Leitmann's direct method of optimization for absolute extrema of certain problems of the calculus of variations on time scales^{*}

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Abstract

The fundamental problem of the calculus of variations on time scales concerns the minimization of a delta-integral over all trajectories satisfying given boundary conditions. This includes the discrete-time, the quantum, and the continuous/classical calculus of variations as particular cases. In this note we follow Leitmann's direct method to give explicit solutions for some concrete optimal control problems on an arbitrary time scale.

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1 Introduction

The calculus on time scales is a recent field that unifies the theories of difference and differential equations. It has found applications in several contexts that require simultaneous modeling of discrete and continuous data, and is nowadays under strong current research in several different areas [6, 7].

The area of the calculus of variations on time scales, which we are concerned in this paper, was born in 2004 [5] and is now receiving a lot of attention, both for theoretical and practical reasons - see [1, 2, 3, 4, 12,13, 21, 22, 23 and references therein. Although the theory is already well developed in many directions, a crucial problem still persists: solving Euler-Lagrange delta-differential equations on arbitrary time scales is difficult or even impossible. As a consequence, there is a lack of concrete variational problems for which a solution is known. In this paper we follow a different approach. We show that the direct method introduced by Leitmann in the sixties of the XX century [15] can also be applied to variational problems on time scales. Leitmann's method is a venerable forty years old method that has shown through the times to be an universal and useful method in several different contexts – see, e.g., [8, 9, 10, 11, 16, 17, 18, 19, 20, 24, 25, 26]. Here we provide concrete examples of problems of the calculus of variations on time scales for which a global minimizer is easily found by the application of Leitmann's direct approach.

2 Preliminaries on time scales

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the set \mathbb{R} of real numbers. It is a model of time. Besides standard cases of \mathbb{R} (continuous time) and \mathbb{Z} (discrete time), many different models are used. For each time scale \mathbb{T} the following operators are used:

- the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}, \ \sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t < \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$;
- the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}, \ \rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t > \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$;
- the graininess function $\mu : \mathbb{T} \to [0, \infty), \, \mu(t) := \sigma(t) t.$

For $\mathbb{T} = \mathbb{R}$ one has $\sigma(t) = t = \rho(t)$ and $\mu(t) \equiv 0$ for any $t \in \mathbb{R}$. For $\mathbb{T} = \mathbb{Z}$ one has $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) \equiv 1$ for every $t \in \mathbb{Z}$.

A point $t \in \mathbb{T}$ is called: (i) right-scattered if $\sigma(t) > t$, (ii) right-dense if $\sigma(t) = t$, (iii) left-scattered if $\rho(t) < t$, (iv) left-dense if $\rho(t) = t$, (v) isolated if it is both left-scattered and right-scattered, (vi) dense if it is both left-dense and right-dense. If $\sup \mathbb{T}$ is finite and left-scattered we set $\mathbb{T}^{\kappa} := \mathbb{T} \setminus \{\sup \mathbb{T}\};$ otherwise, $\mathbb{T}^{\kappa} := \mathbb{T}$.

We assume that a time scale \mathbb{T} has the topology that it inherits from the real numbers with the standard topology. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. The *delta derivative* of f at t is the real number $f^{\Delta}(t)$ with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t such that

$$|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. We say that f is *delta-differentiable* on \mathbb{T} provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

We note that if $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if f is differentiable in the ordinary sense at t. Then, $f^{\Delta}(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^{\Delta}(t) = f(t+1) - f(t)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at the right-dense points in \mathbb{T} and its left-sided limits exist at all left-dense points in \mathbb{T} . The set of all rd-continuous functions is denoted by C_{rd} . Similarly, C_{rd}^1 will denote the set of functions from C_{rd} whose delta derivative belongs to C_{rd} . A continuous function f is *piecewise rd-continuously delta-differentiable* (we write $f \in C_{prd}^1$) if f is continuous and f^{Δ} exists for all, except possibly at finitely many $t \in \mathbb{T}^{\kappa}$, and $f^{\Delta} \in C_{rd}$. It is known that piecewise rd-continuous functions possess an *antiderivative*, i.e., there exists a function F with $F^{\Delta} = f$, and in this case the *delta-integral* is defined by $\int_c^d f(t)\Delta t = F(d) - F(c)$ for all $c, d \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then

$$\int_{a}^{b} f(t)\Delta t = \int_{a}^{b} f(t)dt,$$

where the integral on the right hand side is the usual Riemann integral; if $\mathbb{T} = h\mathbb{Z}, h > 0$, and a < b, then

$$\int_{a}^{b} f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h \cdot f(kh)$$

For $f: \mathbb{T} \to X$, where X is an arbitrary set, we define $f^{\sigma} := f \circ \sigma$.

3 Leitmann's direct method on time scales

Let \mathbb{T} be a time scale with at least two points. Throughout we let $a, b \in \mathbb{T}$ with a < b. For an interval $[a, b] \cap \mathbb{T}$ we simply write [a, b].

The problem of the calculus of variations on time scales consists in minimizing

$$\mathcal{L}[x] = \int_{a}^{b} L(t, x^{\sigma}(t), x^{\Delta}(t)) \Delta t$$

over all $x \in C^1_{prd}([a, b], \mathbb{R})$ satisfying the boundary conditions

$$x(a) = \alpha, \ x(b) = \beta, \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$ and $L : [a, b]^{\kappa} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We assume that $(t, y, v) \to L(t, y, v)$ has partial continuous derivatives L_y and L_v , respectively with respect to the second and third arguments, for all $t \in [a, b]^{\kappa}$, and $L(\cdot, y, v)$, $L_y(\cdot, y, v)$ and $L_v(\cdot, y, v)$ are piecewise rd-continuous in t for all $x \in C^1_{prd}([a, b], \mathbb{R})$. A function $x \in C^1_{prd}([a, b], \mathbb{R})$ is said to be admissible if it satisfies the boundary conditions (1).

Let $L : [a, b]^{\kappa} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We assume that $(t, y, v) \to L(t, y, v)$ has partial continuous derivatives \tilde{L}_y and \tilde{L}_v , respectively with respect to the second and third arguments, for all $t \in [a, b]^{\kappa}$, and $\tilde{L}(\cdot, y, v)$, $\tilde{L}_y(\cdot, y, v)$ and $\tilde{L}_v(\cdot, y, v)$ are piecewise rd-continuous in t for all $x \in C^1_{prd}([a, b], \mathbb{R})$. Consider the integral

$$\tilde{\mathcal{L}}[\tilde{x}] = \int_{a}^{b} \tilde{L}(t, \tilde{x}^{\sigma}(t), \tilde{x}^{\Delta}(t)) \Delta t$$

Lemma 3.1 (Leitmann's fundamental lemma). Let $x = z(t, \tilde{x})$ be a transformation having an unique inverse $\tilde{x} = \tilde{z}(t, x)$ for all $t \in [a, b]$ such that there is a one-to-one correspondence

$$x(t) \Leftrightarrow \tilde{x}(t),$$

for all functions $x \in C^1_{prd}([a, b], \mathbb{R})$ satisfying (1) and all functions $\tilde{x} \in C^1_{prd}([a, b], \mathbb{R})$ satisfying

$$\tilde{x} = \tilde{z}(a, \alpha), \quad \tilde{x} = \tilde{z}(b, \beta).$$
 (2)

If the transformation $x = z(t, \tilde{x})$ is such that there exists a function $G : [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfying the functional identity

$$L(t, x^{\sigma}(t), x^{\Delta}(t)) - \tilde{L}(t, \tilde{x}^{\sigma}(t), \tilde{x}^{\Delta}(t)) = G^{\Delta}(t, \tilde{x}(t)), \qquad (3)$$

then if \tilde{x}^* yields the extremum of $\tilde{\mathcal{L}}$ with \tilde{x}^* satisfying (2), $x^* = z(t, \tilde{x}^*)$ yields the extremum of \mathcal{L} for x^* satisfying (1).

Proof. The proof is similar in spirit to Leitmann's proof [15, 16, 17, 20]. Let $x \in C_{prd}^1([a, b], \mathbb{R})$ satisfy (1) and define functions $\tilde{x} \in C_{prd}^1([a, b], \mathbb{R})$ through the formula $\tilde{x} = \tilde{z}(t, x), a \leq t \leq b$. Then $\tilde{x} \in C_{prd}^1([a, b], \mathbb{R})$ and satisfies (2). Moreover, as a result of (3), it follows that

$$\begin{aligned} \mathcal{L}[x] - \tilde{\mathcal{L}}[\tilde{x}] &= \int_{a}^{b} L(t, x^{\sigma}(t), x^{\Delta}(t)) \Delta t - \int_{a}^{b} \tilde{L}(t, \tilde{x}^{\sigma}(t), \tilde{x}^{\Delta}(t)) \Delta t \\ &= \int_{a}^{b} G^{\Delta}(t, \tilde{x}(t)) \Delta t = G(b, \tilde{x}(b)) - G(a, \tilde{x}(a)) \\ &= G(b, \tilde{z}(b, \beta)) - G(a, \tilde{z}(a, \beta)), \end{aligned}$$

from which the desired conclusion follows immediately since the right-hand side of the above equality is a constant depending only on the fixed-endpoint conditions (1). \Box

4 An illustrative example

Let $a, b \in \mathbb{T}$, a < b, and α and β be two given reals, $\alpha \neq \beta$. We consider the following problem of the calculus of variations on time scales:

minimize
$$\mathcal{L}[x] = \int_{a}^{b} \left((x^{\Delta}(t))^{2} + x^{\sigma}(t) + tx^{\Delta}(t) \right) \Delta t,$$

 $x(a) = \alpha, \quad x(b) = \beta.$
(4)

We transform problem (4) into the trivial problem

minimize
$$\tilde{\mathcal{L}}[\tilde{x}] = \int_{a}^{b} (\tilde{x}^{\Delta}(t))^{2} \Delta t$$
,
 $\tilde{x}(a) = 0$, $\tilde{x}(b) = 0$,

which has the solution $\tilde{x} \equiv 0$. For that we consider the transformation

$$x(t) = \tilde{x}(t) + ct + d, \quad c, d \in \mathbb{R},$$

where constants c and d will be chosen later. According to the above, we have

$$x^{\Delta}(t) = \tilde{x}^{\Delta}(t) + c, \quad x^{\sigma}(t) = \tilde{x}^{\sigma}(t) + c\sigma(t) + d$$

and

$$\begin{split} L(t, x^{\sigma}(t), x^{\Delta}(t)) &= (x^{\Delta}(t))^{2} + x^{\sigma}(t) + tx^{\Delta}(t) \\ &= (\tilde{x}^{\Delta}(t))^{2} + 2c\tilde{x}^{\Delta}(t) + c^{2} + \tilde{x}^{\sigma}(t) + c\sigma(t) + d + t\tilde{x}^{\Delta}(t) + ct \\ &= \tilde{L}(t, \tilde{x}^{\sigma}(t), \tilde{x}^{\Delta}(t)) + [2c\tilde{x}(t) + t\tilde{x}(t) + ct^{2} + (c^{2} + d)t]^{\Delta}. \end{split}$$

In order to obtain the solution to the original problem, it suffices to chose c and d so that

$$\begin{cases} ca+d=\alpha\\ cb+d=\beta \,. \end{cases}$$
(5)

Solving the system of equations (5) we obtain $c = \frac{\alpha - \beta}{a - b}$ and $d = \frac{\beta a - b\alpha}{a - b}$. Hence, the global minimizer to problem (4) is

$$x(t) = \frac{\alpha - \beta}{a - b}t + \frac{\beta a - b\alpha}{a - b}.$$

5 Optimal control on time scales

The study of more general problems of optimal control on time scales is in its infancy, and results are rare (see [22, 27] for some preliminary results). Similar to the calculus of variations on time scales, there is a lack of examples with known solution. Here we solve an optimal control problem on an arbitrary time scale using the idea of Leitmann's direct method. Consider the global minimum problem

minimize
$$\mathcal{L}[u_1, u_2] = \int_0^1 \left((u_1(t))^2 + u_2(t))^2 \right) \Delta t$$
 (6)

subject to the control system

$$\begin{cases} x_1^{\Delta}(t) = \exp(u_1(t)) + u_1(t) + u_2(t), \\ x_2^{\Delta}(t) = u_2(t), \end{cases}$$
(7)

and conditions

$$x_1(0) = 0, \quad x_1(1) = 2, \quad x_2(0) = 0, \quad x_2(1) = 1, u_1(t), u_2(t) \in \Omega = [-1, 1].$$
(8)

This example is inspired in [25]. It is worth to mention that a theory based on necessary optimality conditions on time scales to solve problem (6)-(8) does not exist at the moment.

We begin noticing that problem (6)-(8) is variationally invariant according to [14] under the one-parameter transformations¹

$$x_1^s = x_1 + st$$
, $x_2^s = x_2 + st$, $u_2^s = u_2 + s$ $(t^s = t \text{ and } u_1^s = u_1)$. (9)

To prove this, we need to show that both the functional integral $\mathcal{L}[\cdot]$ and the control system stay invariant under the *s*-parameter transformations (9). This is easily seen by direct calculations:

$$\mathcal{L}^{s}[u_{1}^{s}, u_{2}^{s}] = \int_{0}^{1} (u_{1}^{s}(t))^{2} + (u_{2}^{s}(t))^{2} \Delta t$$

$$= \int_{0}^{1} u_{1}(t)^{2} + (u_{2}(t) + s)^{2} \Delta t$$

$$= \int_{0}^{1} (u_{1}(t)^{2} + u_{2}(t)^{2} + [s^{2}t + 2sx_{2}(t)]^{\Delta}) \Delta t$$

$$= \mathcal{L}[u_{1}, u_{2}] + s^{2} + 2s.$$
(10)

We remark that \mathcal{L}^s and \mathcal{L} have the same minimizers: adding a constant $s^2 + 2s$ to the functional \mathcal{L} does not change the minimizer of \mathcal{L} . It remains to prove that the control system also remains invariant under transformations (9):

$$(x_1^s(t))^{\Delta} = (x_1(t) + st)^{\Delta} = x_1^{\Delta}(t) + s = \exp(u_1(t)) + u_1(t) + u_2(t) + s$$

= $\exp(u_1^s(t)) + u_1^s(t) + u_2^s(t)$, (11)
 $(x_2^s(t))^{\Delta} = (x_2(t) + st)^{\Delta} = x_2^{\Delta}(t) + s = u_2(t) + s = u_2^s(t)$.

Conditions (10) and (11) prove that problem (6)-(8) is invariant under the *s*-parameter transformations (9) up to $(s^2t + 2sx_2)^{\Delta}$. Using the invariance

¹A computer algebra package that can be used to find the invariance transformations is available from the *Maple Application Center* at http://www.maplesoft.com/applications/view.aspx?SID=4805

transformations (9), we generalize problem (6)-(8) to a s-parameter family of problems, $s \in \mathbb{R}$, which include the original problem for s = 0:

minimize
$$\mathcal{L}^{s}[u_{1}^{s}, u_{2}^{s}] = \int_{0}^{1} (u_{1}^{s}(t))^{2} + (u_{2}^{s}(t))^{2} \Delta t$$

subject to the control system

$$\begin{cases} (x_1^s(t))^{\Delta} = \exp(u_1^s(t)) + u_1^s(t) + u_2^s(t), \\ (x_2^s(t))^{\Delta} = u_2^s(t), \end{cases}$$

and conditions

$$\begin{aligned} x_1^s(0) &= 0 \,, \quad x_1^s(1) = 2 + s \,, \quad x_2^s(0) = 0 \,, \quad x_2^s(1) = 1 + s \,, \\ u_1^s(t) &\in [-1,1] \,, \quad u_2^s(t) \in [-1 + s, 1 + s] \,. \end{aligned}$$

It is clear that $\mathcal{L}^s \geq 0$ and that $\mathcal{L}^s = 0$ if $u_1^s(t) = u_2^s(t) \equiv 0$. The control equations, the boundary conditions, and the constraints on the values of the controls, imply that $u_1^s(t) = u_2^s(t) \equiv 0$ is admissible only if s = -1: $x_1^{s=-1}(t) = t$, $x_2^{s=-1}(t) \equiv 0$. Hence, for s = -1 the global minimum to \mathcal{L}^s is 0 and the minimizing trajectory is given by

$$\tilde{u}_1^s(t) \equiv 0$$
, $\tilde{u}_2^s(t) \equiv 0$, $\tilde{x}_1^s(t) = t$, $\tilde{x}_2^s(t) \equiv 0$.

Since for any s one has by (10) that $\mathcal{L}[u_1, u_2] = \mathcal{L}^s[u_1^s, u_2^s] - s^2 - 2s$, we conclude that the global minimum for problem $\mathcal{L}[u_1, u_2]$ is 1. Thus, using the inverse functions of the variational symmetries (9),

$$u_1(t) = u_1^s(t)$$
, $u_2(t) = u_2^s(t) - s$, $x_1(t) = x_1^s(t) - st$, $x_2(t) = x_2^s(t) - st$

The absolute minimizer for problem (6)-(8) is

$$\tilde{u}_1(t) = 0$$
, $\tilde{u}_2(t) = 1$, $\tilde{x}_1(t) = 2t$, $\tilde{x}_2(t) = t$.

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