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On the numerical solution of Korteweg–de Vries equation by the iterative splitting method

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A R T I C L E   I N F O

Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

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Nonlinear wave equation
KdV equation
Iterative splitting method

A B S T R A C T

In this paper, we apply the method of iterative operator splitting on the Korteweg–de Vries (KdV) equation. The method is based on first, splitting the complex problem into simpler sub-problems. Then each sub-equation is combined with iterative schemes and solved with suitable integrators. Von Neumann analysis is performed to achieve stability criteria for the proposed method applied to the KdV equation. The numerical results obtained by iterative splitting method for various initial conditions are compared with the exact solutions. It is seen that they are in a good agreement with each other.

1. Introduction

Nonlinear wave equations are widely used to describe complex phenomena in various sciences such as fundamental particle physics, plasma and fluid dynamics, statistical mechanics, protein dynamics, condensed matter, biophysics, nonlinear optics, quantum field theory, see [14,3,1,6]. The wide applicability of these equations is the main reason why they have attracted so much attention from many mathematicians. However, they are usually very difficult to solve, either numerically or analytically.

During the past four decades, both mathematicians and physicists have devoted considerable effort to the study of exact and numerical solutions of the nonlinear partial differential equations corresponding to the nonlinear problems. Many powerful methods have been presented, for instance, Darboux transformation method [9], Adomians decomposition method [15,11], He's perturbation method [16], Operator splitting method [10], Iterative splitting method [7].

In this paper, we consider the nonlinear Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

which was found to admit soliton solutions and be able to model the propagation of solitary wave on water surface. Its phenomena was first discovered by Russell in 1834 [13] and Korteweg–de Vries formulated the mathematical model equation to provide explanation of the phenomena. In [11,16,2], KdV equation has been solved with Adomain's decomposition (ADM), He's perturbation method (HPM) and a particle method (based on diffusion-velocity method) analytically and numerically. Here, we use iterative operator splitting method to study on the nonlinear KdV equation.

The iterative splitting is a recent popular technique which is based on first splitting the complex problem into simpler differential equations. Then each sub-equation is combined with the iterative schemes, each of which is efficiently solved with suitable integrators [5,4,8,7].

Furthermore, this study explicitly derives the stability criteria for iterative splitting method using Fourier analysis, which based on KdV equation [12].

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The structure of the paper is as follows: In Section 2, outline of the iterative splitting method is given. Stability analysis of the method which based on KdV equation is derived in Section 3. In Section 4, applications of the method on KdV equation is done. Finally, we have numerical results and conclusion part.

2. Outline of the method

Consider the abstract Cauchy problem

$$u'(t) = (A + B)u(t), \quad t \in [0, T], \tag{2}$$

$$u(0) = u_0, \tag{3}$$

where A and B are bounded linear operators and u_0 is initial condition. For such problem, the exact solution can be given as

$$u(t) = \exp((A + B))u_0, \quad t \in [0, T]. \tag{4}$$

The method is based on iteration by fixing the splitting discretization step size Δt on time interval $[t^n, t^{n+1}]$. The following algorithms are then solved consecutively for $i = 1, 3, \ldots, 2m + 1$.

$$u_i'(t) = Au_i(t) + Bu_{i-1}(t) \text{ with } u_i(t^n) = u^n, \tag{5}$$

$$u_{i+1}'(t) = Au_i(t) + Bu_{i+1}(t) \text{ with } u_{i+1}(t^n) = u^n, \tag{6}$$

where u^n is the known split approximation at time level $t = t^n$ and $u_0 \equiv 0$ is the initial guess. The split approximation at the time-level $t = t^{n+1}$ is defined as $u^{n+1} = u_{2m+2}(t^n)$.

3. Stability analysis of iterative splitting method on KdV equation via von Neumann

In this section, we will investigate the stability analysis of iterative splitting method for KdV equation via von Neumann approach. Consider again the KdV equation of the form

$$u_t + 6uu_x + u_{xxx} = 0. \tag{7}$$

Firstly, split Eq. (7) into two parts

$$u_t = -u_{xxx} \text{ and } u_t = -6uu_x \tag{8}$$

and apply iterative splitting schemes, then have the following algorithms:

$$u_i' = -(u_i)_{xxx} + 6u_{i-1}(u_{i-1})_x, \tag{9}$$

$$u_{i+1}' = -(u_i)_{xxx} + 6u_i(u_{i+1})_x, \tag{10}$$

where $i = 1, 3, \ldots, 2m + 1$.

Note that, in this approach, it is not necessary to specify a spatial discretization technique.

Rearrangement of algorithms (19) and (20) with a linearization about steady state $6u_{i-1} = k_1$, $6u_i = k_2$ yields

$$u_i' = L_1u_i + k_1L_2u_{i-1}, \tag{11}$$

$$u_{i+1}' = L_1u_i + k_2L_2u_{i+1}, \tag{12}$$

where $L_1 = -\frac{\partial^3}{\partial x^3}$, $L_2 = -\frac{\partial}{\partial x}$ and $i = 1, 3, \ldots, 2m + 1$.

Secondly, combine algorithms (11) and (12) with the second order midpoint rule then have

$$\begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} u_i^n \\ u_{i+1}^n \end{pmatrix} + \Delta t \begin{pmatrix} L_1 \frac{u_i^n + u_i^{n+1}}{2} + k_1L_1 \frac{u_{i-1}^n + u_{i-1}^{n+1}}{2} \\ L_1 \frac{u_i^n + u_i^{n+1}}{2} + k_2L_2 \frac{u_{i+1}^n + u_{i+1}^{n+1}}{2} \end{pmatrix}, \tag{13}$$

where Δt is the time step on $[t^n, t^{n+1}]$ interval.

Finally, Eq. (13) can be put in the following matrix form

$$\begin{pmatrix} \tilde{u}_i^{n+1} \\ \tilde{u}_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1+\frac{\Delta t}{2}w^3 i}{1-\frac{\Delta t}{2}w^3 i} & 0 \\ \frac{\frac{\Delta t}{2}w^3 i\left(1+\frac{\Delta t}{2}w^3 i\right)}{\left(1-\frac{\Delta t}{2}w^3 i\right)\left(1+\frac{\Delta t}{2}k_2 wi\right)} + \frac{\frac{\Delta t}{2}w^3 i}{1+\frac{\Delta t}{2}k_2 wi} & \frac{1-\frac{\Delta t}{2}k_2 wi}{1+\frac{\Delta t}{2}k_2 wi} \end{pmatrix} \begin{pmatrix} \tilde{u}_i^n \\ \tilde{u}_{i+1}^n \end{pmatrix} = \widetilde{A} \begin{pmatrix} \tilde{u}_i^n \\ \tilde{u}_{i+1}^n \end{pmatrix}, \tag{14}$$

by taking the fourier transform according to the formula

$$\hat{u}(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-iwx} u(x)dx. \tag{15}$$

The eigenvalues of $\widetilde{A}$ are $\lambda_1 = \frac{1+\frac{\Delta t}{2}w^3 i}{1-\frac{\Delta t}{2}w^3 i}$, $\lambda_2 = \frac{1-\frac{\Delta t}{2}k_2 wi}{1+\frac{\Delta t}{2}k_2 wi}$ and for stability eigenvalues must be $|\lambda_i| \leqslant 1$, $i = 1, 2$. One can easily deduce that $|\lambda_1| = |\lambda_2| = 1$, however this holds for any choice of Δt, w and k_2. Hence the method is unconditionally stable. Note that this approach gives advantage since the nonlinear problem can be analyzed as a linear problem. The consistency of the method can be seen in [7].

4. Application

Example 4.1. For purpose of illustration of the iterative splitting method for solving KdV Eq. (1), consider the following initial boundary value problem

$$u_t + 6uu_x + u_{xxx} = 0, \quad u(x,0) = \frac{1}{2}\, sech^2\left(\frac{1}{2}x\right), \quad x \in (l_1, l_2), \tag{16}$$

$$u(x,t)|_{l_1} = 0, \quad u(x,t)|_{l_2} = 0, \quad t \in (0,T], \tag{17}$$

where the analytic solution is

$$u_{analy}(x,t) = \frac{1}{2}\, sech^2\left(\frac{1}{2}(x-t)\right). \tag{18}$$

After splitting and applying iterative schemes, we have the iterative splitting algorithms as in Section 3:

$$u_i' = -(u_i)_{xxx} + 6u_{i-1}(u_{i-1})_x, \tag{19}$$

$$u_{i+1}' = -(u_i)_{xxx} + 6u_i(u_{i+1})_x, \tag{20}$$

where $i = 1, 3, \ldots, 2m + 1$. To solve these iterative schemes, we need to discretize the initial and boundary conditions: For initial condition we have

$$u_m = \frac{1}{2}\, sech^2\left(\frac{1}{2}x_m\right), \quad 1 \leqslant m \leqslant N+1 \tag{21}$$

and for boundary conditions (17) we have

$$u_1 = 0, \quad 0 \leqslant t, \tag{22}$$

$$u_{N+1} = 0, \quad 0 \leqslant t, \tag{23}$$

where m, N define the spatial discretization step and number.

We derive the second order discretization for u_{xxx} term and central difference expansion is taken into account for u_x term as follows:

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{(x_m,t)} = \frac{1}{2h^3}(u_{m+2}(t) - 2u_{m+1}(t) + 2u_{m-1}(t) - u_{m-2}(t)) \tag{24}$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{(x_m,t)} = \frac{1}{2h}(u_{m+1}(t) - u_{m-1}(t)), \tag{25}$$

where h is the spatial stepping and $m = 1, 2, \ldots, N + 1$.

After assembling the unknowns of (24) and (25) for each m, we have the following systems of equations in matrix form as follows:

$$u_{xxx} = Au, \quad u_x = Bu. \tag{26}$$

We fix the nonlinear term $u \simeq \tilde{u}$ at each discretization points $m = 1, 2 \ldots, N + 1$ and have

$$\tilde{u} = \begin{pmatrix} u_2(t) & 0 & 0 & \ldots & 0 & 0 \\ 0 & u_3(t) & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ldots & \vdots & 0 \\ 0 & 0 & 0 & \ldots & u_{N-1}(t) & 0 \\ 0 & 0 & 0 & \ldots & 0 & u_N(t) \end{pmatrix}_{(N-1)\times(N-1)}.$$

Redefining Eqs. (19) and (20) we have

$$u_i' = -Au_i - 6\tilde{u}_{i-1}Bu_{i-1}, \tag{27}$$

$$u_{i+1}' = -Au_i - 6\tilde{u}_i Bu_{i+1}, \tag{28}$$

then solve Eqs. (27), (28) by using the second order midpoint method on each subinterval $[t^n, t^{n+1}]$, $n = 0, 1, \ldots, M$. Thus algorithms can be read as:

$$u_i^{n+1} = \left(I + \frac{\Delta t}{2}A\right)^{-1}\left(\left(I - \frac{\Delta t}{2}A\right)u_i^n - \frac{\Delta t}{2}(6\tilde{u}_{i-1}^n B u_{i-1}^n + 6\tilde{u}_{i-1}^{n+1} B u_{i-1}^{n+1})\right), \tag{29}$$

$$u_{i+1}^{n+1} = \left(I + \frac{\Delta t}{2}6\tilde{u}_i^{n+1}B\right)^{-1}\left((I - \frac{\Delta t}{2}6\tilde{u}_i^n B)u_{i+1}^n - \frac{\Delta t}{2}A(u_i^n + u_{i+1}^n)\right), \tag{30}$$

where Δt is time discretization step and iteration starts with $i = 1$, initial guess $u_0(t) = 0$, initial conditions $u_1(t) = u_0$ and $u_2(t) = u_0$.

Example 4.2. As a second example we present a two-soliton problem such that

$$u_t + 6uu_x + u_{xxx} = 0, \quad u(x,0) = 6\,sech^2(x), \quad x \in (l_1, l_2), \tag{31}$$

$$u(x,t)|_{l_1} = 0, \quad u(x,t)|_{l_2} = 0, \quad t \in (0, T], \tag{32}$$

where the analytic solution is

$$u_{analy}(x,t) = 12\frac{3 + 4cosh(2x - 8t) + cosh(4x - 64t)}{(3cosh(x - 28t) + cosh(3x - 36t))^2}. \tag{33}$$

Example 4.3. As a third example we present a double-soliton problem such that

$$u_t + 6uu_x + u_{xxx} = 0, \quad u(x,0) = \frac{1}{2}sech^2\left(\frac{1}{2}x\right) + 6\,sech^2(x), \quad x \in (l_1, l_2), \tag{34}$$

$$u(x,t)|_{l_1} = 0, \quad u(x,t)|_{l_2} = 0, \quad t \in (0, T]. \tag{35}$$

5. Numerical results and conclusion

The iterative splitting method has been successfully applied to finding the numerical solution of KdV equations with different initial conditions. Also, stability analysis of the method based on KdV equation was studied via von Neumann approach. It gave advantages since the nonlinear problem could be analyzed as a linear problem.

In Figs. 1 and 2, one soliton solutions were obtained at different times. The results showed that iterative splitting and exact soliton solutions behaved in similar way. We compared the errors taken with different methods in Tables 1 and 2, it was seen that iterative splitting method was the best one. In Fig. 3, we plotted the exact and iterative splitting two soliton solutions at different times. Finally, we computed the double soliton collision by taking the initial condition as a sum of two

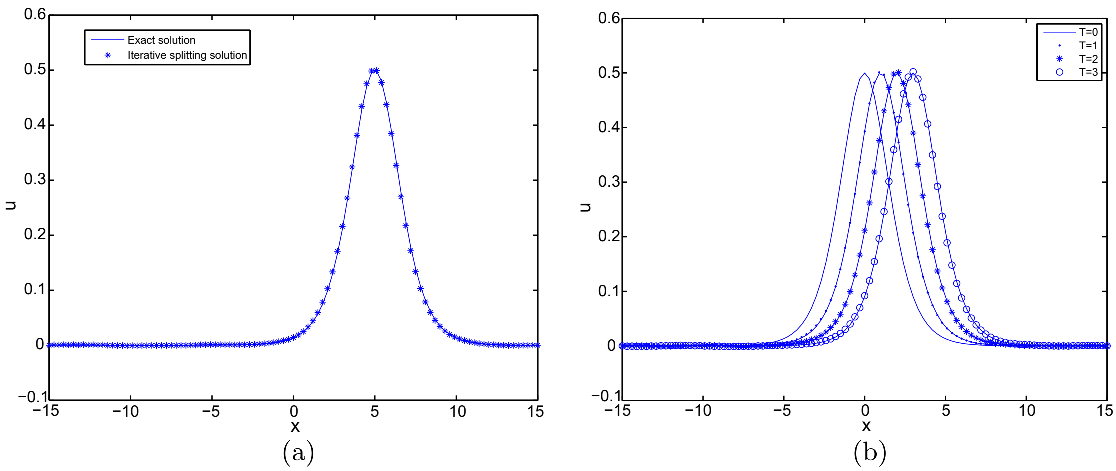


Fig. 1. (a) Comparision of numerical and exact solutions of KdV Eqs. (16) and (17) on $-15 \leqslant x \leqslant 15$ interval at $T = 5$ time, (b) the solution of KdV Eqs. (16) and (17) on $-15 \leqslant x \leqslant 15$ interval at different times. The points represent the location of iterative splitting solutions. The solid lines represent exact solution (18).

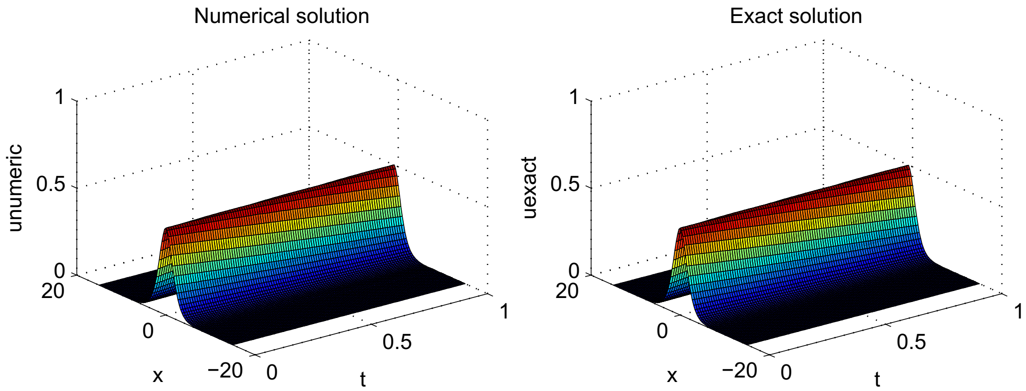


Fig. 2. Comparision of numerical and exact solutions of KdV Eqs. (16) and (17) on $-15 \leqslant x \leqslant 15$ interval at $T = 5$ time.

Table 1
The errors of KdV Eqs. (16) and (17) on $[-15,15]$ interval for $h = 0.3$, $\Delta t = 0.0005$ at $T = 5$.

	err_{L_∞}	err_{L_2}
Iterative splitting	0.0093	0.0288
Difference method	0.0102	0.0290

Table 2
The errors of KdV Eqs. (16) and (17) on $[-15,15]$ interval for $h = 0.3$, $\Delta t = 0.05$ at $T = 5$.

	err_{L_∞}	err_{L_2}
Iterative splitting	0.0098	0.0301
Lie-Trotter splitting	0.0503	0.01434

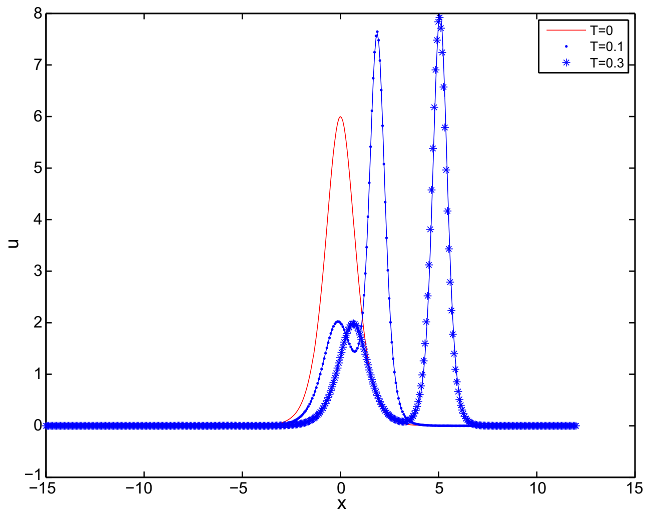
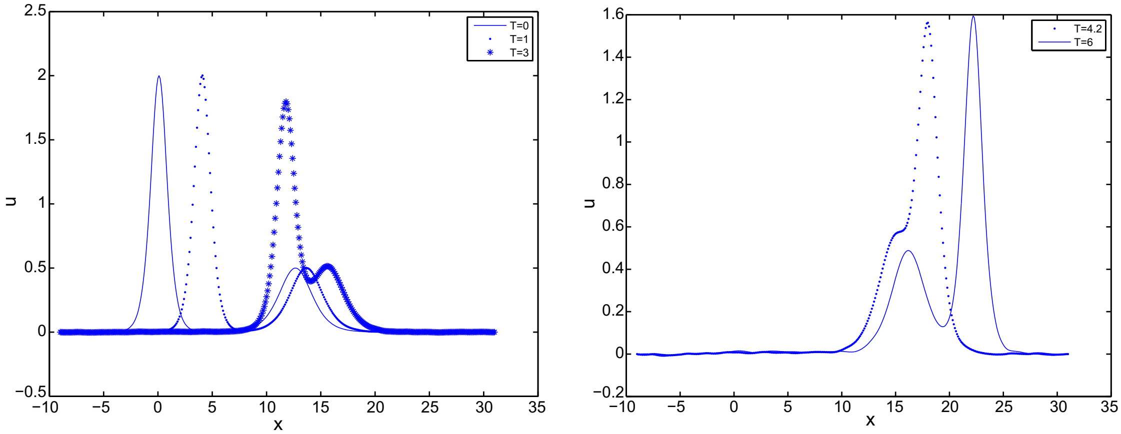


Fig. 3. The solutions of KdV Eqs. (31) and (32) on $-15 \leqslant x \leqslant 12$ interval at different times. The points represent the location of iterative splitting solutions. The solid lines represent exact solution (33).

solitons. The results were presented in Fig. 4. As one could see, the higher soliton that traveled with higher velocity, passed through the lower soliton which traveled slower after going through a nonlinear interaction.

Finally, we can say that the iterative splitting method is extremely simple, easy to use and is accurate for solving nonlinear evolution equations in various sciences. Its applications are worth further studying.



(a) (b)

Fig. 4. (a) The iterative splitting solutions of KdV Eqs. (34) and (35) on $-9 \leqslant x \leqslant 31$ interval at different times. (b) The figure goes on.

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