# Construction a new generating function of Bernstein type polynomials 

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#### Abstract

Main purpose of this paper is to reconstruct generating function of the Bernstein type polynomials. Some properties this generating functions are given. By applying this generating function, not only derivative of these polynomials but also recurrence relations of these polynomials are found. Interpolation function of these polynomials is also constructed via Mellin Transformation. This function interpolates these polynomials at negative integers which are given explicitly. Moreover, relations between these polynomials, the generalized Stirling numbers, and Bernoulli polynomials of higher order are given. Furthermore some applications associated with B'ezier curve are given.


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## 1. Introduction, Definitions and Preliminaries

The Bernstein polynomials, recently, have been defined by many different ways, for examples in $q$-series, by complex function and many algorithms. These polynomials are used not only approximations of functions in various, but also in the other fields such as smoothing in statistics, numerical analysis, the solution of the differential equations, and constructing B'ezier curve and in Computer Aided Design cf. ([2], 8, [3, 4, [7, [10, 1]), and see also the references cited in each of these earlier works.

By the same motivation of Ozden' [6] paper, which is related to the unification of the Bernoulli, Euler and Genocchi polynomials, we, in this paper, construct a generating function of the Bernstein polynomials which unify generating function in [10], [1].

## 2. Construction generating functions of Bernstein type polynomials

In this section we unify generating function of the Bernstein polynomials. We define

$$
\mathcal{F}(t, b, s: x)=\frac{2^{b} x^{b s}\left(\frac{t}{2}\right)^{b s} e^{t(1-x)}}{(b s)!}
$$

where $b, s \in \mathbb{Z}^{+}:=\{1,2,3, \cdots\}, t \in \mathbb{C}$ and $x \in[0,1]$. This function is generating function of the polynomials $\mathfrak{S}_{n}(b s, x)$ :

$$
\begin{equation*}
\mathcal{F}(t, b, s: x)=\sum_{n=0}^{\infty} \mathfrak{S}_{n}(b s, x) \frac{t^{n}}{n!}, \tag{2.1}
\end{equation*}
$$

where $\mathfrak{S}_{0}(b s, x)=\cdots=\mathfrak{S}_{b s-1}(b s, x)=0$.
Remark 1. If we set $s=1$ in (2.1), we obtain

$$
\frac{(x t)^{b} e^{t(1-x)}}{b!}=\sum_{n=0}^{\infty} B_{n}(b, x) \frac{t^{n}}{n!}
$$

and $\mathfrak{S}_{n}(b, x)=B_{n}(b, x)$, which denotes the Bernstein polynomials cf. ([2], [3], [4], 8], [10], [1]).

By using Taylor expansion of $e^{t}$ in (2.1), we arrive at the following theorem:
Theorem 1. Let $x, y \in[0,1]$. Let $b, n$ and $s$ be nonnegative integers. If $n \geq b s$, then we have

$$
\mathfrak{S}_{n}(b s, x)=\binom{n}{b s} \frac{x^{b s}(1-x)^{n-b s}}{2^{b(s-1)}}
$$

Remark 1. Setting $s=1$ in Theorem 11, one can see that the polynomials

$$
\mathfrak{S}_{n}(b, x)=\binom{n}{b} x^{b}(1-x)^{n-b}
$$

which give us the Bernstein polynomials cf. ([10], [1]). Consequently, the polynomials $\mathfrak{S}_{n}(b s, x)$ are unification of the Bernstein polynomials.

By using Theorem [1, we easily obtain the following results.
Corollary 1. Let $b$, $n$ and $s$ be nonnegative integers with $n \geq b s$. Then we have

$$
\binom{n}{b s} \mathfrak{S}_{n-b s}(b s ; x)=\binom{n+b s}{n} \mathfrak{S}_{n}(b s ; x)
$$

Setting

$$
\mathfrak{g}_{n}(b s, x)=2^{b(s-1)} \mathfrak{S}_{n}(b s, x),
$$

where, for $b s=j$,

$$
\sum_{j=0}^{n} \mathfrak{g}_{n}(j, x)=1
$$

Let $f$ be a continuous function on $[0,1]$. Then we define unification Bernstein type operator as follows:

$$
\begin{equation*}
\mathbb{S}_{n}(f(x))=\sum_{j=0}^{n} f\left(\frac{j}{n}\right) \mathfrak{g}_{n}(j ; x) \tag{2.2}
\end{equation*}
$$

where $x \in[0,1], n$ is positive integer.
Setting $f(x)=x$ in (2.2), then we have

$$
\mathbb{S}_{n}(x)=\sum_{j=0}^{n} \frac{j}{n}\binom{n}{j} x^{j}(1-x)^{n-j}
$$

From the above, we get

$$
\mathbb{S}_{n}(x)=x \sum_{j=0}^{n} \mathfrak{g}_{n-1}(j-1, x)
$$

## 3. Fundamental relations of the polynomials $\mathfrak{S}_{n}(b s, x)$

By using generating function of $\mathfrak{S}_{n}(b s, x)$, in this section we give derivative of $\mathfrak{S}_{n}(b s, x)$ and recurrence relation of $\mathfrak{S}_{n}(b s, x)$.
Theorem 2. Let $x \in[0,1]$. Let $b, n$ and $s$ be nonnegative integers with $n \geq b s$. Then we have

$$
\begin{equation*}
\frac{d}{d x} \mathfrak{S}_{n}(b s, x)=n\left(\mathfrak{S}_{n-1}(b s-1, x)-\mathfrak{S}_{n}(b s, x)\right) \tag{3.1}
\end{equation*}
$$

Proof. By using the partial derivative of a function in (2.1) with respect to the variable $x$, we have

$$
\sum_{n=0}^{\infty} \frac{\partial}{\partial x}\left(\mathfrak{S}_{n}(b s, x)\right) \frac{t^{n}}{n!}=t \sum_{n=0}^{\infty} \mathfrak{S}_{n}(b s-1, x) \frac{t^{n}}{n!}-t \sum_{n=0}^{\infty} \mathfrak{S}_{n}(b s, x) \frac{t^{n}}{n!}
$$

From the above, we obtain

$$
\sum_{n=0}^{\infty}\left(\frac{d}{d x} \mathfrak{S}_{n}(b s, x)\right) \frac{t^{n}}{n!}=\sum_{n=0}^{\infty} n \mathfrak{S}_{n-1}(b s-1, x) \frac{t^{n}}{n!}-\sum_{n=0}^{\infty} n \mathfrak{S}_{n-1}(b s, x) \frac{t^{n}}{n!}
$$

By using the partial derivative of a function in (2.1) with respect to the variable $t$, we arrive at the following theorem:
Theorem 3. Let $x \in[0,1]$. Let $b, n$ and $s$ be nonnegative integers with $n \geq b s$. Then we have

$$
\begin{equation*}
\mathfrak{S}_{n}(b s, x)=x \mathfrak{S}_{n-1}(b s-1, x)+(1-x) \mathfrak{S}_{n-1}(b s, x) \tag{3.2}
\end{equation*}
$$

Remark 3. If setting $s=1$, then (3.2) reduces to a recursive relation of the Bernstein polynomials

$$
B_{n}(b, x)=(1-x) B_{n-1}(b, x)+x B_{n-1}(b-1, x)
$$

and (3.1) reduces to derivative of the Bernstein polynomials

$$
\frac{d}{d x} B_{n}(j, x)=n\left(B_{n-1}(j-1, x)-B_{n-1}(j, x)\right)
$$

respectively.
By the umbral calculus convention in (2.1), we get

$$
\frac{2^{b} x^{b s}\left(\frac{t}{2}\right)^{b s}}{(b s)!}=e^{(\mathfrak{S}(b s, x)-(1-x)) t}
$$

where $\mathfrak{S}^{n}(b s ; x)$ is replaced by $\mathfrak{S}_{n}(b s ; x)$. After some elementary calculation, we arrive at the following theorem.

Theorem 4. If $n=b s$, then we have

$$
2^{b(1-s)} x^{b s}=\sum_{j=0}^{b s}\binom{b s}{j}(-1)^{b s-j}(1-x)^{b s-j} \mathfrak{S}_{j}(b s, x) .
$$

If $n>b s$, then we have

$$
\sum_{j=b s+1}^{n}\binom{n}{j}(-1)^{n-j}(1-x)^{n-j} \mathfrak{S}_{j}(b s, x)=0
$$

Relations between the polynomials the polynomial $\mathfrak{S}_{n}(b s, x)$, Bernoulli polynomial of higher order and Stirling numbers of second kind is given by the following theorem:

Theorem 5. Let $b, n$ and $s$ be nonnegative integers with $n \geq b s$. Then we have

$$
\mathfrak{S}_{n}(b s, x)=2^{b(1-s)} x^{b s} \sum_{j=0}^{n}\binom{n}{j} S(j, b s) B_{n-j}^{(b s)}(1-x),
$$

where $B_{n}^{(v)}(x)$ and $S(n, j)$ denote Bernoulli polynomial of higher order and Stirling numbers of second kind, which are given by means of the following generating function, respectively

$$
\frac{t^{v} e^{x t}}{\left(e^{t}-1\right)^{v}}=\sum_{n=0}^{\infty} B_{n}^{(v)}(x) \frac{t^{n}}{n!}, \quad(|t|<2 \pi)
$$

and

$$
(-1)^{v} \frac{\left(1-e^{t}\right)^{v}}{v!}=\sum_{n=0}^{\infty} S(n, v) \frac{t^{n}}{n!}
$$

Proof. By (2.1), we have

$$
2^{b(1-s)} x^{b s}\left(\frac{(-1)^{b s}\left(e^{t}-1\right)^{b s}}{(b s)!}\right)\left(\frac{t^{b s} e^{(1-x) t}}{\left(e^{t}-1\right)^{b s}}\right)=\sum_{n=0}^{\infty} \mathfrak{S}_{n}(b s, x) \frac{t^{n}}{n!}
$$

From the above, we have

$$
\sum_{n=0}^{\infty} \mathfrak{S}_{n}(b s, x) \frac{t^{n}}{n!}=2^{b(1-s)} x^{b s}\left(\sum_{n=0}^{\infty} B_{n}^{(b s)}(1-x) \frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty} S(n, k) \frac{t^{n}}{n!}\right)
$$

By Cauchy product in the above, after some calculation, we find the desired result.

By using same method of Lopez and Temme' [5], we give contour integral representation of $\mathfrak{S}_{n}(b s, x)$ as follows:

$$
\mathfrak{S}_{n}(b s, x)=\frac{\Gamma(m+1)}{\Gamma(k+1)} \frac{1}{2 \pi i} \int_{\mathcal{C}} \mathcal{F}(t, b, s: x) \frac{d z}{z^{m+1}},
$$

where $\mathcal{C}$ is a circle around the origin and the integration is in positive direction.

## 4. Interpolation Function of the polynomials $\mathfrak{S}_{n}(b s, x)$

In this section, we construct meromorphic function. This function interpolates $\mathfrak{S}_{n}(b s ; x)$ at negative integers. These values are given explicitly in Theorem 6.

For $z \in \mathbb{C}$, by applying the Mellin transformation to (2.1), we obtain

$$
\mathfrak{B}(z, b s ; x)=\frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{z-1} \mathcal{F}(-t, b, s: x) d t
$$

where $\Gamma(z)$ is Euler gamma function. From the above, we define the following interpolation function.

Definition 1. Let $z \in \mathbb{C}$ with $\Re(z)>0$ and $x \neq 1$. Let $b$ and $s$ be nonnegative integers. Then we define

$$
\begin{equation*}
\mathfrak{B}(z, b s ; x)=(-1)^{b s} \frac{\Gamma(z+b s)}{\Gamma(b s+1) \Gamma(z)} \frac{2^{b(1-s)} x^{b s}}{(1-x)^{z+b s}}, \tag{4.1}
\end{equation*}
$$

Remark 4. By the well-known identity $\Gamma(b s+1)=b s \Gamma(b s)$, for $\Re(z)>0$ we have

$$
\mathfrak{B}(z, k ; x)=\frac{(-1)^{b s} 2^{b(1-s)} x^{b s}}{b s B(z, k)(1-x)^{z+b s}},
$$

where $B(z, k)$ denotes the beta function. Observe that if $x=1$, then

$$
\mathfrak{B}(z, b s, 1)=\infty .
$$

Theorem 6. Let $b, n$ and $s$ be nonnegative integers with $n \geq$ bs and $x \in[0,1]$. Then we have

$$
\mathfrak{B}(-n, b s ; x)=\mathfrak{S}_{n}(b s, x)
$$

Proof. Let $n$ and $b$, and $s$ be positive integers with $b s \leq n . \Gamma(z)$ has simple poles at $z=-n=0,-1,-2,-3, \cdots$. The residue of $\Gamma(z)$ is

$$
\operatorname{Res}(\Gamma(z),-n)=\frac{(-1)^{n}}{n!}
$$

Taking $z \rightarrow-n$ into (4.1) and using the above relations, the desired result can be obtained.

Observe that if we set $s=1$ in Theorem 6, we arrive at

$$
\mathfrak{B}(-n, b ; x)=B_{n}(b, x) .
$$

## 5. Further Remarks on $\mathrm{B}^{\prime}$ ezier curves

The Bernstein polynomials are used to construct B'ezier curves. B'ezier was an engineer with the Renault car company and set out in the early 1960's to develop a curve formulation which would lend itself to shape design. Engineers may find it most understandable to think of B'ezier curves in terms of the center of mass of a set of point masses cf. [13], for example, consider the four masses $m_{0}, m_{1}, m_{2}$, and $m_{3}$ located at points $P_{0}, P_{1}, P_{2}, P_{3}$. The center of mass of these four point masses is given by the equation

$$
P=\frac{m_{0} P_{0}+m_{1} P_{1}+m_{2} P_{2}+m_{3} P_{3}}{m_{0}+m_{1}+m_{2}+m_{3}} .
$$

Next, imagine that instead of being fixed, constant values, each mass varies as a function of some parameter $x$. In specific case, let $m_{0}=(1-x)^{3}, m_{1}=3 t(1-x)^{2}, m_{2}=3 t^{2}(1-x)$ and $m_{3}=x^{3}$. The values of these masses are a function of $x$. For each value of $x$, the masses assume different weights and their center of mass changes continuously. As $x$ varies between 0 and 1 , a curve is swept out by the center of masses. This curve is a cubic $\mathrm{B}^{\prime}$ ezier curve. For any value of $x$, this $\mathrm{B}^{\prime}$ ezier curve is

$$
P=m_{0} P_{0}+m_{1} P_{1}+m_{2} P_{2}+m_{3} P_{3}
$$

where $m_{0}+m_{1}+m_{2}+m_{3} \equiv 1$. These variable masses $m_{i}$ are normally called blending functions and their locations $P_{i}$ are known as control points or B'ezier points. The blending functions, in the case of B'ezier curves, are known as Bernstein polynomials. This curve is used in computer graphics and related fields and also in the time domain, particularly in animation and interface design cf. ([3, 4], [13]).

The B'ezier curve of degree $n$ can be generalized as follows. Given points $P_{0}, P_{1}, P_{2}, \cdots$, $P_{n}$ the $\mathrm{B}^{\prime}$ ezier curve is

$$
\begin{equation*}
B(x)=\sum_{k=0}^{n} P_{k} B_{n}(k, x), \tag{5.1}
\end{equation*}
$$

where $x \in[0,1]$ and $B_{n}(k, t)$ denotes Bernstein polynomials.
We now unify the B'ezier curve in (5.1) by the polynomials $\mathfrak{g}_{n}(b s, x)$ as follows

$$
\mathbb{B}_{n}(x, y)=\sum_{k=0}^{n} P_{k} \mathfrak{g}_{n}(k ; x)
$$

with control points $P_{k}$.
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