

SHARP BOUNDS FOR HARMONIC NUMBERS

FENG QI AND BAI-NI GUO

ABSTRACT. In the paper, we first survey some results on inequalities for bounding harmonic numbers or Euler-Mascheroni constant, and then we establish a new sharp double inequality for bounding harmonic numbers as follows: For $n \in \mathbb{N}$, the double inequality

$$-\frac{1}{12n^2 + 2(7 - 12\gamma)/(2\gamma - 1)} \leq H(n) - \ln n - \frac{1}{2n} - \gamma < -\frac{1}{12n^2 + 6/5}$$

is valid, with equality in the left-hand side only when $n = 1$, where the scalars $\frac{2(7-12\gamma)}{2\gamma-1}$ and $\frac{6}{5}$ are the best possible.

1. INTRODUCTION

The series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \quad (1)$$

is called harmonic series. The n -th harmonic number $H(n)$ for $n \in \mathbb{N}$, the sum of the first n terms of the harmonic series, may be given analytically by

$$H(n) = \sum_{i=1}^n \frac{1}{i} = \gamma + \psi(n+1), \quad (2)$$

see [1, p. 258, 6.3.2], where $\gamma = 0.57721566 \cdots$ is Euler-Mascheroni constant and $\psi(x)$ denotes the psi function, the logarithmic derivative $\frac{\Gamma'(x)}{\Gamma(x)}$ of the classical Euler gamma function $\Gamma(x)$ which may be defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0. \quad (3)$$

In [17], the so-called Franel's inequality in literature was given by

$$\frac{1}{2n} - \frac{1}{8n^2} < H(n) - \ln n - \gamma < \frac{1}{2n}, \quad n \in \mathbb{N}. \quad (4)$$

In [11, pp. 105–106], by considering

$$I_n = \int_{1/n}^1 \left(\frac{1}{x} - \left[\frac{1}{x} \right] \right) dx = \ln n - H(n) \quad (5)$$

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and $0 < I_n < \frac{1}{2}$, where $[t]$ denotes the largest integer less than or equal to t , it was established that

$$\frac{1}{2} < H(n) - \ln n < 1, \quad n \in \mathbb{N}. \quad (6)$$

In [11, pp. 128–129, Problem 65], it was verified that

$$\frac{1}{2} \ln(2n+1) < \sum_{k=1}^n \frac{1}{2k-1} < 1 + \frac{1}{2} \ln(2n-1), \quad n \in \mathbb{N}. \quad (7)$$

In [27], it was obtained that

$$\frac{1}{2(n+1)} < H(n) - \ln n - \gamma < \frac{1}{2n}, \quad n \in \mathbb{N}. \quad (8)$$

In [8], it was proved that

$$\frac{1}{24(n+1)^2} < H(n) - \ln\left(n + \frac{1}{2}\right) - \gamma < \frac{1}{24n^2}, \quad n \in \mathbb{N}. \quad (9)$$

In [21], the following problems were proposed:

- (1) Prove that for every positive integer n we have

$$\frac{1}{2n+2/5} < H(n) - \ln n - \gamma < \frac{1}{2n+1/3}. \quad (10)$$

- (2) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but $\frac{1}{3}$ cannot be replaced by a slightly larger number.

In [10], by using

$$H(n) = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4} \quad (11)$$

for $0 < \varepsilon_n < 1$, these problems were answered affirmatively. The editorial comment in [10] said that the number $\frac{2}{5}$ in (10) can be replaced by $\frac{2\gamma-1}{1-\gamma}$ and equality holds only when $n = 1$. This means that

$$\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq H(n) - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \quad n \in \mathbb{N}. \quad (12)$$

This double inequality was recovered and sharpened in [6, 7] and [18, Theorem 2].

In [26], basing on an improved Euler-Maclaurin summation formula, some general inequalities for the n -th harmonic number $H(n)$ are established, including recovery of the inequality (10).

In [25], the problems above-mentioned was solved once again by employing

$$H(n) = \ln n + \gamma + \frac{1}{2n} - \frac{1}{2} \sum_{i=1}^{q-1} \frac{B_{2i}}{in^{2i}} - \int_n^\infty \frac{B_{2q}(x)}{x^{2q}} dx \quad (13)$$

and

$$\int_n^\infty \frac{B_{2q-1}(x)}{x^{2q}} dx < \frac{(-1)^q B_{2q}}{2qn^{2q}}, \quad (14)$$

where n and q are positive integers, $B_i(x)$ are Bernoulli polynomials and $B_{2i} = B_{2i}(0)$ denote Bernoulli numbers for $i \in \mathbb{N}$. For definitions of $B_i(x)$ and B_{2i} , please refer to [1, p. 804].

In [23], the inequality (10) was verified again by calculus.

In [13], by utilizing Euler-Maclaurin summation formula, the following general result was obtained:

$$H(n) = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{2} \sum_{i=3}^m \frac{B_{2(i-1)}}{(i-1)n^{2(i-1)}} + O\left(\frac{1}{n^{2m}}\right). \quad (15)$$

See also [15, p. 77]. In fact, this is equivalent to the formula in [1, p. 259, 6.3.18].

In [22], by considering the decreasing monotonicity of the sequence

$$x_n = \frac{1}{\left| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k} \right|} - 2n, \quad (16)$$

it was shown that the best constants a and b such that

$$\frac{1}{2n+a} \leq \left| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k} \right| < \frac{1}{2n+b} \quad (17)$$

for $n \geq 1$ are $a = \frac{1}{1-\ln 2} - 2$ and $b = 1$.

In [4, Theorem 2.8] and [19], alternative sharp bounds for $H(n)$ were presented: For $n \in \mathbb{N}$,

$$1 + \ln(\sqrt{e} - 1) - \ln(e^{1/(n+1)} - 1) \leq H(n) < \gamma - \ln(e^{1/(n+1)} - 1). \quad (18)$$

The constants $1 + \ln(\sqrt{e} - 1)$ and γ in (18) are the best possible. This improves the result in [3, pp. 386–387].

In [20], it was established that

$$\ln\left(n + \frac{1}{2}\right) + \gamma < H(n) \leq \ln(n + e^{1-\gamma} - 1) + \gamma, \quad n \in \mathbb{N}. \quad (19)$$

In [5], it was obtained that

$$\frac{1}{24\{n + 1/2\sqrt{6[1-\gamma-\ln(3/2)]}\}^2} \leq H(n) - \ln\left(n + \frac{1}{2}\right) - \gamma < \frac{1}{24(n + 1/2)^2} \quad (20)$$

for $n \in \mathbb{N}$, where the constants

$$\frac{1}{2\sqrt{6[1-\gamma-\ln(3/2)]}}$$

and $\frac{1}{2}$ are the best possible.

For more information on estimates of harmonic numbers $H(n)$, please refer to [9, 24], [14, pp. 68–86], [15, pp. 75–79] and closely-related references therein.

The aim of this paper is to establish a double inequality for bounding harmonic numbers, which is sharp and refines those inequalities above-mentioned.

Theorem 1. For $n \in \mathbb{N}$, the double inequality

$$-\frac{1}{12n^2 + 2(7-12\gamma)/(2\gamma-1)} \leq H(n) - \ln n - \frac{1}{2n} - \gamma < -\frac{1}{12n^2 + 6/5} \quad (21)$$

is valid, with equality in the left-hand side of (21) only when $n = 1$, where the scalars $\frac{2(7-12\gamma)}{2\gamma-1}$ and $\frac{6}{5}$ in (21) are the best possible.

Remark 1. When $n \geq 2$, the double inequality (21) refines (20) and those mentioned before it.

2. PROOF OF THEOREM 1

We now are in a position to prove Theorem 1.

Let

$$f(x) = \frac{1}{\ln x + 1/2x - \psi(x+1)} - 12x^2 \quad (22)$$

for $x \in (0, \infty)$. An easy computation gives

$$f'(x) = \frac{4x^2\psi'(x+1) - 4x + 2}{[2x \ln x - 2x\psi(x+1) + 1]^2} - 24x = \frac{4x^2g(x)}{[2x \ln x - 2x\psi(x+1) + 1]^2},$$

where

$$g(x) = \psi'(x+1) - \frac{1}{x} + \frac{1}{2x^2} - 24x \left[\psi(x+1) - \ln x - \frac{1}{2x} \right]^2. \quad (23)$$

In [2, Theorem 8], the functions

$$F_n(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) - \sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \quad (24)$$

and

$$G_n(x) = -\ln \Gamma(x) + \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) + \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \quad (25)$$

for $n \geq 0$ were proved to be completely monotonic on $(0, \infty)$. This generalizes [16, Theorem 1] which states that the functions $F_n(x)$ and $G_n(x)$ are convex on $(0, \infty)$. The complete monotonicity of $F_n(x)$ and $G_n(x)$ was proved in [12, Theorem 2] once again. In particular, the functions

$$\begin{aligned} F_2(x) &= \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) \\ &\quad - \frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5} + \frac{1}{1680x^7} \end{aligned} \quad (26)$$

and

$$\begin{aligned} G_1(x) &= -\ln \Gamma(x) + \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) \\ &\quad + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} \end{aligned} \quad (27)$$

are completely monotonic on $(0, \infty)$. Therefore, we have

$$\begin{aligned} \ln x - \frac{1260x^5 + 210x^4 - 21x^2 + 10}{2520x^6} &< \psi(x) \\ &< \ln x - \frac{2520x^7 + 420x^6 - 42x^4 + 20x^2 - 21}{5040x^8} \end{aligned} \quad (28)$$

and

$$\begin{aligned} \frac{210x^8 + 105x^7 + 35x^6 - 7x^4 + 5x^2 - 7}{210x^9} &< \psi'(x) \\ &< \frac{210x^6 + 105x^5 + 35x^4 - 7x^2 + 5}{210x^7} \end{aligned} \quad (29)$$

on $(0, \infty)$. From this, it follows that

$$\begin{aligned} \ln x + \frac{1}{2x} - \psi(x+1) &= \ln x - \frac{1}{2x} - \psi(x) \\ &< \frac{1260x^5 + 210x^4 - 21x^2 + 10}{2520x^6} - \frac{1}{2x} = \frac{10 - 21x^2 + 210x^4}{2520x^6} \end{aligned} \quad (30)$$

and

$$\begin{aligned} g(x) &> \psi'(x) - \frac{1}{x^2} - \frac{1}{x} + \frac{1}{2x^2} - \frac{(10 - 21x^2 + 210x^4)^2}{264600x^{11}} \\ &> \frac{210x^8 + 105x^7 + 35x^6 - 7x^4 + 5x^2 - 7}{210x^9} - \frac{1}{x^2} - \frac{1}{x} \\ &\quad + \frac{1}{2x^2} - \frac{(10 - 21x^2 + 210x^4)^2}{264600x^{11}} \\ &= \frac{1659x^4 - 8400x^2 - 100}{264600x^{11}} \\ &= \frac{1659(x-3)^4 + 19908(x-3)^3 + 81186(x-3)^2 + 128772(x-3) + 58679}{264600x^{11}}. \end{aligned}$$

Hence, the function $g(x)$ is positive on $[3, \infty)$. So the derivative $f'(x) > 0$ on $[3, \infty)$, that is, the function $f(x)$ is strictly increasing on $[3, \infty)$.

It is easy to obtain

$$\begin{aligned} f(1) &= \frac{2(7 - 12\gamma)}{2\gamma - 1} = 0.9507\dots, \\ f(2) &= \frac{4(48\gamma + 48 \ln 2 - 61)}{5 - 4\gamma - 4 \ln 2} = 1.1090\dots, \\ f(3) &= \frac{3(108\gamma + 108 \ln 3 - 181)}{5 - 3\gamma - 3 \ln 3} = 1.1549\dots \end{aligned}$$

This means that the sequence $f(n)$ for $n \in \mathbb{N}$ is strictly increasing.

Employing the inequality (30) yields

$$f(x) > \frac{2520x^6}{10 - 21x^2 + 210x^4} - 12x^2 = \frac{12x^2(21x^2 - 10)}{10 - 21x^2 + 210x^4} \rightarrow \frac{6}{5}$$

as $x \rightarrow \infty$. Utilizing the right-hand side inequality in (28) leads to

$$\begin{aligned} f(x) &= \frac{1}{\ln x - 1/2x - \psi(x)} - 12x^2 \\ &< \frac{1}{(2520x^7 + 420x^6 - 42x^4 + 20x^2 - 21)/5040x^8 - 1/2x} - 12x^2 \\ &= \frac{12x^2(42x^4 - 20x^2 + 21)}{420x^6 - 42x^4 + 20x^2 - 21} \\ &\rightarrow \frac{6}{5} \end{aligned}$$

as $x \rightarrow \infty$. As a result, it follows that $\lim_{x \rightarrow \infty} f(x) = \frac{6}{5}$. Therefore, it is derived that $f(1) \leq f(n) < \frac{6}{5}$ for $n \in \mathbb{N}$, equivalently,

$$\frac{2(7 - 12\gamma)}{2\gamma - 1} \leq \frac{1}{\ln n + 1/2n - \psi(n+1)} - 12n^2 < \frac{6}{5}$$

which can be rearranged as

$$\frac{1}{12n^2 + 2(7 - 12\gamma)/(2\gamma - 1)} \geq \ln n + \frac{1}{2n} - \psi(n + 1) > \frac{1}{12n^2 + 6/5}.$$

Combining this with (2) yields (21). The proof of Theorem 1 is proved.

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