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Fredholm equations for non-symmetric kernels, with applications to iterated integral operators.

by

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Abstract: We give the Jordan form and the Singular Value Decomposition for an integral operator $\mathcal{N}$ with a non-symmetric kernel $N(y,z)$. This is used to give solutions of Fredholm equations for non-symmetric kernels, and to determine the behaviour of $\mathcal{N}^n$ and $(\mathcal{N}\mathcal{N}^*)^n$ for large n.

1 Introduction and summary

Suppose that $\Omega \subset R^p$ and that μ is a σ-finite measure on Ω. Consider a $s_1 \times s_2$ complex matrix function $N(y,z)$ on $\Omega \times \Omega$. Generally we shall assume that $N \in L_2(\mu \times \mu)$ and is non-trivial, that is,

$$0 < ||\mathcal{N}||_2^2 = \int\int ||N(y,z)||^2 d\mu(y) d\mu(z) < \infty.$$

The integral operator asociated with (N,μ) is $\mathcal{N}$ defined by

$$\mathcal{N}q(y) = \int N(y,z)q(z)d\mu(z),\ \ p(z)^*\mathcal{N} = \int p(y)^* N(y,z)d\mu(y) \tag{1.1}$$

where $p : \Omega \to C^{s_1}$ and $q : \Omega \to C^{s_2}$ are any functions for which these integrals exist, for example $p,q \in L_2(\mu)$, that is $\int |p|^2 d\mu < \infty$, and similarly for q. (All integrals are over Ω. * denotes the transpose of the complex conjugate.)

Section 2 reviews Fredholm theory for Hermitian kernels, that is, when $N(y,z)^* = N(z,y)$, so that $s_1 = s_2$. For this case, $\mathcal{N}^n = O(r_1^n)$ for large n where r_1 is the magnitude of the largest eigenvalue.

Section 3 extends this to non-Hermitian kernels for the case of diagonal Jordan form. Again $\mathcal{N}^n = O(r_1^n)$ for large n for r_1 as before.

Section 4 deals with the case of non-diagonal Jordan form. In this case $\mathcal{N}^n = O(r_1^n n^{M-1})$ for large n for r_1 as before where M is the largest multiplicity of those eigenvalues with modulus r_1.

Section 5 gives the Singular Value Decomposition (SVD) for a non-symmetric kernel $N(y,z)$. In this case one has results such as $(\mathcal{N}\mathcal{N}^*)^n = O(\theta_1^{2n})$ for large n where θ_1 is the largest singular value..

2 Hermitian kernels

Matrix theory
First consider a Hermitian matrix $N^* = N \in C^{s\times s}$. Its eigenvalues $\nu_1, \cdots, \nu_s$ are the roots of $\det(N - \nu I) = 0$. They are real. Corresponding to ν_j is an eigenvector p_j satisfying $Np_j = \nu_j p_j$. These are orthonormal: $p_j^* p_k = \delta_{jk}$ where $\delta_{jj} = 1$ and $\delta_{jk} = 0$ for $j \neq k$. Set $P = (p_1, \cdots, p_s)$. If N and its eigenvalues are real, then P can be taken to be real. The *spectral decomposition* of N in terms of its eigenvalues and eigenvectors is

$$\begin{aligned} N &= P\Lambda P^* = \sum_{j=1}^{s} \nu_j p_j p_j^* \text{ where } \Lambda = \mathrm{diag}(\nu_1, \cdots, \nu_s), \qquad (2.1)\\ \sum_{j=1}^{s} \nu_j p_j p_j^* &= PP^* = I_s = P^*P = (p_j^* p_k). \end{aligned}$$

So for $\alpha \in C$

$$N^\alpha = P\Lambda^\alpha P^* = \sum_{j=1}^{s} \nu_j^\alpha p_j p_j^*, \qquad (2.2)$$

provided that if $\det(N) = 0$, then α has non-negative real part. So for $\det(N) \neq 0$,

$$Nf = g \Rightarrow f = N^{-1}g = \sum_{j=1}^{s} \nu_j^{-1} p_j (p_j^* g).$$

Similarly for ν not an eigenvalue and $f, g \in C^s$,

$$(\nu I - N)f = g \Rightarrow f = (\nu I - N)^{-1} g = \sum_{j=1}^{s} (\nu - \nu_j)^{-1} p_j (p_j^* g).$$

For large n, if

$$r_1 = |\nu_1| = \cdots = |\nu_M| > r_0 = \max_{j=M+1}^{s} |\nu_j| \qquad (2.3)$$

then

$$N^n = r_1^n C_n + O(r_0^n) \text{ where } C_n = \sum_{j=1}^{M} [\text{sign } (\nu_j)]^n p_j p_j^* = O(1) \qquad (2.4)$$

assuming that s does not depend on n.

Function theory
Now consider a function $N(y,z) : \Omega^2 \to C^{s\times s}$ and a σ-finite measure μ on $\Omega \subset R^p$. Its integral operator with respect to μ is $\mathcal{N}$ defined by (1.1). Suppose that the *kernel* N is Hermitian, that is,

$$N(y,z)^* = N(z,y).$$

Then the analogues of the matrix results above are as follows. Suppose that $||\mathcal{N}||_2^2 > 0$ and

$$\sum_{j=1}^{s} \int |N_{jj}(x,x) d\mu(x) < \infty.$$

The *spectral decomposition* of N in terms of its eigenvalues and vector eigenfunctions $\{p_j(y)\} : \Omega \to C^s$, is

$$\begin{aligned} N(y,z) &= P(y)\Lambda P(z)^* = \sum_{j=1}^{\infty} \nu_j p_j(y) p_j(z)^* \qquad (2.5) \\ \text{where } \Lambda &= \text{diag}(\nu_1, \nu_2, \cdots),\ P(y) = (p_1(y), p_2(y), \cdots). \end{aligned}$$

The eigenfunctions are orthonormal with respect to μ:

$$\int p_j^* p_k d\mu = \delta_{jk}.$$

In Fredholm theory the convention is to call $\{\lambda_j = \nu_j^{-1}\}$ the eigenvalues, rather than $\{\nu_j\}$.

$\mathcal{I}_{yz} = P(y)P(z)^*$ is generally divergent but can be thought of as a generalized Dirac function: $\int \mathcal{I}_{yz} f(z) d\mu(z) = f(y)$. $P(z)^* P(y) = (p_j(z)^* p_k(y))$ satisfies $\int P(z)^* P(z) d\mu(z) = I_\infty$.

For $n = 1, 2, \cdots,\ N_n(y,z) = \mathcal{N}^{n-1} N(y,z)$ satisfies

$$N_n(y,z) = P(y)\Lambda^n P(z)^* = \sum_{j=1}^{s} \nu_j^n p_j(y) p_j(z)^*.$$

For large n, if (2.3) holds then

$$N_n(y,z) = r_1^n C_n(y,z) + O(r_0^n) \text{ where } C_n(y,z) = \sum_{j=1}^{M} [\text{sign } (\nu_j)]^n p_j(y) p_j(z)^* = O(1).$$

Conditions for (2.5) to hold pointwise and uniformly are given in Withers (1974, 1975, 1978.) It is known as Mercer's Theorem.

The resolvent

Given functions $f, g : \Omega \to C^s$, the *Fredholm integral equation of the second kind,*

$$p(y) - \lambda \mathcal{N} p(y) = f(y), \qquad (2.6)$$

can be solved for λ not an eigenvalue using

$$(I - \lambda\mathcal{N})^{-1} = I + \lambda \mathcal{N}_\lambda, \text{ that is, } \mathcal{N}_\lambda = (I - \lambda\mathcal{N})^{-1}\mathcal{N}$$

where

$$\mathcal{N}_\lambda f(y) = \int N_\lambda(y,z) f(z) d\mu(z),\ g(z)\mathcal{N}_\lambda = \int g(y) N_\lambda(y,z) d\mu(y),$$

and the *resolvent* of $\mathcal{N}$,

$$N_\lambda(y,z) = (I - \lambda\mathcal{N})^{-1} N(y,z) : \mathcal{C} \times \Omega^2 \to \mathcal{C}^{s\times s}$$

with operator $\mathcal{N}_\lambda$ is the unique solution of

$$(I - \lambda\mathcal{N})\mathcal{N}_\lambda = \mathcal{N} = \mathcal{N}_\lambda(I - \lambda\mathcal{N}),$$

that is,

$$\lambda\mathcal{N} N_\lambda(y,z) = N(y,z) - N_\lambda(y,z) = \lambda N_\lambda(y,z)\mathcal{N}.$$

If this can be solved analytically or numerically, then one has a solution of (2.6) without the need to compute the eigenvalues and eigenfunctions of $\mathcal{N}$.

The resolvent satisfies

$$N_\lambda(y,z) = \sum_{j=1}^{\infty} p_j(y)p_j(z)^*/(\lambda_j - \lambda). \tag{2.7}$$

Conditions for this to hold are given by Corollary 3 of Withers (1975). The Fredholm equation of the second kind, (2.6), has solution

$$p(y) = f(y) + \lambda \sum_{j=1}^{\infty} p_j(y) \int p_j^* f d\mu/(\lambda_j - \lambda).$$

The resolvent exists except for $\lambda = \lambda_j$, an eigenvalue. The eigenvalues of $\mathcal{N}$ are the zeros of its *Fredholm determinant*

$$D(\lambda) = \Pi_{j=1}^{\infty}(1 - \lambda/\lambda_j) = \exp\{-\int_0^\lambda d\lambda \int \text{trace } N_\lambda(x,x)d\mu(x)\}. \tag{2.8}$$

The *Fredholm integral equation of the first kind*

$$\lambda\mathcal{N}p(x) = p(x)$$

has a solution provided that λ is an eigenvalue. For ν an eigenvalue, its general solution $p(x)$ is a linear combination of the eigenfunctions $\{p_j(x)\}$ corresponding to $\lambda_j = \lambda$.

Example 2.1 *Suppose that $Y, Z \in R$ and $\binom{Y}{Z} \sim \mathcal{N}_2(0, V)$ where $V = \begin{pmatrix} I & r \\ r & I \end{pmatrix}$. So V is the correlation matrix for $\binom{Y}{Z}$. For $j \in N_+,\ x \in R$ set $p_j(x) = H_j(x)/j!^{1/2}$ where $H_j(x)$ is the standard univariate Hermite polynomial. Then $\int p_j p_k \phi_I = \delta_{jk}$ and*

$$\sum_{j=0}^{\infty} r^j p_j(y)p_j(z) = \phi_C(y,z)/\phi_I(y)\phi_I(z).$$

This is Mehler's expansion for the standard bivariate normal distribution. Pearson gave an integrated version and Kibble extended it to an expansion for $\phi_V(x)/\phi_I(x)$ for $x \in R^k$ and V a correlation matrix. See for example (45.52) p127 and p321-2 of Kotz, Balakrishnan and Johnson (2000).

3 Functions of two variables with diagonal Jordan form

Diagonal Jordan form for matrices
Consider $N \in C^{s\times s}$ with eigenvalues $\nu_1, \cdots, \nu_s$, the roots of $\det(N - \nu I) = 0$. N is said to have *diagonal Jordan form* (DJF) if

$$N = P\Lambda Q^* \text{ where } PQ^* = I \text{ and } \Lambda = \mathrm{diag}(\nu_1, \cdots, \nu_s).$$

So

$$\begin{aligned} N &= \sum_{j=1}^{s} \nu_j p_j q_j^*, \ q_j^* p_k = \delta_{jk} \text{ where } P = (p_1, \cdots, p_s), \ Q = (q_1, \cdots, q_s), \\ \sum_{j=1}^{s} p_j q_j^* &= PQ^* = I_s = Q^*P = (q_j^* p_k). \end{aligned}$$

If N, Λ are real then P, Q can be taken as real. Also

$$NP = P\Lambda, \ Np_j = \nu_j p_j, \ N^*Q = Q\overline{\Lambda}, \ N^* q_j = \overline{\nu}_j q_j, \tag{3.1}$$

and for any complex α,

$$N^\alpha = P\Lambda^\alpha Q^* = \sum_{j=1}^{s} \nu_j^\alpha p_j q_j^*$$

provided that if $\det(N) = 0$, then α has non-negative real part. Suppose that

$$\nu_j = r_j e^{i\theta_j} \text{ and } r_1 = \cdots = r_R > r_0 = \max_{j=R+1}^{s} r_j. \tag{3.2}$$

Then

$$N^n = r_1^n C_n + O(r_0^n) \text{ where } C_n = \sum_{j=1}^{R} e^{in\theta_j} p_j q_j^* = O(1) \tag{3.3}$$

Taking $\alpha = -1$ gives the inverse of N when this exists.

Diagonal Jordan form for functions
Now consider a function $N(y, z) : \Omega^2 \to C^{s\times s}$. When N has diagonal Jordan form (for example when its eigenvalues are all different), then *the Fredholm equations of the first kind*,

$$\lambda \mathcal{N} p(y) = p(y), \ \bar{\lambda} \mathcal{N}^* q(z) = q(z),$$

or equivalently for $\nu = \lambda^{-1}$,

$$\mathcal{N} p(y) = \nu p(y), \ \mathcal{N}^* q(z) = \bar{\nu} q(z),$$

also have only a countable number of solutions, say $\{\lambda_j = \nu_j^{-1}, p_j(y), q_j(z), \ j \geq 1\}$ up to *arbitrary* constant multipliers for $\{p_j(y), \ j \geq 1\}$, satisfying the *bi-orthogonal* conditions

$$\int q_j^* p_k d\mu = \delta_{jk}.$$

These are called the *eigenvalues* and *right and left eigenfunctions* of (N, μ) or $\mathcal{N}$. Also

$$\begin{aligned} N(y,z) &= \sum_{j=1}^{\infty} \nu_j p_j(y) q_j(z)^* = P(y) \Lambda Q(z)^* \\ \text{where } \Lambda &= \text{diag}(\nu_1, \nu_2, \cdots),\ P(y) = (p_1(y), p_2(y), \cdots),\ Q(z) = (q_1(y), q_2(y), \cdots). \end{aligned} \quad (3.4)$$

with convergence in $L_2(\mu \times \mu)$, or pointwise and uniform under stronger conditions. If N is a real function and Λ is real, then P, Q can be taken as real functions

For $n \geq 1$,

$$N_n(y,z) = \mathcal{N}^{n-1} N(y,z) \quad (3.5)$$

satisfies

$$\begin{aligned} N_n(y,z) &= \sum_{j=1}^{\infty} \nu_j^n p_j(y) q_j(z)^* = P(y) \Lambda^n Q(z)^*, \quad (3.6) \\ \mathcal{N}^n p(y) &= \sum_{j=1}^{\infty} \nu_j^n p_j(y) \int q_j^* p d\mu. \end{aligned}$$

If (3.2) holds then

$$\begin{aligned} N_n(y,z) &= r_1^n C_n + O(r_0^n) \text{ where } C_n = \sum_{j=1}^{R} e^{in\theta_j} p_j(y) q_j(z)^* = O(1) \quad (3.7) \\ \mathcal{N}^n p(y) &= r_1^n c_n + O(r_0^n) \text{ where } c_n = \sum_{j=1}^{R} e^{in\theta_j} p_j(y) \int q_j^* p d\mu = O(1). \end{aligned}$$

The resolvent satisfies the equations of Section 2 except that (2.7) is replaced by

$$N_\lambda(y,z) = \sum_{j=1}^{\infty} p_j(y) q_j(z)^* / (\lambda_j - \lambda). \quad (3.8)$$

The *Fredholm determinant* is again given by (2.8). If only a finite number of eigenvalues are non-zero, the *kernel* $N(y,z)$ is said to be *degenerate*. (For example this holds if μ puts weight only at n points.) If $R = 1$, that is,

$$|\lambda_1| < |\lambda_j| \text{ for } j > 1, \quad (3.9)$$

then as $n \to \infty$,

$$\mathcal{N}^{n+1} f(y) / \mathcal{N}^n f(y) \to \lambda_1^{-1},\ f(y) \mathcal{N}^{n+1} / f(y) \mathcal{N}^n \to \lambda_1^{-1}.$$

This is one way to obtain the first eigenvalue λ_1 arbitrarily closely. Another is to use

$$\lambda_1^{-1} = \sup\{ \int g \mathcal{N} h d\mu : \int g h d\mu = 1 \} \text{ if } \lambda_1 > 0, \quad (3.10)$$

$$\lambda_1^{-1} = \inf\{ \int g \mathcal{N} h d\mu : \int g h d\mu = 1 \} \text{ if } \lambda_1 < 0. \quad (3.11)$$

The maximising/minimising functions are the first eigenfunctions $g = g_1, h = h_1$. These are unique up to a constant multiplier if (3.9) holds. If λ_1 is known, one can use

$$(\lambda_1 \mathcal{N})^n f(y) \to p_1(y) \int q_1^* f d\mu, \ f(y)^* (\lambda_1 \mathcal{N})^n \to q_1(y) \int f^* p_1 d\mu,$$

for any function $f : \Omega \to \mathcal{C}^s$, to approximate $p_1(y), q_1(y)$. One may now repeat the procedure on the operator $\mathcal{N}_1$ corresponding to

$$N_1(y,z) = N(y,z) - \nu_1 p_1(y) q_1(z)^*$$

to approximate $\lambda_2, p_2(y), q_2(z)$ assuming that the next eigenvalue in magnitude, λ_2, has multiplicity 1.

For further details see Withers (1974, 1975, 1978) and references.

4 Fredholm theory for non-diagonal Jordan form

Non-diagonal Jordan form for matrices

For $N \neq N^*$ a matrix in $C^{s \times s}$, its general Jordan form is

$$\begin{aligned} N &= PJP^{-1} \text{ where } J = diag(J_1, \cdots, J_r), \ J_j = J_{m_j}(\lambda_j), \ \sum_{j=1}^r m_j = s, \\ J_m(\lambda) &= \lambda I_m + U_m = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}, \end{aligned} \tag{4.1}$$

for some matrix P, and U_m is the $m \times m$ matrix with 1s on the superdiagonal and 0s elsewhere:

$$(U_m)_{jk} = \delta_{j,k-1}.$$

(See [1] for example. If N and its eigenvalues are real, then P can be taken as real.) So for $n \geq 1$,

$$N^n = PJ^nP^{-1} \text{ where } J^n = diag(J_1^n, \cdots, J_r^n).$$

By the Binomial Theorem,

$$J_m(\lambda)^n = \sum_{a=0}^n \binom{n}{a} \lambda^{n-a} U_m^a \text{ and } (U_m^a)_{jk} = \delta_{j,k-a}.$$

So $U_m^m = 0$. For example

$$J_2(\lambda)^n = \lambda^n I_2 + n\lambda^{n-1} U_2 = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

So N^n can be expanded in block matrix form

$$(N^n)_{jk} = \sum_{c=1}^r P_{jc} \ J_c^n \ P^{ck}$$

where we partition P and its inverse as

$$P = (P_{jk}: \ j,k = 1,\cdots,r), \ P^{-1} = (P^{jk}: \ j,k = 1,\cdots,r)$$

with elements P_{jk} and P^{jk} matrices in $C^{m_j \times m_k}$.

Alternatively setting

$$Q^* = P^{-1}, \ (P_1,\cdots,P_r) = P, \ (Q_1,\cdots,Q_r) = Q,$$

with $P_j, Q_j \in C^{s\times m_j}$, we have

$$\begin{aligned} N^n &= PJ^nQ^* = \sum_{j=1}^{r} P_j J_j^n Q_j^*, &(4.2)\\ \sum_{j=1}^{r} P_j Q_j^* &= PQ^* = I_s = Q^*P = (Q_j^* P_k) \end{aligned}$$

so that

$$Q_j^* P_j = I_{m_j}, \ Q_j^* P_k = 0 \in C^{m_j\times m_k} \text{ if } j \neq k.$$

P can be obtained as follows. Let p_{jk} be the kth column of P_j for $k = 1,\cdots,m_j$. Then

$$NP = PJ \Rightarrow NP_j = P_jJ_j \Rightarrow Np_{jk} = \lambda_j p_{jk} + p_{j,k-1} \text{ where } p_{j0} = 0. \tag{4.3}$$

So one first obtains p_{j1}, the right eigenvector of N, then $p_{j2},\cdots,p_{jm_j}$. This is called *the Jordan chain*. Q can either be obtained by inverting P or using

$$N^*Q = QJ^* \Rightarrow N^*Q_j = Q_jJ_j^* \Rightarrow N^*q_{jk} = \bar{\lambda}_j q_{jk} + q_{j,k+1} \text{ where } q_{j,m_j+1} = 0. \tag{4.4}$$

So one first computes q_{j,m_j}, the right eigenvector of N^* then $q_{j,m_j-1},\cdots,q_{j1}$. For large n and $\lambda \neq 0$,

$$J_n(\lambda)^n = \binom{n}{m-1}\lambda^{n-m+1}[U_m^{m-1} + O(1)]$$

and U_m^{m-1} is a matrix of 0's except for a 1 in its upper right corner. So if (3.2) holds and

$$m_1 = \cdots = m_M > \max_{j=M+1}^{R} m_j,$$

then

$$\begin{aligned} (N^n)_{jk} &= \binom{n}{M-1} r_1^{n-M+1}[D_n + O(n^{-1})] \\ \text{where } D_n &= \sum_{c=1}^{M} P_{jc}\ e^{i(n-M+1)\theta_c} U_M^{M-1} P^{ck} = \sum_{c=1}^{M} P_{jc}\ e^{i(n-M+1)\theta_c} U_M^{M-1} P^{ck} = O(1). \end{aligned}$$

See Withers and Nadarajah (2008) for more details.

Non-diagonal Jordan form for functions

Now consider $N: \Omega^2 \to C^{s\times s}$. Suppose that μ is a σ-finite measure on Ω and that N is not Hermitian, that is $N(y,z)^* \neq N(z,y)$. Its Jordan form is

$$N(y,z) = P(y)JP(z)^{-1} = \text{ where } J = diag(J_1, J_2,\cdots), \ J_j = J_{m_j}(\lambda_j) \tag{4.5}$$

for $P(y) : \Omega \to C^{s\times\infty}$ and $J_m(\lambda)$ of (4.1) above. So partitioning

$$P(y) = (P_{jk}(y) :\ j,k = 1,2,\cdots),\ P(z)^{-1} = (P^{jk}(z) :\ j,k = 1,2,\cdots),$$

with elements $P_{jk}(y)$ and $P^{jk}(z)$ matrix functions in $\Omega \to C^{m_j\times m_k}$, we can partition the nth iterated kernel, $N_n(y,z) = \mathcal{N}^{n-1}N(y,z)$ as

$$[N_n(y,z)]_{jk} = \sum_{c=1}^{\infty} P_{jc}(y)\ J_c^n\ P^{ck}(z).$$

Alternatively setting

$$Q(z)^* = P(z)^{-1},\ (P_1(y), P_2(y),\cdots) = P(y),\ (Q_1(z), Q_2(z),\cdots) = Q(z),$$

with $P_j(y), Q_j(z) : \Omega \to C^{s\times m_j}$, we have

$$N_n(y,z) \quad = \quad P(y)J^nQ(z)^* = \sum_{j=1}^{\infty} P_j(y)J_j^nQ_j(z)^*. \tag{4.6}$$

$P(y)$ can be obtained as follows. Let $p_{jk}(y)$ be the kth column of $P_j(y)$ for $k = 1,\cdots,m_j$. Then

$$\mathcal{N}P(y) = P(y)J \Rightarrow \mathcal{N}P_j(y) = P_j(y)J_j \Rightarrow \mathcal{N}p_{jk}(y) = \lambda_j p_{jk}(y) + p_{j,k-1}(y) \tag{4.7}$$

where $p_{j0}(y) = 0$. So one first obtains $p_{j1}(y)$, the right eigenfunction of N, then $p_{j2}(y),\cdots,p_{jm_j}(y)$. $Q(z)$ can either be obtained by inverting $P(z)$ or using

$$\mathcal{N}^*Q(z) = Q(z)J^* \Rightarrow \mathcal{N}^*Q_j(z) = Q_j(z)J_j^* \Rightarrow \mathcal{N}^*q_{jk}(z) = \bar{\lambda}_j q_{jk}(z) + q_{j,k+1}(z) \tag{4.8}$$

where $q_{j,m_j+1}(z) = 0$. So one first computes $q_{j,m_j}(z)$, the right eigenfunction of N^* then $q_{j,m_j-1}(z),\cdots,q_{j1}(z)$.

So if (3.2) holds and

$$m_1 = \cdots = m_M > \max_{j=M+1}^{R} m_j,$$

then

$$\begin{aligned}(N_n(y,z))_{jk} \quad &= \quad \binom{n}{M-1} r_1^{n-M+1}(D_{jkn}(y,z) + O(n^{-1}))\\ \text{where } D_{jkn}(y,z) \quad &= \quad \sum_{c=1}^{M}\ e^{i(n-M+1)\theta_c}P_{jc}(y)U_M^{M-1}P^{ck}(z) = O(1)\end{aligned}$$

has (a,b) element $\sum_{c=1}^{M}\ e^{i(n-M+1)\theta_c}[P_{jc}(y)]_{a1}[P^{ck}(z)]_{Mb}$.

Example 4.1

5 The SVD for functions of two variables

The SVD for matrices
Suppose that $N \in C^{s_1\times s_2}$. That is, N is a $s_1 \times s_2$ complex matrix. Denote its complex conjugate transpose by N^*. Its SVD is

$$N \quad = \quad PDQ^* = \sum_{j=1}^{r} \theta_j p_j q_j^* \text{ where } PP^* = I,\ QQ^* = I,\ r = \min(s_1,s_2), \tag{5.1}$$

$$P \quad = \quad (p_1,\cdots,p_{s_1}) \in C^{s_1\times s_1},\ Q = (q_1,\cdots,q_{s_2}) \in C^{s_2\times s_2},\ \theta_1 \geq \cdots \geq \theta_r > 0$$

and for $s_1 = s_2,\ s_1 > s_2,\ s_1 < s_2$

$$D = \Lambda,\ \begin{pmatrix}\Lambda\\ 0\end{pmatrix},\ (\Lambda, 0) \text{ respectively where } \Lambda = \mathrm{diag}(\theta_1, \cdots, \theta_r).$$

If N is real, then so are P and Q.

So for $s_1 > s_2$,

$$DD^* = \begin{pmatrix}\Lambda^2 & 0\\ 0 & 0\end{pmatrix},\ D^*D = \Lambda^2$$

and for $s_1 < s_2$,

$$DD^* = \Lambda^2,\ D^*D = \begin{pmatrix}\Lambda^2 & 0\\ 0 & 0\end{pmatrix}.$$

Compare this with (3.1). Also for $1 \leq j \leq r$,

$$Nq_j = \theta_j p_j,\ N^*p_j = \theta_j q_j,$$

for $r < j \leq s_1,\ Nq_j = 0$, and for $r < j \leq s_2,\ N^*p_j = 0$. Also since

$$NN^*P = PDD^*,\ N^*NQ = QD^*D,$$

the p_j is a right eigenvector of NN^* with eigenvalue θ_j^2 (or 0 if $r < j \leq s_1$) and the q_j is a right eigenvector of N^*N with eigenvalue θ_j^2 (or 0 if $r < j \leq s_2$). So (or by Section 2),

$$\begin{aligned} (NN^*)^n &= \sum_{j=1}^r \theta_j^{2n} p_j p_j^*,\ (N^*N)^n = \sum_{j=1}^r \theta_j^{2n} q_j q_j^* \text{ for } n \geq 1,\\ (NN^*)^n N &= \sum_{j=1}^r \theta_j^{2n+1} p_j q_j^*,\ (N^*N)^n N^* = \sum_{j=1}^r \theta_j^{2n+1} q_j p_j^* \text{ for } n \geq 0. \end{aligned} \tag{5.2}$$

These do not depend on the vectors $\{p_j, q_j, j \geq r\}$.

So if $\theta_1 = \cdots = \theta_M > \theta_{M+1}$, then we have the approximations for $n \geq 0$,

$$\begin{aligned} (NN^*)^n &= \theta_1^{2n} \sum_{j=1}^M p_j p_j^* + O(\theta_{M+1}^{2n}),\ (N^*N)^n = \theta_1^{2n} \sum_{j=1}^M q_j q_j^* + O(\theta_{M+1}^{2n}) \text{ for } n \geq 1,\\ (NN^*)^n N &= \theta_1^{2n+1} \sum_{j=1}^r p_j q_j^* + O(\theta_{M+1}^{2n+1}),\ (N^*N)^n N^* = \theta_1^{2n+1} \sum_{j=1}^r q_j p_j^* + O(\theta_{M+1}^{2n+1}). \end{aligned} \tag{5.3}$$

But

$$I_{s_1} = (NN^*)^0 = \sum_{j=1}^{s_1} p_j p_j^*,\ I_{s_1} = (N^*N)^0 = \sum_{j=1}^{s_2} q_j q_j^*.$$

If $s_1 = s_2$ and N is non-singular, its inverse is

$$N^{-1} = Q\Lambda^{-1}P^*.$$

However unlike Jordan form, the SVD does not give a nice form for powers of N.

Now suppose $\Omega \subset R^p$ and that μ is a σ-finite measure on Ω. Consider a function $N(y, z) : \Omega^2 \to C^{s_1 \times s_2}$.

The equations

$$\mathcal{N}q(y) = \theta p(y),\ \mathcal{N}^* p(z) = \theta q(z),$$

have a countable number of solutions, say $\{\theta_j, p_j(y), q_j(z),\ j \geq 1\}$ satisfying

$$\int p_j^* p_k d\mu = \int q_j^* q_k d\mu = \delta_{jk}.$$

The *singular values* $\{\theta_j\}$ may be taken as real, non-negative and non-increasing. (For convenience we have included $\theta_j = 0$.) $\{p_j(y)\}$ and $\{q_j(y)\}$ are the right eigenfunctions of $\mathcal{N}\mathcal{N}^*$ and $\mathcal{N}^*\mathcal{N}$ respectively, with eigenvalues $\{\theta_j^2\}$. Also in $L_2(\mu \times \mu)$

$$N(y, z) = \sum_{j=1}^{\infty} \theta_j p_j(y) q_j(z)^*. \tag{5.4}$$

If N is real, then so are $\{p_j, q_j\}$. By (5.4), for $n \geq 0$,

$$\begin{aligned}
(\mathcal{N}\mathcal{N}^*)^n N(y, z) &= \sum_{j=1}^{\infty} \theta_j^{2n+1} p_j(y) q_j(z)^*, \\
(\mathcal{N}^*\mathcal{N})^n N(y, z)^* &= \sum_{j=1}^{\infty} \theta_j^{2n+1} q_j(z) p_j(y)^*, \\
\mathcal{N}^*(\mathcal{N}\mathcal{N}^*)^n N(y, z) &= \sum_{j=1}^{\infty} \theta_j^{2n+2} q_j(y) q_j(z)^*, \\
\mathcal{N}(\mathcal{N}^*\mathcal{N})^n N(y, z)^* &= \sum_{j=1}^{\infty} \theta_j^{2n+2} p_j(z) p_j(y)^*,
\end{aligned}$$

$$\begin{aligned}
(\mathcal{N}\mathcal{N}^*)^n p(y) &= \sum_{j=1}^{\infty} \theta_j^{2n} p_j(y) \int p_j^* p d\mu, \\
(\mathcal{N}^*\mathcal{N})^n q(z) &= \sum_{j=1}^{\infty} \theta_j^{2n} q_j(z) \int q_j^* q d\mu, \\
(\mathcal{N}\mathcal{N}^*)^n \mathcal{N} q(y) &= \sum_{j=1}^{\infty} \theta_j^{2n+1} p_j(y) \int q_j^* q d\mu, \\
(\mathcal{N}^*\mathcal{N})^n \mathcal{N}^* p(y) &= \sum_{j=1}^{\infty} \theta_j^{2n+1} q_j(y) \int p_j^* p d\mu, \\
\int \text{trace } \mathcal{N}^*(\mathcal{N}\mathcal{N}^*)^n N(y, z)|_{z=y} d\mu(y) &= \int \text{trace } \mathcal{N}(\mathcal{N}^*\mathcal{N})^n N(y, z)|_{z=y} d\mu(y) = \sum_{j=1}^{\infty} \theta_j^{2n+2}.
\end{aligned}$$

So if $\theta_1 = \cdots = \theta_M > \theta_{M+1}$, then we have approximations such as

$$(\mathcal{N}\mathcal{N}^*)^n p(y) = \theta_1^{2n} \sum_{j=1}^{\infty} p_j(y) \int p_j^* p d\mu + O(\theta_{M+1}^{2n})$$

if $p_j(y) \int p_j^* p d\mu = O(1)$ for $j > M$. So for iterations of $\mathcal{N}^*\mathcal{N}$ or $\mathcal{N}\mathcal{N}^*$ the most important parameter is the largest singular value.

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