathcal{F}_2 , and so on. Since the maps f_i are continuous and convergence is preserved under continuity, then it is a straight forward to check that this construction provides a UA geometric cycle of minimal period q for some $q \in \mathcal{A}_{p,r}$. Conversely, whenever $x_{n+1} = f_n(x_n)$ has a UA q -cycle, $q \in \mathcal{A}_{p,r}$, we take the r points on the fiber \mathcal{F}_0 , that provide an attracting r -cycle for the autonomous equation $x_{n+1} = g(x_n)$.

From steps one and two, and by counting the two boundary points of the domain X , the p -periodic difference equation $x_{n+1} = f_{n \bmod p}(x_n)$ has at most $2 + \sum_{i=0}^{p-1} m_i$ UA cycles.

Corollary 9 *The p -periodic logistic equation in (1) has at most p attracting cycles.*

PROOF. Since $f'_i(x) = 0$ at one point $x = \frac{1}{2}$ for each $i = 0, 1, \dots, p-1$, and the boundary points 0 and 1 are attracted to the origin, then equation (1) has at most p attracting cycles in the interval $(0, 1)$.

3 The case $p = 2$

Throughout this section we consider $p = 2$; i.e. $f_0(x) = \mu_0 x(1-x)$ and $f_1(x) = \mu_1 x(1-x)$. This can be thought of as a model for species with two non-overlapping generations each year, one for the winter and one for the summer. Since $\mu_0 = \mu_1$ reduces equation (1) to the autonomous case, we consider $\mu_0 \neq \mu_1$. Also, observe that $\mu_0 < \mu_1$ and $\mu_1 < \mu_0$ exhibit the same dynamics.

In equation (1), the origin $x^* = 0$ is GAS with respect to the domain $[0, 1]$ as long as $\mu_0 \mu_1 \leq 1$. This case can be considered trivial, thus we focus our attention on the domain $(0, 1)$, and consider $\mu_0 \mu_1 > 1$. It is well known that the autonomous equation $x_{n+1} = \mu x_n(1-x_n)$ has a GAS fixed point when $1 < \mu < 3$, namely $x^* = \frac{\mu-1}{\mu}$, and an asymptotically stable 2-cycle when $3 < \mu < 1 + \sqrt{6}$. By Theorem 2, equation (1) has a GAS 2-cycle for all $\mu_0, \mu_1 \in [\mu - \epsilon_\mu, \mu + \epsilon_\mu]$, $1 < \mu < 3$, and two UAS 2-cycles when $\mu_0, \mu_1 \in$

$[\mu - \epsilon_\mu, \mu + \epsilon_\mu]$, $3 < \mu < 1 + \sqrt{6}$. But Theorem 2 does not help in the case of large perturbations, so we need to approach these 2-cycles differently. Using resultants, and MAPLE, we solve the following systems of equations for μ_0 and μ_1

$$\begin{cases} x_1 = f_0(x_0) \\ x_0 = f_1(x_1) \\ f'_0(x_0)f'_1(x_1) = 1, \end{cases}$$

and

$$\begin{cases} x_1 = f_0(x_0) \\ x_0 = f_1(x_1) \\ f'_0(x_0)f'_1(x_1) = -1. \end{cases}$$

The first system comes from the idea of a tangent bifurcation. After solving and simplifying, we get $\mu_0\mu_1 - 1 = 0$ and

$$-4\mu_1\mu_0^2 + \mu_1^2\mu_0^2 - 4\mu_1^2\mu_0 + 18\mu_1\mu_0 - 27 = 0. \quad (2)$$

Denote the curve branches of equation (2) by Γ_{1L} and Γ_{1U} . Then a GAS 2-cycle is born at $\mu_0\mu_1 = 1$, and it changes from a GAS to a UAS when a new UAS 2-cycle is created at Γ_{1L} and Γ_{1U} , see Figures 1 and 2,3. The second system comes from the idea of a saddle-node bifurcation. After solving and simplifying, we get

$$-4\mu_1^2\mu_0^3 + 12\mu_1\mu_0^2 - 85\mu_1\mu_0 + 15\mu_1^2\mu_0^2 + 12\mu_1^2\mu_0 + \mu_1^3\mu_0^3 - 4\mu_1^3\mu_0^2 + 125 = 0, \quad (3)$$

we denote the curve branches of this equation by Γ_{2L} and Γ_{2U} , see Figure 1. A 2-cycle loses stability to a 4-cycle when it bifurcates into a 4-cycle at Γ_{2U} and Γ_{2L} . Thus in the region bounded by the curves $\Gamma_{2L}, \Gamma_{2U}, \Gamma_{1U}, \Gamma_{1L}$ and $3 < \mu_0 < 1 + \sqrt{6}, \mu_0 \neq \mu_1$, we have two UAS 2-cycles, see Figure 3.

Next, the autonomous equation has an asymptotically stable 4-cycle for $1 + \sqrt{6} < \mu < \gamma$, where gamma is a positive real number less than 4. It is not easy to find the exact value of γ ; however, a numerical approximation shows that $\gamma \approx 3.544090360$. Using MAPLE, we find that the exact value of γ can be written in an interesting way. The equation

$$\mu^6 - 12\mu^5 + 47\mu^4 - 188\mu^3 + 517\mu^2 + 4913 = 0$$

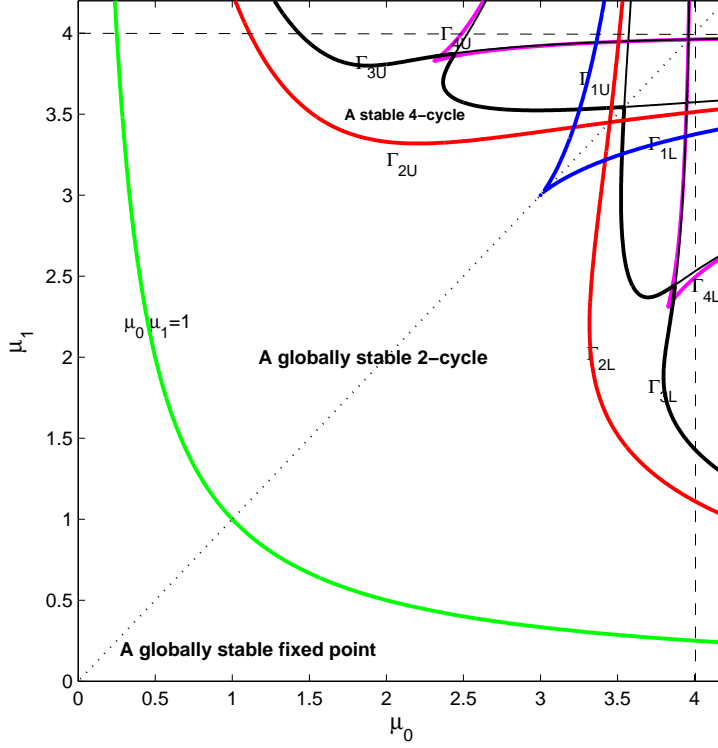


Fig. 1. The bifurcation curves for the stable 2-cycles and 4-cycles in the 2-periodic logistic equation

has a real zero between 5 and 6, which we denote by α . Then $\gamma = 1 + \sqrt{1 + \alpha}$. By Theorem 2, equation (1) has two UAS 4-cycles for some $\mu_0, \mu_1 \in [\mu - \epsilon_\mu, \mu + \epsilon_\mu]$ and $1 + \sqrt{6} < \mu < 1 + \sqrt{1 + \alpha}$. We get a better understanding of these 4-cycles by solving the following equations for μ_0 and μ_1 .

$$\begin{cases} x_1 = f_0(x_0) \\ x_2 = f_1(x_1) \\ x_3 = f_0(x_2) \\ x_4 = f_1(x_3) \\ f'_0(x_0)f'_1(x_1)f'_0(x_2)f'_1(x_3) = 1, \end{cases}$$

and

$$\begin{cases} x_1 = f_0(x_0) \\ x_2 = f_1(x_1) \\ x_3 = f_0(x_2) \\ x_4 = f_1(x_3) \\ f'_0(x_0)f'_1(x_1)f'_0(x_2)f'_1(x_3) = -1, \end{cases}$$

Again the first system comes from the idea of a tangent bifurcation and the second system comes from the idea of a saddle-node bifurcation. After some MAPLE manipulation, the first system provides

$$\begin{aligned} & x^8y^8 - 8x^8y^7 - 8x^7y^8 + 16x^8y^6 + 64x^7y^7 + 16x^6y^8 - 112x^7y^6 - 112x^6y^7 - 64x^7y^5 + \\ & 8x^6y^6 - 64x^5y^7 + 720x^6y^5 + 720x^5y^6 - 1504x^5y^5 + 256x^6y^3 - 832x^5y^4 - 832x^4y^5 + \\ & 256x^3y^6 - 2176x^5y^3 + 1682x^4y^4 - 2176x^3y^5 + 4792x^4y^3 + 4792x^3y^4 + 3328x^4y^2 + \\ & 4896x^3y^3 + 3328x^2y^4 - 13680x^3y^2 - 13680x^2y^3 - 432x^2y^2 - 19872x^2y - 19872xy^2 + \\ & 77760xy + 91125 = 0, \end{aligned}$$

where (x, y) is used in place (μ_0, μ_1) to shorten the equation, we denote its curve branches by Γ_{4U} and Γ_{4L} . The second system provides a lengthy polynomial equation in μ_0 and μ_1 of degree 24, we denote its curve branches by Γ_{3U} and Γ_{3L} (see Figure 1). In Figures 2 and 3 we use MATLAB to plot the stable cycles of the 2-periodic logistic equation along parametric curves in the μ_0, μ_1 plane. Now, we summarize

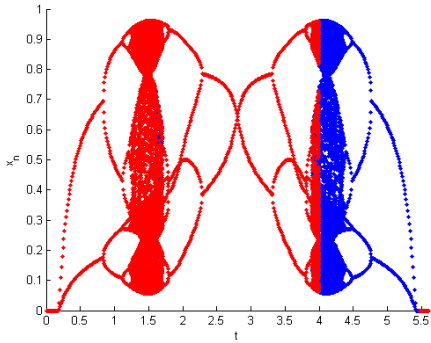


Fig. 2. The stable cycles along the line $\mu_0 = t, \mu_1 = 5.6 - \mu_0, 0 \leq t \leq 5.6$

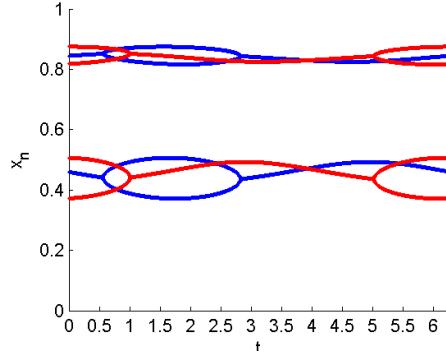


Fig. 3. The stable cycles along $\mu_0 = 0.1 \cos(t) + 3.4, \mu_1 = 0.1 \sin(t) + 3.4, 0 < t < 2\pi$.

- A UAS 4-cycle is born at Γ_{2U} and Γ_{2L} , another one is born at Γ_{4U} and Γ_{4L} .
- A UAS 4-cycle bifurcates into a UAS 8-cycle at Γ_{3U} and Γ_{3L} .

- In the region bounded by $\Gamma_{2U}, \Gamma_{1U}, \Gamma_{3U}$ and Γ_{2L} , there is a UAS 2-cycle and a UAS 4-cycle. The situation is similar for the region bounded by $\Gamma_{2L}, \Gamma_{1L}, \Gamma_{3L}$ and Γ_{2U} . See Figures 3 and 1.
- In the region bounded by Γ_{2L}, Γ_{3U} and $\mu_1 = \mu_0$, we have two UAS 4-cycles. The situation is similar in the region bounded by Γ_{2U}, Γ_{3L} and $\mu_1 = \mu_0$. See Figures 3 and 1.

Remark 10 (i) *In general, the dynamics of the 2-periodic equation changes dramatically to the right or above the bold parts of the curves Γ_{3L} and Γ_{3U} . So we give no description in that region. However, the same method used above can be used to locate the region in the μ_0, μ_1 plane where UAS 8-cycles exist.*

(ii) *Our analysis does not capture those cycles that are born unstable and stay unstable, see Example 11.*

We end this section by the following specific example.

Example 11 *Fix $\mu_0 = 2.29$ and $\mu_1 = 3.8174$, and observe the location of $(2.29, 3.8174)$ in Figure 1. Numerical calculations reveal the existence of the following.*

- A UAS 2-cycle $\{\bar{x}_0, \bar{x}_1\} = \{0.831606, 0.320685\}$.
- A UAS 4-cycle $\{\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3\} = \{0.373333, 0.535758, 0.949469, 0.109869\}$.
- A UAS 4-cycle $\{\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3\} = \{0.423867, 0.559227, 0.940959, 0.127221\}$.
- An unstable 4-cycle $\{\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3\} = \{0.383656, 0.541503, 0.947775, 0.113350\}$.

4 The case $p = 3$

Throughout this section we concentrate on the 3-periodic logistic equation; i.e.

$$f_0(x) = \mu_0 x(1 - x), \quad f_1(x) = \mu_1 x(1 - x), \quad f_2(x) = \mu_2 x(1 - x).$$

The three parameters in this case make it more complex. Nevertheless, we find it possible to determine the regions in the μ_0, μ_1, μ_2 space, where a GAS 3-cycle exist. Also, we determine the region where three UAS 3-cycles exist. Let us recall that the autonomous equation $x_{n+1} = \mu x_n(1 - x_n)$ has a GAS fixed point for $1 < \mu < 3$, and an asymptotically stable 3-cycle for $1 + 2\sqrt{2} < \mu < \gamma^* \approx 3.841499008$, where γ^* is the positive root of

$$\mu^2 - 2\mu + \frac{1}{480}(7660 + 540\sqrt{201})^{\frac{2}{3}}(27\sqrt{201} - 383) - \frac{1}{6}(7660 + 540\sqrt{201})^{\frac{1}{3}} - \frac{8}{3} = 0$$

(finding γ^* is not obvious, nevertheless, MAPLE calculations helped in getting this form). By Theorem 2, there exist $\mu_0, \mu_1, \mu_2 \in [\mu - \epsilon, \mu + \epsilon], \mu \in (1, 3)$ such that equation (1) has a GAS 3-cycle, and $\mu_0, \mu_1, \mu_2 \in [\mu - \epsilon, \mu + \epsilon], \mu \in$

$(1 + 2\sqrt{2}, \gamma^*)$ such that equation (1) has three UAS 3-cycles. Recall from Theorem 3, the cascade of cycles in this case is given by

$$9 \prec 15 \prec \dots \prec 12 \prec 6 \prec 3.$$

The 3-cycles are the most simple ones to deal with in this case. In fact, they are the most interesting ones here. We give an explicit description of these cycles by solving the following systems of equations for μ_0, μ_1, μ_2 .

$$\begin{cases} x_1 = f_0(x_0) \\ x_2 = f_1(x_1) \\ x_0 = f_2(x_2) \\ f'_0(x_0)f'_1(x_1)f'_2(x_2) = 1, \end{cases}$$

and

$$\begin{cases} x_1 = f_0(x_0) \\ x_2 = f_1(x_1) \\ x_0 = f_2(x_2) \\ f'_0(x_0)f'_1(x_1)f'_2(x_2) = -1. \end{cases}$$

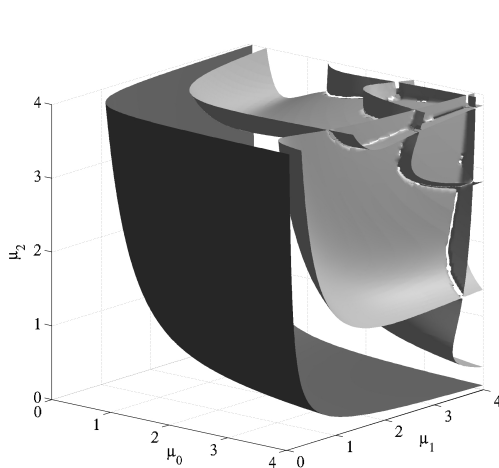


Fig. 4. The surfaces $\mu_0\mu_1\mu_2 = 1$ (left) and S_2 (right)

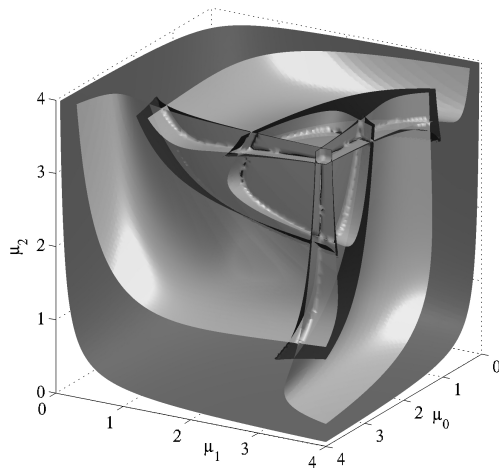


Fig. 5. The surfaces $\mu_0\mu_1\mu_2 = 1$, S_1 and S_2 .

The first system of equations gives two surfaces. The first is $\mu_0\mu_1\mu_2 = 1$ which is the place where a GAS 3-cycle is born. The second, which we call S_1 is the place where three UAS 3-cycles are born. The second system of equations gives one complex surface, which we denote by S_2 , Figure 4 shows the region

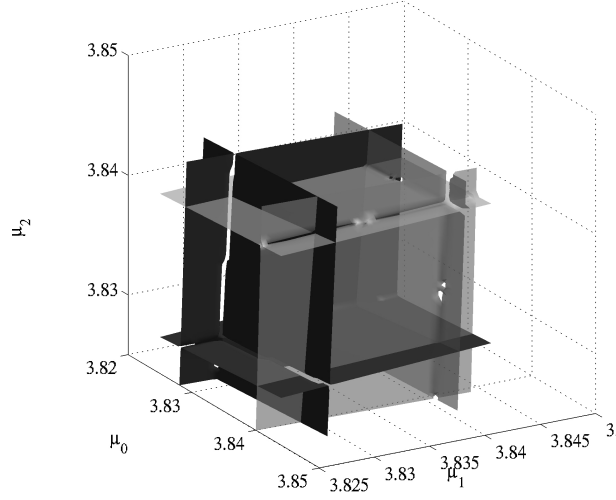


Fig. 6. The region between S_1 and S_2 where three UAS 3-cycles exit

between the surfaces $\mu_0\mu_1\mu_2 = 1$ and S_2 , where a GAS 3-cycle exist. Exiting this region through the surface S_2 contributes to the bifurcation of the GAS 3-cycle into a UAS 6-cycle, then the period doubling phenomenon takes over. As we see in Figure 5, the surfaces S_1 and S_2 intersect to form a “chamber”, we magnify this chamber in Figure 4. Entering the chamber through a positive direction in μ_0, μ_1 or μ_2 contribute to the creation of a UAS 3-cycle. Thus inside the chamber we have three UAS 3-cycles. Of course now exiting the chamber in a positive direction in μ_0, μ_1 or μ_2 contribute to the bifurcation of a UAS 3-cycle into a UAS 6-cycle, then the period doubling phenomenon takes over again. Figures 7 and 8 show the stable cycles along two parametric curves in the μ_0, μ_1, μ_2 space.

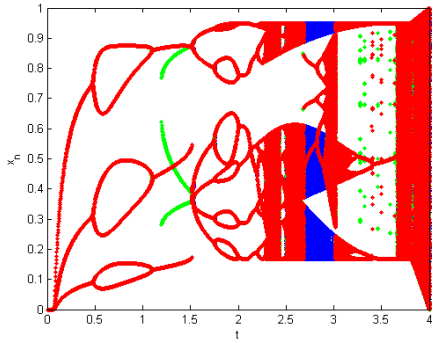


Fig. 7. The stable cycles along $\mu_0 = 3.8, \mu_1 = 3.5, \mu_2 = t, 0 < t < 4$.

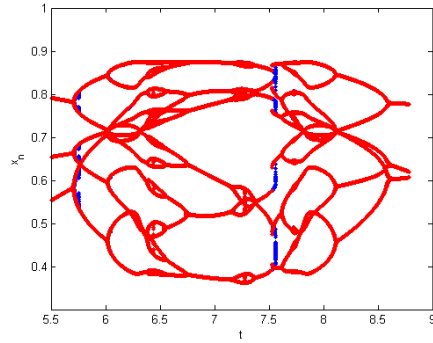


Fig. 8. The stable cycles along $\mu_0 = 0.5 \cos(t) + 3, \mu_1 = 0.5 \sin(t) + 3, \mu_2 = 3.5, 5.5 < t < 2\pi + 2.5$.

References

- [1] Z. AlSharawi, J. Angelos, S. Elaydi, and L. Rakesh, An extension of Sharkovsky's theorem to periodic difference equations, *J. Math. Anal. Appl.*, to appear.
- [2] R. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley, Reading, MA, 1992.
- [3] S. Elaydi, *An Introduction to Difference Equations*, Third Edition, Springer-Verlag, New York, 2005.
- [4] S. Elaydi, *Discrete Chaos*, Chapman & Hall/CRC, Boca Raton, 2000.
- [5] S. Elaydi and R. Sacker, Global stability of periodic orbits of nonautonomous difference equations and population biology, *J. Differential Equations* **208** (2005), 258-273.
- [6] S. Elaydi and R. Sacker, Global stability of periodic orbits of nonautonomous difference equations in population biology and the Cushing-Henson conjectures, *Proceedings of the 8th International Conference on Difference Equations*, Brno, 2003.
- [7] J.E. Franke and J.F. Selgrade, Attractors for periodic dynamical systems, *J. Math. Anal. Appl.* **286** (2003), 64-79.
- [8] J. Gleick, *Chaos: Making a New Science*, Penguin, 1987.
- [9] M. Grinfeld, P.A. Knight and H. Lamba, On the periodically perturbed logistic equation, *J. Phys. A: Math. Gen.* **29** (1996), 8635-8040.
- [10] B. Hao, and W. Zheng, *Applied Symbolic Dynamics and Chaos*, World Scientific, 1998.
- [11] S.M. Henson, Multiple attractors and resonance in periodically forced population, *Physica D.* **140** (2000), 33-49.
- [12] M. Kot and W. M. Schaffer, The effects of seasonality on discrete models of population growth, *Theoretical Population Biology* **26** (1984) 340-360.
- [13] J. Li, Periodic solutions of population models in a periodically fluctuating environment, *Math. Biosci.* **110** (1992), 17-25.
- [14] M. Martelli, *Introduction to Discrete Dynamical Systems and Chaos*, Wiley-Interscience, 1999.
- [15] R.M. May, Simple mathematical models with very complicated dynamics, *Nature* **261** (1976) 459-467.
- [16] H. Peitgen, H. Jurgens and D. Saupe, *Fractals for the Classroom*, Springer-Verlag, 1992.

- [17] J.F. Selgrade and H.D. Roberds, On the structure of attractors for discrete periodically forced systems with applications to population models, *Physica D.* **158** (2001), 69–82.
- [18] D. Singer, Stable orbits and bifurcation of maps of the interval, *SIAM J. Appl. Math.* **35** (1978) 260-267.