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Derivative Polynomials and Closed-Form Higher Derivative Formulae

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Abstract. In a recent paper, Adamchik [V.S. Adamchik, On the Hurwitz function for rational arguments, *Appl. Math. Comp.*, **187** (2007), 3–12] expressed in a closed form symbolic derivatives of four functions belonging to the class of functions whose derivatives are polynomials in terms of the same functions. In this sequel, simple closed-form higher derivative formulae which involve the Carlitz-Scoville higher order tangent and secant numbers are derived for eight trigonometric and hyperbolic functions.

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1. Introduction

Recently, Adamchik [1, Eqs. 26, 30, 31 and 32] expressed in a closed form symbolic derivatives of four functions belonging to the class of functions whose derivatives are polynomials in terms of the same functions. In particular, he completely solved a long-standing problem of finding a closed-form expression for the higher derivatives of the cotangent function (see, for instance, [2–5] and [6, p. 161]) by showing that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\cot(x)=(2\imath)^n\big(\cot(x)-\imath\big)\sum_{k=1}^{n}\frac{k!}{2}\left\{{n \atop k}\right\}\big(\imath\cot(x)-1\big)^k,$$

where $\left\{{n \atop k}\right\}$ are the Stirling subset numbers.

In this sequel to the work of Adamchik, by using the derivative polynomials introduced by Hoffmann [4, 7] (but see also references [2, 8]), we further investigate the aforementioned class of functions and derive simple explicit closed-form higher derivative formulae for cotangent, tangent, cosecant and secant functions and their hyperbolic analogs. The formulae obtained involve the Carlitz-Scoville higher order tangent and secant numbers [9, 10].

2. Derivative Polynomials

Following Hoffman [4, 7] we, by means of the exponential generating functions, define two sequences of polynomials, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$, $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, where $\mathbb{N}:=\{1,2,3,\ldots\}$, which are here referred to as the derivative polynomials for tangent

$$P(x,t):=\frac{x+\tan(t)}{1-x\tan(t)}=\sum_{n=0}^{\infty}P_n(x)\,\frac{t^n}{n!} \tag{2.1}$$

and the derivative polynomials for secant

$$Q(x,t) := \frac{\sec(t)}{1 - x\tan(t)} = \sum_{n=0}^{\infty} Q_n(x)\,\frac{t^n}{n!}. \tag{2.2}$$

Equivalently, they may be defined by the formulae

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\tan(x) = P_n\big(\tan(x)\big) \tag{2.1*}$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\sec(x) = \sec(x)\,Q_n\big(\tan(x)\big), \tag{2.2*}$$

as it can be easily shown that

$$\begin{aligned} P_n\big(\tan(x)\big) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\,P\big(\tan(x),t\big)\Big|_{t=0} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\tan\,(x+t)\Big|_{t=0} \\ &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\tan\,(x+t)\Big|_{t=0} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\tan(x) \end{aligned}$$

and

$$\begin{aligned} &\sec(x)\,Q_n\big(\tan(x)\big) = \sec(x)\;\frac{\mathrm{d}^n}{\mathrm{d}t^n}\,\frac{\sec\big[\arctan\big(\tan(x)\big)+t\big]}{\sec\big[\arctan\big(\tan(x)\big)\big]}\Bigg|_{t=0} \\ &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\,\sec\,(x+t)\Big|_{t=0} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\sec\,(x+t)\Big|_{t=0} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\sec(x). \end{aligned}$$

By making use of the chain rule it follows from (2.1*) that $P_n(x)$ satisfy

$$P_0(x) = x, \qquad P_n(x) = (1+x^2)\,P_{n-1}^{'}(x) \qquad (n \in \mathbb{N}) \tag{2.1**}$$

and, similarly, from (2.2*) that

$$Q_0(x) = 1, \qquad Q_n(x) = (1+x^2)\,Q_{n-1}^{'}(x) + x\,Q_{n-1}(x) \qquad (n \in \mathbb{N}). \tag{2.2**}$$

Another important and readily deducible property of $P_n(x)$ and $Q_n(x)$ is

$$P_n(-x) = (-1)^{n+1}\,P_n(x) \quad \text{and} \quad Q_n(-x) = (-1)^n Q_n(x) \quad (n \in \mathbb{N}_0). \tag{2.3}$$

Upon noting that $\tan(x+\frac{\pi}{2}) = -\cot(x)$ and $\sec(x+\frac{\pi}{2}) = -\csc(x)$ and using (2.1*) and (2.2*) in conjunction with (2.5), we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\cot(x) = (-1)^n\,P_n\big(\cot(x)\big) \tag{2.4}$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\csc(x) = (-1)^n\,\csc(x)\,Q_n\big(\cot(x)\big). \tag{2.5}$$

We also consider the hyperbolic analogs of the derivative polynomials and define the derivative polynomials for hyperbolic tangent

$$\boldsymbol{\mathcal{P}}(x,t) := \frac{x + \tanh(t)}{1 + x\tanh(t)} = \sum_{n=0}^{\infty} \boldsymbol{\mathcal{P}}_n(x)\,\frac{t^n}{n!} \tag{2.6}$$

and for hyperbolic secant

$$\boldsymbol{\mathcal{Q}}(x,t) := \frac{\operatorname{sech}(t)}{1 + x\tanh(t)} = \sum_{n=0}^{\infty} \boldsymbol{\mathcal{Q}}_n(x)\,\frac{t^n}{n!}, \tag{2.7}$$

or, alternatively, as follows

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\tanh(x) = \boldsymbol{\mathcal{P}}_n\big(\tanh(x)\big) \tag{2.6*}$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\operatorname{sech}(x) = \operatorname{sech}(x)\,\boldsymbol{\mathcal{Q}}_n\big(\tanh(x)\big). \tag{2.7*}$$

These polynomials also may be generated by recurrence relations

$$\boldsymbol{\mathcal{P}}_0(x) = x, \qquad \boldsymbol{\mathcal{P}}_n(x) = (1-x^2)\,\boldsymbol{\mathcal{P}}'_{n-1}(x) \qquad (n\in\mathbb{N}) \tag{2.6**}$$

and

$$\boldsymbol{\mathcal{Q}}_0(x) = 1, \qquad \boldsymbol{\mathcal{Q}}_n(x) = (1-x^2)\,\boldsymbol{\mathcal{Q}}'_{n-1}(x) - x\,\boldsymbol{\mathcal{Q}}_{n-1}(x) \qquad (n\in\mathbb{N}) \tag{2.7**}$$

and they satisfy the symmetry relations

$$\boldsymbol{\mathcal{P}}_n(-x) = (-1)^{n+1}\,\boldsymbol{\mathcal{P}}_n(x) \quad\text{and}\quad \boldsymbol{\mathcal{Q}}_n(-x) = (-1)^n\,\boldsymbol{\mathcal{Q}}_n(x) \quad (n\in\mathbb{N}_0). \tag{2.8}$$

Note that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\coth(x) = \boldsymbol{\mathcal{P}}_n\big(\coth(x)\big) \quad\text{and}\quad \frac{\mathrm{d}^n}{\mathrm{d}x^n}\operatorname{csch}(x) = \operatorname{csch}(x)\,\boldsymbol{\mathcal{Q}}_n\big(\coth(x)\big). \tag{2.9}$$

3. Higher Derivative Formulae

The *tangent numbers (of order k)* $T(n,k)$ and *secant numbers (of order k)* $S(n,k)$ are respectively defined by (see [9, p. 428] and [10, p. 305])

$$\tan^k(t) = \sum_{n=k}^{\infty} T(n,k)\,\frac{t^n}{n!} \qquad (k\in\mathbb{N}) \tag{3.1}$$

and

$$\sec(t)\tan^k(t) = \sum_{n=k}^{\infty} S(n,k)\,\frac{t^n}{n!} \qquad (k\in\mathbb{N}_0). \tag{3.2}$$

It is obvious, by parity considerations, that $T(n,k)\neq 0$ is only when $1\le k\le n$ and either both n and k are even or both n and k are odd. The same applies to $S(n,k)$ when $0\le k\le n$. Observe that, $T(n,1)$ and $S(n,0)$ are, in fact, well-known the tangent and Euler numbers. Moreover, by (2.1) and (3.1), we have

$$P_n(0) = T(n,1) \qquad\text{and}\qquad Q_n(0) = S(n,0). \tag{3.3}$$

Our main results are as follows.

Theorem 1. *Assume that n and k are nonnegative integers and let $P_n(x)$ and $Q_n(x)$ be the polynomials as defined by* (2.1) *and* (2.2). *Then, in terms of the tangent numbers of order k, $T(n,k)$, given by* (3.1), *we have:*

$$P_n(x) = T(n,1) + \sum_{k=1}^{n+1} \frac{1}{k}\,T(n+1,k)\,x^k, \tag{3.4}$$

and, in terms of the secant numbers of order k, $S(n,k)$, given by (3.2), we have:

$$Q_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k. \tag{3.5}$$

Theorem 2. *Let $\boldsymbol{\mathcal{P}}_n(x)$ and $\boldsymbol{\mathcal{Q}}_n(x)$ be the polynomials defined by (2.6) and (2.7). Then:*

$$\boldsymbol{\mathcal{P}}_n(x) = (-1)^{\frac{n-1}{2}}\, T(n,1) + \sum_{k=1}^{n+1} \frac{(-1)^{\frac{n+k-1}{2}}}{k}\, T(n+1,k)\, x^k \tag{3.6}$$

and

$$\boldsymbol{\mathcal{Q}}_n(x) = \sum_{k=0}^{n} (-1)^{\frac{n+k}{2}}\, S(n,k)\, x^k, \tag{3.7}$$

where $T(n,k)$ and $S(n,k)$ are the numbers (3.1) and (3.2).

Corollary 1. *In terms of the tangent and secant numbers of order k, $T(n,k)$ and $S(n,k)$, for $n \in \mathbb{N}_0$, we have:*

(a) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \tan(x) = T(n,1) + \sum_{k=1}^{n+1} \frac{1}{k}\, T(n+1,k) \tan^k(x);$$

(b) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \sec(x) = \sec(x) \sum_{k=0}^{n} S(n,k)\, \tan^k(x);$$

(c) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \cot(x) = (-1)^n \left[T(n,1) + \sum_{k=1}^{n+1} \frac{1}{k}\, T(n+1,k)\, \cot^k(x) \right];$$

(d) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \csc(x) = (-1)^n \csc(x) \sum_{k=0}^{n} S(n,k)\, \cot^k(x).$$

Corollary 2. *In terms of the tangent and secant numbers of order k, $T(n,k)$ and $S(n,k)$, for $n \in \mathbb{N}_0$, we have:*

(a) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \tanh(x) = (-1)^{\frac{n-1}{2}}\, T(n,1) + \sum_{k=1}^{n+1} \frac{(-1)^{\frac{n+k-1}{2}}}{k}\, T(n+1,k) \tanh^k(x);$$

(b) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \operatorname{sech}(x) = \operatorname{sech}(x) \sum_{k=0}^{n} (-1)^{\frac{n+k}{2}}\, S(n,k)\, \tanh^k(x);$$

(c) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \coth(x) = (-1)^{\frac{n-1}{2}}\, T(n,1) + \sum_{k=1}^{n+1} \frac{(-1)^{\frac{n+k-1}{2}}}{k}\, T(n+1,k)\, \coth^k(x);$$

(d) $$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \operatorname{csch}(x) = \operatorname{csch}(x) \sum_{k=0}^{n} (-1)^{\frac{n+k}{2}}\, S(n,k)\, \coth^k(x).$$

Remark 1. We remark that, since, as detailed above, $T(n,k)$ and $S(n,k)$ are nonzero only under certain conditions, then the above given formulae can be written (for computational purposes) in somewhat simplified form. For instance, we have

$$P_{2m-1}(x) = T(2m-1,1) + \sum_{r=1}^{m} \frac{1}{2\,r}\, T(2m,2r)\, x^{2r} \qquad (m \in \mathbb{N})$$

and

$$P_{2m}(x) = \sum_{r=0}^{m} \frac{1}{2\,r+1}\, T(2m+1,2r+1)\, x^{2r+1} \qquad (m \in \mathbb{N}_0).$$

Proof of Theorem 1. In order to prove the formula (3.4) we first note that the generating function of the polynomials $P_n(x)$ can be rewritten as

$$P(x,t) = \big(x + \tan(t)\big) \sum_{k=0}^{\infty} \big(x\,\tan(t)\big)^k = x + \big(1+x^2\big) \sum_{k=1}^{\infty} x^{k-1} \tan^k(t)$$

which, by making use of the definition of $T(n,k)$ in (3.1) and the elementary double series identities [11, p. 57, Eq. (2)]

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} A(k,n) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A(k,n+k) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} A(k,n),$$

becomes

$$\begin{aligned} P(x,t) &= x + \big(1+x^2\big) \sum_{k=1}^{\infty} x^{k-1} \sum_{n=k}^{\infty} T(n,k)\,\frac{t^n}{n!} \\ &= x + \sum_{n=1}^{\infty} \big(1+x^2\big) \left(\sum_{k=1}^{n} T(n,k)\,x^{k-1}\right) \frac{t^n}{n!}. \end{aligned} \tag{3.8}$$

On the other hand, by (2.1) in conjunction with the recurrence (2.1**), we have

$$P(x,t) = P_0(x) + \sum_{n=1}^{\infty} P_n(x)\,\frac{t^n}{n!} = P_0(x) + \sum_{n=1}^{\infty} \big(1+x^2\big)\, P'_{n-1}(x)\,\frac{t^n}{n!} \tag{3.9}$$

and thus comparing (3.8) with (3.9) clearly yields

$$P'_{n-1}(x) = \sum_{k=1}^{n} T(n,k)\,x^{k-1}$$

so that we find by integration that

$$P_n(x) = P_n(0) + \sum_{k=1}^{n+1} \frac{1}{k}\, T(n+1,k)\, x^k. \tag{3.10}$$

Now, in view of (3.3), the desired result (3.4) follows from (3.10).

Similarly, along the same lines, we have

$$\begin{aligned} Q(x,t) &= \sum_{k=0}^{\infty} \sec(t)\tan^k(t)\,x^k = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} S(n,k)\,\frac{t^n}{n!}\right) x^k \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S(n,k)\,x^k\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Q_n(x)\,\frac{t^n}{n!}, \end{aligned}$$

and in this way we arrive at the second needed result (3.5).

Proof of Theorem 2. We first verify that

$$\boldsymbol{\mathcal{P}}_n(x) = \imath^{n-1} P_n(\imath\, x) \qquad \text{and} \qquad \boldsymbol{\mathcal{Q}}_n(x) = \imath^n Q_n(\imath\, x),$$

and then, upon applying these last identities and Theorem 1, the two assertions of Theorem 2 follow.

Proof of Corollaries 1 and 2. The parts (a)–(d) of Corollary 1, in view of Theorem 1, are direct consequences of, respectively, the formulae (2.1*), (2.2*), (2.4) and (2.5). Similarly, the parts (a)–(d) of Corollary 2 follow by Theorem 2 and, respectively, formulae (2.6*), (2.7*) and (2.9).

4. Concluding Remarks

We have explicitly expressed the trigonometric and hyperbolic derivative polynomials in a closed form in the terms of the higher tangent and secant numbers, $T(n,k)$ and $S(n,k)$. The first few derivative polynomials are

$$\left.\begin{matrix} P_0(x) \\ \boldsymbol{\mathcal{P}}_0(x) \end{matrix}\right\} = x, \qquad \left.\begin{matrix} P_1(x) \\ \boldsymbol{\mathcal{P}}_1(x) \end{matrix}\right\} = \pm x^2 + 1, \qquad \left.\begin{matrix} P_2(x) \\ \boldsymbol{\mathcal{P}}_2(x) \end{matrix}\right\} = 2x^3 \pm 2x,$$

$$\left.\begin{matrix} P_3(x) \\ \boldsymbol{\mathcal{P}}_3(x) \end{matrix}\right\} = \pm 6x^4 + 8x^2 \pm 2, \qquad \left.\begin{matrix} P_4(x) \\ \boldsymbol{\mathcal{P}}_4(x) \end{matrix}\right\} = 24x^5 \pm 40x^3 + 16x,$$

$$\left.\begin{matrix} P_5(x) \\ \boldsymbol{\mathcal{P}}_5(x) \end{matrix}\right\} = \pm 120x^6 + 240x^4 \pm 136x^2 + 16,$$

$$\left.\begin{matrix} P_6(x) \\ \boldsymbol{\mathcal{P}}_6(x) \end{matrix}\right\} = 720x^7 \pm 1680x^5 + 1232x^3 \pm 272x,$$

and

$$\left.\begin{matrix} Q_0(x) \\ \boldsymbol{\mathcal{Q}}_0(x) \end{matrix}\right\} = 1, \qquad \left.\begin{matrix} Q_1(x) \\ \boldsymbol{\mathcal{Q}}_1(x) \end{matrix}\right\} = \pm x, \qquad \left.\begin{matrix} Q_2(x) \\ \boldsymbol{\mathcal{Q}}_2(x) \end{matrix}\right\} = 2x^2 \pm 1,$$

$$\left.\begin{matrix} Q_3(x) \\ \boldsymbol{\mathcal{Q}}_3(x) \end{matrix}\right\} = \pm 6x^3 + 5x, \qquad \left.\begin{matrix} Q_4(x) \\ \boldsymbol{\mathcal{Q}}_4(x) \end{matrix}\right\} = 24x^4 \pm 28x^2 + 5,$$

$$\left.\begin{matrix} Q_5(x) \\ \boldsymbol{\mathcal{Q}}_5(x) \end{matrix}\right\} = \pm 120x^5 + 180x^3 \pm 61x,$$

$$\left.\begin{matrix} Q_6(x) \\ \boldsymbol{\mathcal{Q}}_6(x) \end{matrix}\right\} = 720x^6 \pm 1320x^4 + 662x^2 \pm 61.$$

It should be noted that the numbers $T(n,k)$ and $S(n,k)$ appear to be insufficiently investigated but simplicity of the the above-found formulae suggests that they would be interesting ones and well worthy of further study.

In conclusion, we confine ourselves to give only one of numerous consequences of the results presented in Section 3. The following formula

$$\begin{aligned} &\zeta\big(n, 1-x\big) + (-1)^n\, \zeta\big(n, 1-x\big) \\ &= \frac{(-1)^n \pi^n}{(n-1)!} \left[T(n-1,1) + \sum_{k=1}^{n} \frac{1}{k}\, T(n,k)\, \cot^k(\pi x) \right] (n \in \mathbb{N} \setminus \{1\}; 0 < x < 1) \end{aligned}$$

(*c.f.* [1, p. 8, Theorem 2.2]) is obtained from our Corollary 1(c) and the reflection formula for the Hurwitz zeta function $\zeta(s,a)$ (see, for instance, [12, Sec. 2.2]).

Remark. Since submitting this paper (June 7, 2008) the author has learned of the following related publications: Refs. $[13-15]$

References

[1] V.S. Adamchik, On the Hurwitz function for rational arguments, *Appl. Math. Comp.*, **187** (2007), 3–12.

[2] D.E. Knuth and T.J. Buckholtz, Computation of tangent, Euler, and Bernoulli numbers, *Math. Comp.*, **21** (1967), 663-688.

[3] T.M. Apostol, Dirichlet L-functions and character power sums, *J. Numb. Theory*, **2** (1970), 223–234.

[4] M.E. Hoffman, Derivative polynomials for tangent and secant, *Amer. Math. Monthly*, **102** (1995), 23–30.

[5] K.S. Kölbig, The polygamma function and the derivatives of the cotangent function for rational arguments, CERN-IT-Reports, CERN-CN-96-005, 1996.

[6] B.C. Berndt, *Ramanujan's Notebooks*, Part I, Springer Verlag, New York, 1985.

[7] M.E. Hoffman, Derivative polynomials, Euler polynomials, and associated integer sequences, *Electron. J. Combin.*, **6** (1999), R21.

[8] C. Krishanamachary and R.M. Bhimsenrao, On a table for calculating Eulerian numbers based on a new method, *Proc. London Math. Soc.*, Ser. 2, **22** (1923), 73–80.

[9] L. Carlitz and R. Scoville, Tangent numbers and operators, *Duke Math. J.*, **39** (1972), 413–429.

[10] L. Carlitz, Permutations, sequences and special functions, *SIAM Review*, **17** (1975), 298–322.

[11] E.D. Rainville, *Special functions*, Macmillan, New York, 1960.

[12] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[13] K.N. Boyadzhiev, Derivative polynomials for tanh, tan, sech and sec in explict form, Fibonacci Quart. 45 (2007),291-303.

[14] G.R. Franssens, Functions with derivatives given by polynomials in the function itself or a related function, Analysis Mathematica 33 (2007), 17-36.

[15] C.-H. Chang, C.-W. Ha, Central factorial numbers and values of Bernoulli and Euler polynomials at rationals, Numer. Funct. Anal. Optimiz. 30 (2009), 214-226.