Backward doubly stochastic differential equations with weak assumptions on the coefficients

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Abstract

In this paper, we deal with one dimensional backward doubly stochastic differential equations (BDSDEs). We obtain existence theorems and comparison theorems for solutions of BDSDEs with weak assumptions on the coefficients.

Keywords: backward doubly stochastic differential equations; backward stochastic differential equations; comparison theorem; existence theorem.

1 Introduction

Pardoux and Peng [14] introduced the following nonlinear backward stochastic differential equations (BSDEs):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ t \in [0, T].$$

They obtained the existence and uniqueness of solutions under the Lipschitz condition. Since then, the theory of BSDEs has been developed by many researchers and there are many works attempting to weaken the Lipschitz condition in order to obtain the existence and uniqueness results of BDSDEs (see e.g., Bahlali [1], Briand and Confortola [3], Darling and Pardoux [5], El Karoui and Huang [6], Hamadène [7], Jia [8], Kobylanski [9], Lepeltier and San Martin [10] and the references therein). Today the BSDE has become a powerful tool in the study of partial differential equations, risk measures, mathematical finance, as well as stochastic optimal controls and stochastic differential games.

After the nonlinear BSDEs were introduced, Pardoux and Peng [15] brought forward BDSDEs with two different directions of stochastic integrals, i.e., the equations involve both a standard stochastic Itô's integral and a backward stochastic Itô's integral:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s,\ t \in [0, T], \tag{1.1}$$

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the integral with respect to $\{B_t\}$ is a backward Itô's integral and the integral with respect to $\{W_t\}$ is a standard forward Itô's integral. By virtue of this kind of BDSDE, Pardoux and Peng [15] established the connections between certain quasi-linear stochastic partial differential equations and BDSDEs, and obtained a probabilistic representation for a class of quasi-linear stochastic partial differential equations. They established the existence and uniqueness results for solutions of BDSDEs under the Lipschitz condition on the coefficients. This kind of BDSDEs has a practical background in finance. The extra noise B can be regarded as some extra information, which can not be detected in the financial market, but is available to the particular investors.

Since the work of Pardoux and Peng [15], there are only several works attempting to relax the Lipschitz condition to get the existence and uniqueness results for one dimensional BDSDEs. Shi et al. [16] obtained that one dimensional BDSDE (1.1) has at least one solution if f is continuous and of linear growth in (y, z), and $\{f(t, 0, 0)\}_{t\in[0,T]}$ is bounded. Under the assumptions that f is bounded, left continuous and non-decreasing in y and Lipschitz in z, Lin [11] established an existence theorem for one dimensional BDSDE (1.1). Lin [12] proved that one dimensional BDSDE (1.1) has at least one solution if the coefficient f is left Lipschitz and left continuous in y, and Lipschitz in z. Lin and Wu [13] obtained a uniqueness result for one dimensional BDSDE (1.1) under the conditions that f is Lipschitz in y and uniformly continuous in z.

Motivated by the above results, one of the objectives of this paper is to get an existence theorem for one dimensional BDSDE (1.1), which generalizes the result in Shi et al. [16] by the condition of the square integrability of $\{f(t, 0, 0)\}_{t\in[0,T]}$ instead of the boundedness of $\{f(t, 0, 0)\}_{t\in[0,T]}$. The other objective of this paper is to generalize the existence result in Lin [12]. We consider the following BDSDE:

$$Y_t = \xi + \int_t^T \left(sgn(Y_s)Y_s^2 + \sqrt{Z_s 1_{\{Z_s\geq 0\}}} \right) ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s,\ t \in [0, T].$$

Since $\sqrt{z1_{\{z\geq 0\}}}$ is not Lipschitz in z, then we can not apply the existence result in Lin [12] to get the existence theorem of the above BDSDE. We shall investigate an existence result for one dimensional BDSDE (1.1) where f is left Lipschitz and left continuous in y and uniformly continuous in z, which improves the result in Lin [12]. Since f is uniformly continuous in z, then we can not apply comparison theorems for solutions of BDSDEs in [16] and [12]. In order to get the existence theorem for solutions of BDSDEs we shall first establish a comparison theorem for solutions of BDSDEs when f is Lipschitz in y and uniformly continuous in z, which plays an important role.

This paper is organized as follows: In section 2, we give some preliminaries and notations, which will be useful in what follows. In section 3, we obtain an existence theorem for the solutions of BDSDEs with continuous coefficients. In section 4, we establish an existence theorem and a comparison theorem for the solutions of a class of BDSDEs with discontinuous coefficients.

2 Preliminaries and Notations

Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{W_t\}_{0\leq t\leq T}$ and $\{B_t\}_{0\leq t\leq T}$ be two mutually independent standard Brownian motion processes, with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ denote the class of $\mathcal{P}$-null sets of $\mathcal{F}$. Then, we define

$$\mathcal{F}_t \doteq \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T}^B, \quad t \in [0, T],$$

where for any process $\{\eta_t\}$, $\mathcal{F}^{\eta}_{s,t} = \sigma\{\eta_r - \eta_s, s \leq r \leq t\} \vee \mathcal{N}$. Let us point out that $\mathcal{F}^W_{0,t}$ is increasing and $\mathcal{F}^B_{t,T}$ is decreasing in t, but $\mathcal{F}_t$ is neither increasing nor decreasing in t.

Let us introduce the following spaces:

- $L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \doteq \Big\{\xi : \mathcal{F}_T-$measurable random variable such that $\mathbb{E}[|\xi|^2] < \infty\Big\}$.

- $S^2(0,T;\mathbb{R}) \doteq \Big\{\varphi : \varphi$ is a continuous process with value in $\mathbb{R}$ such that $\|\varphi\|^2_{S^2} = \mathbb{E}[\sup_{0\leq t\leq T} |\varphi_t|^2] < \infty$, and φ_t is $\mathcal{F}_t-$measurable, for all $t \in [0,T]$ $\Big\}$.

- $M^2(0,T;\mathbb{R}^d) \doteq \Big\{\varphi : \varphi$ is a jointly measurable process with value in $\mathbb{R}^d$ such that $\|\varphi\|^2_{M^2} = \mathbb{E}[\int_0^T |\varphi_t|^2 dt] < \infty$, and φ_t is $\mathcal{F}_t-$measurable, for all $t \in [0,T]$ $\Big\}$.

Let

$$g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^l.$$

In this paper, we suppose that $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and g always satisfies the following assumptions:

$(H1)$ (Lipschitz condition): There exist constants $C > 0$ and $0 < \alpha < 1$ such that, for all $(t, y_i, z_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1,2$

$$|g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2.$$

$(H2)$ For all $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\cdot, y, z) \in M^2(0,T;\mathbb{R}^l)$.

Let

$$f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$

be such that, for all $(t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$, $f(t,y,z)$ is $\mathcal{F}_t-$measurable. We make the following assumptions:

$(H3)$ (Lipschitz condition): There exists a constant $C > 0$ such that, for all $(t, y_i, z_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1,2,$

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

$(H4)$ $f(\cdot,0,0) \in M^2(0,T;\mathbb{R})$.

$(H5)$ $f(t,y,\cdot)$ is uniformly continuous and uniformly with respect to (ω,t,y), i.e., there exists a continuous, sub-additive, non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with linear growth and satisfying $\phi(0) = 0$ such that

$$|f(t,y,z_1) - f(t,y,z_2)| \leq \phi(|z_1 - z_2|),$$

for all $(t,y,z_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1,2$. Here we denote the constant of linear growth for ϕ by C, i.e.,

$$0 \leq \phi(|x|) \leq C(1 + |x|),$$

for all $x \in \mathbb{R}$.

$(H6)$ $f(t,\cdot,z)$ is left continuous and satisfies left Lipschitz condition in y, i.e., for all $(t,y_i,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1,2$ and $y_1 \geq y_2$,

$$f(t,y_1,z) - f(t,y_2,z) \geq -C(y_1 - y_2).$$

$(H7)$ For all $(t,\omega) \in [0,T] \times \Omega$, $f(\omega,t,\cdot,\cdot)$ is continuous.

$(H8)$ There exists a positive constant C such that

$$|f(\omega,t,y,z)| \leq C(1+|y|+|z|), \quad (\omega,t,y,z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d.$$

$(H8')$ There exists a constant $C>0$ and a positive stochastic process $K \in M^2(0,T;\mathbb{R})$ such that, for all $(\omega,t,y,z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(\omega,t,y,z)| \leq C(K_t(\omega)+|y|+|z|).$$

Remark 2.1 *Crandall [4] first used* $(H5)$ *to study viscosity solutions of partial differential equations.*

Remark 2.2 *From* $(H5)$ *and* $(H6)$ *we know that, for* $(t,y_i,z_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $i=1,2$ *and* $y_1 \geq y_2$, *we have*

$$f(t,y_1,z_1) - f(t,y_2,z_2) \geq -C(y_1-y_2) - \phi(|z_1-z_2|).$$

Remark 2.3 *If we take* $\phi(x) = Cx, x \geq 0$, *in* $(H5)$, *where* C *is a positive constant, then combining* $(H6)$ *with some conditions Lin [12] obtained that one dimensional BDSDE has at least one solution.*

Remark 2.4 *Under the assumptions* $(H7)$ *and* $(H8)$ *Shi et al. [16] proved that one dimensional BDSDE has at least one solution.*

Remark 2.5 *It is obvious that* $(H8')$ *implies* $(H8)$.

For $n \in N$, we let

$$\underline{f}_n(t,y,z) = \inf_{u\in\mathbb{R},v\in\mathbb{R}^d} \Big\{ f(t,u,v) + n(|y-u|+|z-v|) \Big\}$$

and

$$\overline{f}_n(t,y,z) = \sup_{u\in\mathbb{R},v\in\mathbb{R}^d} \Big\{ f(t,u,v) - n(|y-u|+|z-v|) \Big\}.$$

Then, we have the following lemma, which was established by Lepeltier and San Martin [10].

Lemma 2.6 *If* f *satisfies* $(H7)$ *and* $(H8)$, *then, for* $n > C$ *and* $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, *we have*

(i) $-C(|y|+|z|+1) \leq \underline{f}_n(t,y,z) \leq f(t,y,z) \leq \overline{f}_n(t,y,z) \leq C(|y|+|z|+1)$.

(ii) $\underline{f}_n(t,y,z)$ *is non-decreasing in* n *and* $\overline{f}_n(t,y,z)$ *is non-increasing in* n.

(iii) For all $(\omega,t,y_i,z_i) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $i=1,2$, *we have*

$$|\underline{f}_n(\omega,t,y_1,z_1) - \underline{f}_n(\omega,t,y_2,z_2)| \leq n(|y_1-y_2|+|z_1-z_2|).$$

The same holds for $\overline{f}_n$.

(iv) If $(y_n,z_n) \to (y,z)$, *as* $n \to \infty$, *then* $\underline{f}_n(t,y_n,z_n) \to f(t,y,z)$, *as* $n \to \infty$. *The same holds for* $\overline{f}_n$.

Given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, we consider the following BDSDE with data (f, g, T, ξ) :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s. \tag{2.1}$$

Definition 2.7 *A pair of processes $(Y, Z) \in \mathbb{R} \times \mathbb{R}^d$ is called a solution of BDSDE (2.1), if $(Y, Z) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$ and satisfies BDSDE (2.1).*

Pardoux and Peng [15] established the following existence and uniqueness for solutions of BDSDE (2.1).

Lemma 2.8 *Under the assumptions $(H1)-(H4)$, BDSDE (2.1) has a unique solution $(Y, Z) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$.*

Finally, we make another assumption, which will be needed in what follows.

$(H9)$ There exist two BDSDEs with data (f_i, g, T, ξ) which have at least one solution (Y^i, Z^i), $i = 1, 2$, respectively. For all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$f_1(t, y, z) \leq f(t, y, z) \leq f_2(t, y, z), \quad Y_t^1 \leq Y_t^2, a.s..$$

Moreover, the processes $\left\{f_i(t, Y_t^i, Z_t^i)\right\}_{t\in[0,T]}$, $i = 1, 2$, are square integrable.

3 Existence theorem for BDSDEs with general continuous coefficients

The objective of this section is to obtain an existence theorem for BDSDEs, which generalizes the corresponding result of Shi et al. [16].

We first give the following useful lemma. For its proof the reader is referred to [2] and [10].

Lemma 3.1 *Let $\underline{f}_n$ and $\overline{f}_n$ be introduced in Section 2. If f satisfies $(H7)$ and $(H8')$, then, for $n > C$ and $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, we have*

(i) $-C(|y| + |z| + K_t) \leq \underline{f}_n(t, y, z) \leq f(t, y, z) \leq \overline{f}_n(t, y, z) \leq C(|y| + |z| + K_t)$.

(ii) $\underline{f}_n(t, y, z)$ is non-decreasing in n and $\overline{f}_n(t, y, z)$ is non-increasing in n.

(iii) For all $(\omega, t, y_i, z_i) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1, 2$, we have

$$|\underline{f}_n(\omega, t, y_1, z_1) - \underline{f}_n(\omega, t, y_2, z_2)| \leq n(|y_1 - y_2| + |z_1 - z_2|).$$

The same holds for $\overline{f}_n$.

(iv) If $(y_n, z_n) \to (y, z)$, as $n \to \infty$, then $\underline{f}_n(t, y_n, z_n) \to f(t, y, z)$, as $n \to \infty$. The same holds for $\overline{f}_n$.

We also need the following comparison theorem obtained in Lin [12].

Lemma 3.2 *Assume BDSDEs (2.1) with data (f^1, g, T, ξ^1) and (f^2, g, T, ξ^2) have solutions (y^1, z^1) and (y^2, z^2), respectively. If f^1 satisfies $(H3)$ and $(H4)$, $\xi^1 \leq \xi^2$, a.s., $f^1(t, y_t^2, z_t^2) \leq f^2(t, y_t^2, z_t^2)$, $d\mathbb{P}dt-$a.s. (resp. f^2 satisfies $(H3)$ and $(H4)$, $f^1(t, y_t^1, z_t^1) \leq f^2(t, y_t^1, z_t^1)$, $d\mathbb{P}dt-$a.s.), then we have $y_t^1 \leq y_t^2$, a.s., for all $t \in [0, T]$.*

We now give the following existence theorem for BDSDEs, which extends the corresponding result in Shi et al. [16] by eliminating the condition that $\{K_t\}_{t\in[0,T]}$ is a bounded process. The coefficient g in the backward Itô's integral will bring the extra estimate difficulty.

Theorem 3.3 *Under the assumptions* $(H7)$ *and* $(H8')$*, BDSDE with data* (f,g,T,ξ) *has a minimal (resp. maximal) solution* $(\underline{y},\underline{z})$ *(resp.* $(\overline{y},\overline{z})$*) of BDSDE with data* (f,g,T,ξ)*, in the sense that, for any other solution* (y,z) *of BDSDE with data* (f,g,T,ξ)*, we have* $\underline{y}\leq y$ *(resp.* $\overline{y}\geq y$*).*

Proof: We only prove that BDSDE (2.1) with data (f,g,T,ξ) has a minimal solution. The other case can be proved similarly. Let

$$h(\omega,t,y,z)=C(K_t(\omega)+|y|+|z|),$$

and $\underline{f}_n$ be introduced in Section 2. Then, $\underline{f}_n\leq h$, and we consider the following BDSDEs:

$$y_t^n=\xi+\int_t^T\underline{f}_n(s,y_s^n,z_s^n)ds+\int_t^T g(s,y_s^n,z_s^n)dB_s-\int_t^T z_s^n dW_s,\quad t\in[0,T], \tag{3.1}$$

and

$$U_t=\xi+\int_t^T h(s,U_s,V_s)ds+\int_t^T g(s,U_s,V_s)dB_s-\int_t^T V_s dW_s,\quad t\in[0,T].$$

From Lemma 2.8 it follows that the above BDSDEs have unique solutions $(y^n,z^n)\in S^2(0,T;\mathbb{R})\times M^2(0,T;\mathbb{R}^d)$, and $(U,V)\in S^2(0,T;\mathbb{R})\times M^2(0,T;\mathbb{R}^d)$, respectively.

By a comparison theorem for BDSDEs (see Lemma 3.2 or [16]) and Lemma 3.1 we have, for $n>C$,

$$y_n\leq y_{n+1}\leq U,\ d\mathbb{P}dt-a.s.$$

Then, there exists a positive constant A independent of n such that

$$\parallel U\parallel_{S^2}\leq A, \parallel V\parallel_{M^2}\leq A,\ \text{and}\ \parallel y_n\parallel_{S^2}\leq A.$$

Therefore, from the dominated convergence theorem it follows that $\{y_n\}$ converges in $S^2(0,T;\mathbb{R})$. We shall denote its limit by $\underline{y}\in S^2(0,T;\mathbb{R})$.

By $(H1)$ and Young inequality we get

$$\begin{aligned}|g(t,y_t^n,z_t^n)|^2 &\leq (1+\frac{1-\alpha}{4\alpha})|g(t,y_t^n,z_t^n)-g(t,0,0)|^2+(1+\frac{4\alpha}{1-\alpha})\mid g(t,0,0)\mid^2\\ &\leq \frac{1+3\alpha}{4\alpha}C|y_t^n|^2+\frac{1+3\alpha}{4}|z_t^n|^2+\frac{1+3\alpha}{1-\alpha}|g(t,0,0)|^2.\end{aligned}$$

By virtue of Lemma 3.1 and using Young inequality we have

$$\begin{aligned}y_t^n\underline{f}_n(t,y_t^n,z_t^n) &\leq C|y_t^n|(K_t+|y_t^n|+|z_t^n|)\\ &\leq (\frac{3C}{2}+\frac{C^2}{2-2\alpha})|y_t^n|^2+\frac{1-\alpha}{2}|z_t^n|^2+\frac{C}{2}|K_t|^2.\end{aligned}$$

Consequently, by the above inequalities and applying Itô's formula to $|y_t^n|^2$ and taking mathematical expectation, we obtain

$$\begin{aligned}\mathbb{E}\int_0^T|z_t^n|^2dt &= \mathbb{E}|\xi|^2-|y_0^n|^2+2\mathbb{E}\int_0^T y_t^n\underline{f}_n(t,y_t^n,z_t^n)dt\\ &\quad+\mathbb{E}\int_0^T|g(t,y_t^n,z_t^n)|^2dt\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}|\xi|^2+\frac{3+\alpha}{4}\mathbb{E}\int_0^T|z_t^n|^2dt+\frac{1+3\alpha}{1-\alpha}\mathbb{E}\int_0^T|g(t,0,0)|^2dt\\
&\quad+\Big(\frac{3C}{2}+\frac{C^2}{2-2\alpha}+\frac{1+3\alpha}{4\alpha}C\Big)\mathbb{E}\int_0^T|y_t^n|^2dt.+\frac{C}{2}E\int_0^T|K_t|^2dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\int_0^T|z_t^n|^2dt &\leq \frac{4}{1-\alpha}E|\xi|^2+\frac{4+12\alpha}{(1-\alpha)^2}\mathbb{E}\int_0^T|g(t,0,0)|^2dt\\
&\quad+\frac{4}{1-\alpha}\Big(\frac{3C}{2}+\frac{C^2}{2-2\alpha}+\frac{1+3\alpha}{4\alpha}C\Big)\mathbb{E}\int_0^T|y_t^n|^2dt+\frac{2C}{1-\alpha}\mathbb{E}\int_0^T|K_t|^2dt\\
&\doteq A,
\end{aligned}
$$

which is bounded and independent of n.

Using Itô's formula to $\mid y_t^n-y_t^m\mid^2$ we obtain

$$
\begin{aligned}
&\mathbb{E}\int_0^T|z_t^n-z_t^m|^2dt+\mid y_0^n-y_0^m\mid^2\\
=\ &2\mathbb{E}[\int_0^T(y_t^n-y_t^m)\Big(\underline{f}_n(t,y_t^n,z_t^n)-\underline{f}_m(t,y_t^m,z_t^m)\Big)dt]\\
&+\mathbb{E}\int_0^T\mid g(t,y_t^n,z_t^n)-g(t,y_t^m,z_t^m)\mid^2dt.
\end{aligned}
$$

From Lemma 3.1, $\parallel z_n\parallel_{M^2}\leq A$ and $\parallel y_n\parallel_{S^2}\leq A$ it follows that there exists a positive constant C_0 independent of n,m such that

$$
\begin{aligned}
&\mathbb{E}[\int_0^T(y_t^n-y_t^m)(\underline{f}_n(t,y_t^n,z_t^n)-\underline{f}_m(t,y_t^m,z_t^m))dt]\\
&\leq C_0(\mathbb{E}\int_0^T|y_t^n-y_t^m|^2dt)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, by virtue of $(H1)$ we get

$$
\begin{aligned}
&\mathbb{E}\int_0^T|z_t^n-z_t^m|^2dt+\mid y_0^n-y_0^m\mid^2\\
\leq\ &C_0\Big\{\mathbb{E}\int_0^T|y_t^n-y_t^m|^2dt\Big\}^{\frac{1}{2}}\\
&+\alpha\mathbb{E}\int_0^T|z_t^n-z_t^m|^2dt+C\mathbb{E}\int_0^T|y_t^n-y_t^m|^2dt.
\end{aligned}
$$

Then, we deduce

$$
\begin{aligned}
&(1-\alpha)\mathbb{E}\int_0^T|z_t^n-z_t^m|^2dt\\
&\leq C_0\Big\{\mathbb{E}\int_0^T|y_t^n-y_t^m|^2dt\Big\}^{\frac{1}{2}}+C\mathbb{E}\int_0^T|y_t^n-y_t^m|^2dt.
\end{aligned}
$$

Therefore, $\{z^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $M^2(0,T;\mathbb{R}^d)$. Then, there exists $\underline{z}\in M^2(0,T;\mathbb{R}^d)$ such that

$$
\lim_{n\to\infty}\mathbb{E}[\int_0^T|z_t^n-\underline{z}_t|^2dt]=0.
$$

Thanks to $(H1)$ and BDG inequality we know that there exists a positive constant C_1 independent of n such that

$$\begin{aligned}
& \mathbb{E}[\sup_{t\in[0,T]} |\int_t^T g(s,\underline{y}_s,\underline{z}_s)dB_s - \int_t^T g(s,y_s^n,z_s^n)dB_s|^2] \\
\leq & \ C_1\mathbb{E}[\int_0^T |g(s,\underline{y}_s,\underline{z}_s) - g(s,y_s^n,z_s^n)|^2 ds] \\
\leq & \ C_1\alpha\mathbb{E}[\int_0^T |z_t^n - \underline{z}_t|^2 dt] + C_1 C\mathbb{E}[\int_0^T |y_t^n - \underline{y}_t|^2 dt] \\
\to & \ 0, \text{ as } n\to\infty.
\end{aligned}$$

For all $N>0$ and $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, from Lemma 3.1 and Dini's Theorem it follows that

$$\lim_{n\to\infty} \sup_{|y|+|z|\leq N} |\underline{f}_n(t,y,z) - f(t,y,z)| = 0, \ dtd\mathbb{P}-a.s.$$

Therefore, by the dominated convergence theorem we have

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |\underline{f}_n(t,y_t^n,z_t^n) - f(t,y_t^n,z_t^n)| 1_{\{|y_t^n|+|z_t^n|\leq N\}} dt = 0.$$

By virtue of $(H7)$ we know that

$$\mathbb{E}\int_0^T |f(t,y_t^n,z_t^n) - f(t,\underline{y}_t,\underline{z}_t)| 1_{\{|y_t^n|+|z_t^n|\leq N\}} dt$$

converges to 0 at least along a subsequence.

From Lemma 3.1 and $(H8')$ it follows that

$$\mathbb{E}\int_0^T |\underline{f}_n(t,y_t^n,z_t^n) - f(t,\underline{y}_t,\underline{z}_t)|(|y_t^n|+|z_t^n|)dt \doteq C_2 < \infty.$$

Here C_2 is a positive constant and independent of n. Consequently,

$$\mathbb{E}\int_0^T |\underline{f}_n(t,y_t^n,z_t^n) - f(t,\underline{y}_t,\underline{z}_t)| 1_{\{|y_t^n|+|z_t^n|>N\}} dt \leq \frac{C_2}{N}.$$

Combining the above inequalities, passing to a subsequence if necessary, we have

$$\begin{aligned}
& \mathbb{E}\int_0^T |\underline{f}_n(t,y_t^n,z_t^n) - f(t,\underline{y}_t,\underline{z}_t)| dt \\
\leq & \ \mathbb{E}\int_0^T |\underline{f}_n(t,y_t^n,z_t^n) - f(t,y_t^n,z_t^n)| 1_{\{|y_t^n|+|z_t^n|\leq N\}} dt \\
& + \mathbb{E}\int_0^T |f(t,y_t^n,z_t^n) - f(t,\underline{y}_t,\underline{z}_t)| 1_{\{|y_t^n|+|z_t^n|\leq N\}} dt + \frac{C_2}{N} \\
\to & \ \frac{C_2}{N},
\end{aligned}$$

as $n\to\infty$. Thus, letting $N\to\infty$, we have

$$\mathbb{E}\int_0^T |\underline{f}_n(t,y_t^n,z_t^n) - f(t,y_t,z_t)| dt \to 0,$$

as $n \to \infty$, passing to a subsequence if necessary. We now pass to the limit on both sides of BDSDE (3.1), passing to a subsequence if necessary, it follows that

$$\underline{y}_t = \xi + \int_t^T f(s, \underline{y}_s, \underline{z}_s)ds + \int_t^T g(s, \underline{y}_s, \underline{z}_s)dB_s - \int_t^T \underline{z}_s dW_s.$$

Consequently, BDSDE with data (f, g, T, ξ) has a solution $(\underline{y}, \underline{z})$.

Let (y', z') be any solution of BDSDE with data (f, g, T, ξ). Then, let us consider the following BDSDEs:

$$y'_t = \xi + \int_t^T f(s, y'_s, z'_s)ds + \int_t^T g(s, y'_s, z'_s)dB_s - \int_t^T z'_s dW_s,\ t \in [0, T],$$

and

$$y^n_t = \xi + \int_t^T \underline{f}_n(s, y^n_s, z^n_s)ds + \int_t^T g(s, y^n_s, z^n_s)dB_s - \int_t^T z^n_s dW_s, \quad t \in [0, T],$$

By virtue of Lemma 3.2 we have $y^n \leq y'$. Consequently, due to the first part of the proof and taking the limit we have $\underline{y} \leq y'$. The proof is complete. $\square$

If $\{K_t\}_{t\in[0,T]}$ is a bounded process, then we have the following corollary, which was obtained by Shi et al. [16].

Corollary 3.4 *Under the assumptions $(H7)$ and $(H8)$, BDSDE with data (f, g, T, ξ) has the minimal solution $(\underline{y}_t, \underline{z}_t)_{0\leq t\leq T}$ (resp. maximal solution $(\overline{y}_t, \overline{z}_t)_{0\leq t\leq T}$). Moreover, for all $t \in [0, T]$,*

$$\underline{y}^n_t \leq \underline{y}^{n+1}_t \leq \underline{y}_t \leq \overline{y}_t \leq \overline{y}^{n+1}_t \leq \overline{y}^n_t.$$

And $(\underline{y}^n, \underline{z}^n) \to (\underline{y}, \underline{z})$ and $(\overline{y}^n, \overline{z}^n) \to (\overline{y}, \overline{z})$ both in $S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$, as $n \to \infty$, where $(\underline{y}^n, \underline{z}^n)$ is the unique solution of BDSDE with data $(\underline{f}_n, g, T, \xi)$ and $(\overline{y}^n, \overline{z}^n)$ is the unique solution of BDSDE with data $(\overline{f}_n, g, T, \xi)$.

4 Existence theorem and comparison theorems for BDSDEs with discontinuous coefficients

The objective of this section is to investigate an existence theorem and a comparison theorem for solutions of BDSDEs with discontinuous coefficients.

We shall give a comparison theorem for BDSDEs (2.1) under the conditions that f is Lipschitz in y and uniformly continuous in z, i.e.,

$(H10)$ There exists a positive constant C such that, for all $(t, y_i, z_i) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1, 2$,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C|y_1 - y_2| + \phi(|z_1 - z_2|),$$

where ϕ is introduced in $(H5)$.

We need the following existence theorem and uniqueness theorem for BDSDEs, which was established in [13].

Lemma 4.1 *Under the assumptions $(H4)$ and $(H10)$, BDSDE (2.1) has a unique solution $(Y, Z) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$.*

Since ϕ is uniformly continuous, then we can not apply comparison theorems for solutions of BDSDEs in [16] and [12] to the proofs of Lemma 4.3 and Theorem 4.4. We now establish a comparison theorem of BDSDEs when f satisfies the condition $(H10)$, which plays an important role in the proofs of Lemma 4.3 and Theorem 4.4.

Theorem 4.2 *Suppose that BDSDEs with data (f^1, g, T, ξ^1) and (f^2, g, T, ξ^2) have solutions (y^1, z^1) and (y^2, z^2), respectively. If f^1 satisfies $(H4)$ and $(H10)$, $\xi^1 \leq \xi^2$, a.s., $f^1(t, y_t^2, z_t^2) \leq f^2(t, y_t^2, z_t^2)$, $d\mathbb{P}dt - a.s.$ (resp. f^2 satisfies $(H4)$ and $(H10)$, $f^1(t, y_t^1, z_t^1) \leq f^2(t, y_t^1, z_t^1)$, $d\mathbb{P}dt - a.s.$), then we have $y_t^1 \leq y_t^2$, a.s., for all $t \in [0, T]$.*

Proof: We only prove the first case, the other case can be proved similarly. For $n \in \mathbb{N}, (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, we let

$$f_n^1(t, y, z) = \inf_{v \in \mathbb{R}^d} \Big\{ f^1(t, y, v) + n|z - v| \Big\}.$$

Then, since $f^1(t, y_t^2, z_t^2) \leq f^2(t, y_t^2, z_t^2)$, $a.s.$, we have $f_n^1(t, y_t^2, z_t^2) \leq f^1(t, y_t^2 . z_t^2) \leq f^2(t, y_t^2, z_t^2)$, $a.s.$

From Lemma 2.6 and $(H10)$ it follows that, for all $(t, y_i, z_i) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1, 2$,

$$|f_n^1(t, y_1, z_1) - f_n^1(t, y_2, z_2)| \leq C|y_1 - y_2| + n|z_1 - z_2|,$$

and we consider the following BDSDE:

$$\underline{y}_t^n = \xi^1 + \int_t^T f_n^1(s, \underline{y}_s^n, \underline{z}_s^n)ds + \int_t^T g(s, \underline{y}_s^n, \underline{z}_s^n)dB_s - \int_t^T \underline{z}_s^n dW_s, \ t \in [0, T],$$

and

$$y_t^2 = \xi^2 + \int_t^T f^2(s, y_s^2, z_s^2)ds + \int_t^T g(s, y_s^2, z_s^2)dB_s - \int_t^T z_s^2 dW_s, \ t \in [0, T].$$

By virtue of Lemma 3.2 we obtain $\underline{y}_t^n \leq y_t^2$, for $n > C$. Theorem 3.3 and Lemma 4.1 yield

$$y_t^1 \leq y_t^2, \quad a.s., \text{ for all } \ t \in [0, T].$$

The proof is complete. □

From now we study an existence theorem and a comparison theorem for solutions of BDSDEs under the conditions $(H5), (H6)$ and $(H9)$.

From $(H9)$ we know that there exist two BDSDEs: $i = 1, 2$,

$$Y_t^i = \xi + \int_t^T f_i(s, Y_s^i, Z_s^i)ds + \int_t^T g(s, Y_s^i, Z_s^i)dB_s - \int_t^T Z_s^i dW_s, \ t \in [0, T]$$

such that $f_i(\cdot, Y_\cdot^i, Z_\cdot^i) \in M^2(0, T; \mathbb{R})$.

We now construct a sequence of BDSDEs as follows:

$$\begin{aligned} y_t^n = \xi + &\int_t^T [f(s, y_s^{n-1}, z_s^{n-1}) - C(y_s^n - y_s^{n-1}) - \phi(|z_s^n - z_s^{n-1}|)]ds \\ &+ \int_t^T g(s, y_s^n, z_s^n)dB_s - \int_t^T z_s^n dW_s, \quad t \in [0, T], \end{aligned} \tag{4.1}$$

where $n = 1, 2, \cdots$, and $(y^0, z^0) = (Y^1, Z^1)$. We have the following lemma:

Lemma 4.3 *Under the assumptions* $(H5), (H6)$ *and* $(H9)$*, for all* $n = 1, 2, \cdots$ *, BDSDE (4.1) has a unique solution* $(y^n, z^n) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$*, and* $Y_t^1 \leq y_t^n \leq y_t^{n+1} \leq Y_t^2$*, a.s., for all* $t \in [0, T]$*.*

Proof: For $n = 1$, from $(H5)$, $(H6)$, $(H9)$ and $Y_t^1 \leq Y_t^2$ it follows that

$$\begin{aligned} & f_2(t, Y_t^2, Z_t^2) - f(t, Y_t^1, Z_t^1) \geq f(t, Y_t^2, Z_t^2) - f(t, Y_t^1, Z_t^1) \\ & \geq -C(Y_t^2 - Y_t^1) - \phi(|Z_t^2 - Z_t^1|). \end{aligned}$$

Then, we have

$$f_2(t, Y_t^2, Z_t^2) + C(Y_t^2 - Y_t^1) + \phi(|Z_t^2 - Z_t^1|) \geq f(t, Y_t^1, Z_t^1) \geq f_1(t, Y_t^1, Z_t^1),$$

and

$$f_2(t, Y_t^2, Z_t^2) \geq f(t, Y_t^1, Z_t^1) - C(Y_t^2 - Y_t^1) - \phi(|Z_t^2 - Z_t^1|).$$

Thus, due to $(H9)$ and linear growth of ϕ we have $f(\cdot, Y_\cdot^1, Z_\cdot^1) \in M^2(0, T; \mathbb{R})$, and from Lemma 4.1 it follows that BDSDE (4.1) has a unique solution $(y^1, z^1) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$, and by virtue of Theorem 4.2 we have $Y_t^1 \leq y_t^1 \leq Y_t^2$, a.s., for all $t \in [0, T]$.

For $n = 2$, by $(H5)$, $(H6)$, $(H9)$ and $Y_\cdot^1 \leq y_\cdot^1 \leq Y_\cdot^2$ we deduce

$$\begin{aligned} f_2(t, Y_t^2, Z_t^2) - f(t, y_t^1, z_t^1) & \geq f(t, Y_t^2, Z_t^2) - f(t, y_t^1, z_t^1) \\ & \geq -C(Y_t^2 - y_t^1) - \phi(|Z_t^2 - z_t^1|), \end{aligned}$$

and

$$\begin{aligned} f(t, y_t^1, z_t^1) - f_1(t, Y_t^1, Z_t^1) & \geq f(t, y_t^1, z_t^1) - f(t, Y_t^1, Z_t^1) \\ & \geq -C(y_t^1 - Y_t^1) - \phi(|z_t^1 - Z_t^1|). \end{aligned}$$

Then, we obtain

$$\begin{aligned} & f_2(t, Y_t^2, Z_t^2) + C(Y_t^2 - y_t^1) + \phi(|Z_t^2 - z_t^1|) \\ & \geq f(t, y_t^1, z_t^1) \geq f_1(t, Y_t^1, Z_t^1) - C(y_t^1 - Y_t^1) - \phi(|z_t^1 - Z_t^1|), \end{aligned}$$

$$f_2(t, Y_t^2, Z_t^2) \geq f(t, y_t^1, z_t^1) - C(Y_t^2 - y_t^1) - \phi(|Z_t^2 - z_t^1|),$$

and

$$f(t, y_t^1, z_t^1) \geq f(t, Y_t^1, Z_t^1) - C(y_t^1 - Y_t^1) - \phi(|z_t^1 - Z_t^1|).$$

Thus, $f(\cdot, y_\cdot^1, z_\cdot^1) \in M^2(0, T; \mathbb{R})$, and by Lemma 4.1 and Theorem 4.2 we know that BDSDE (4.1) has a unique solution $(y^2, z^2) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$, and $Y_t^1 \leq y_t^1 \leq y_t^2 \leq Y_t^2$, a.s., for all $t \in [0, T]$.

For $n > 2$, we suppose that $Y^1 \leq y^{n-1} \leq y^n \leq Y^2$, and $f(\cdot, y_\cdot^{n-1}, z_\cdot^{n-1}) \in M^2(0, T; \mathbb{R})$. Let us consider the following BDSDE:

$$\begin{aligned} y_t^{n+1} = \xi + & \int_t^T \Big[f(s, y_s^n, z_s^n) - C(y_s^{n+1} - y_s^n) - \phi(|z_s^{n+1} - z_s^n|) \Big] ds \\ & + \int_t^T g(s, y_s^{n+1}, z_s^{n+1}) dB_s - \int_t^T z_s^{n+1} dW_s, \ t \in [0, T]. \end{aligned} \tag{4.2}$$

Using the similar argument as $n = 2$ we get

$$\begin{aligned}&f_2(t, Y_t^2, Z_t^2) + C(Y_t^2 - y_t^n) + \phi(|Z_t^2 - z_t^n|) \geq f(t, y_t^n, z_t^n)\\ &\geq f_1(t, Y_t^1, Z_t^1) - C(y_t^n - Y_t^1) - \phi(|z_t^n - Z_t^1|),\end{aligned} \tag{4.3}$$

$$f_2(t, Y_t^2, Z_t^2) \geq f(t, y_t^n, z_t^n) - C(Y_t^2 - y_t^n) - \phi(|Z_t^2 - z_t^n|),$$

and

$$f(t, y_t^n, z_t^n) \geq f(t, y_t^{n-1}, z_t^{n-1}) - C(y_t^n - y_t^{n-1}) - \phi(|z_t^n - z_t^{n-1}|).$$

Consequently, $f(\cdot, y_\cdot^n, z_\cdot^n) \in M^2(0, T; \mathbb{R})$, and using Lemma 4.1 and Theorem 4.2 again we obtain that BDSDE (4.2) has a unique solution $(y^{n+1}, z^{n+1}) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$, and $Y_t^1 \leq y_t^n \leq y_t^{n+1} \leq Y_t^2$, a.s., for all $t \in [0, T]$. The proof is complete. □

We now state and prove the main result in this section.

Theorem 4.4 *Under the assumptions* $(H5), (H6)$ *and* $(H9)$*, BDSDE with data* (f, g, T, ξ) *has a solution. Moreover, if* f_1 *satisfies* $(H4)$ *and* $(H10)$*, then BDSDE with data* (f, g, T, ξ) *has a minimal solution* $(\underline{y}, \underline{z})$*, in the sense that, for any other solution* (y, z) *of BDSDE with data* (f, g, T, ξ)*, we have* $\underline{y} \leq y$*.*

Proof: By Lemma 4.3 we know that $\{y^n\}_{n=1}^{\infty}$ converges to a limit $\underline{y}$ in $S^2(0, T; \mathbb{R})$ and

$$\sup_n \mathbb{E}[\sup_{0\leq t\leq T} |y_t^n|^2] \leq \mathbb{E}[\sup_{0\leq t\leq T} |Y_t^1|^2] + \mathbb{E}[\sup_{0\leq t\leq T} |Y_t^2|^2] < \infty.$$

Let

$$f^n(t, y_t^n, z_t^n) \doteq f(t, y_t^{n-1}, z_t^{n-1}) - C(y_t^n - y_t^{n-1}) - \phi(|z_t^n - z_t^{n-1}|).$$

Then, from $(H5), (H6)$ and (4.3) it follows that

$$\begin{aligned}&| f^n(t, y_t^n, z_t^n)| \leq |f(t, y_t^{n-1}, z_t^{n-1})\ | + C|y_t^n - y_t^{n-1}| + \phi(|z_t^n - z_t^{n-1}|)\\ \leq\ & \sum_{i=1}^{2} \Big[|f_i(t, Y_t^i, Z_t^i)| + C|Y_t^i| + C|Z_t^i|\Big]\\ &+C\Big[|y_t^n| + |z_t^n|\Big] + 3C\Big[|y_t^{n-1}| + |z_t^{n-1}| + 1\Big].\end{aligned}$$

Thanks to $(H1)$ we get

$$\begin{aligned}|g(t, y_t^n, z_t^n)|^2 &\leq (1 + \frac{1-\alpha}{2\alpha})|g(t, y_t^n, z_t^n) - g(t, 0, 0)\ |^2 + (1 + \frac{2\alpha}{1-\alpha})\ |\ g(t, 0, 0)\ |^2\\ &\leq \frac{1+\alpha}{2\alpha} C|y_t^n|^2 + \frac{1+\alpha}{2}|z_t^n|^2 + \frac{1+\alpha}{1-\alpha}\ |\ g(t, 0, 0)\ |^2\ .\end{aligned}$$

We apply Itô's formula to $|\ y_t^n\ |^2$ and obtain

$$\begin{aligned}\mathbb{E}\int_0^T |z_t^n|^2 dt &= \mathbb{E}\ |\ \xi\ |^2 - |\ y_0^n\ |^2 + 2\mathbb{E}\int_0^T y_t^n f^n(t, y_t^n, z_t^n) dt\\ &\quad + \mathbb{E}\int_0^T |\ g(t, y_t^n, z_t^n)\ |^2\ dt\\ &\leq C_1 + \frac{3+\alpha}{4}\mathbb{E}\int_0^T |z_t^n|^2 dt + \frac{1-\alpha}{8} E\int_0^T |z_t^{n-1}|^2 dt,\end{aligned}$$

where

$$
\begin{aligned}
C_1 \quad \doteq \quad & \sup_n \mathbb{E}\Big\{2\int_0^T \sum_{i=1}^{2} |y_t^n| \Big[|f_i(t, Y_t^i, Z_t^i)| + C|Y_t^i| + C|Z_t^i|\Big] dt \\
& + \frac{1+\alpha}{1-\alpha}\mathbb{E}\int_0^T |g(t,0,0)|^2 dt + \Big(2C + \frac{1+\alpha}{2\alpha}C + \frac{76C^2}{1-\alpha}\Big)\int_0^T |y_t^n|^2 dt \\
& + 6C\int_0^T |y_t^n y_t^{n-1}| dt + 6C\int_0^T |y_t^n| dt\Big\} + \mathbb{E}|\xi|^2 < \infty.
\end{aligned}
$$

Then, we deduce

$$
\mathbb{E}\int_0^T |z_t^n|^2 dt \leq \frac{4C_1}{1-\alpha} + \frac{1}{2}E\int_0^T |z_t^{n-1}|^2 dt.
$$

Therefore, we get

$$
\sup_n \mathbb{E}\int_0^T |z_t^n|^2 dt < \infty
$$

and

$$
\sup_n \mathbb{E}\int_0^T |f^n(t, y_t^n, z_t^n)|^2 dt < \infty.
$$

Let

$$
C_2 \doteq \sup_n \mathbb{E}\int_0^T |f^n(t, y_t^n, z_t^n)|^2 dt.
$$

Using Itô's formula to $| y_t^n - y_t^m |^2$ we obtain

$$
\begin{aligned}
& \mathbb{E}\int_0^T |z_t^n - z_t^m|^2 dt + | y_0^n - y_0^m |^2 \\
= \quad & 2\mathbb{E}\int_0^T (y_t^n - y_t^m)\Big(f^n(t, y_t^n, z_t^n) - f^m(t, y_t^m, z_t^m)\Big) dt \\
& + \mathbb{E}\int_0^T | g(t, y_t^n, z_t^n) - g(t, y_t^m, z_t^m) |^2 \, dt.
\end{aligned}
$$

Due to $(H1)$ again it follows that

$$
\begin{aligned}
& \mathbb{E}\int_0^T |z_t^n - z_t^m|^2 dt + | y_0^n - y_0^m |^2 \\
\leq \quad & 4C_2^{\frac{1}{2}}\Big\{\mathbb{E}\int_0^T |y_t^n - y_t^m|^2 dt\Big\}^{\frac{1}{2}} \\
& + \alpha\mathbb{E}\int_0^T |z_t^n - z_t^m|^2 dt + C\mathbb{E}\int_0^T |y_t^n - y_t^m|^2 dt.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& (1-\alpha)\mathbb{E}\int_0^T |z_t^n - z_t^m|^2 dt \\
& \leq 4C_2^{\frac{1}{2}}\Big\{\mathbb{E}\int_0^T |y_t^n - y_t^m|^2 dt\Big\}^{\frac{1}{2}} + C\mathbb{E}\int_0^T |y_t^n - y_t^m|^2 dt.
\end{aligned}
$$

Therefore, $\{z^n\}_{n=1}^\infty$ is a Cauchy sequence in $M^2(0,T;\mathbb{R}^d)$, and there exists $\underline{z} \in M^2(0,T;\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} E\int_0^T |z_t^n - \underline{z}_t|^2 dt = 0.$$

From $(H1),(H2),(H5),(H6),$ the above equality and $\{y^n\}_{n=1}^\infty$ converges to $\underline{y}$ in $S^2(0,T;\mathbb{R})$ it follows that

$$\sup_{t\in[0,T]} |\int_t^T z_s^n dW_s - \int_t^T \underline{z}_s dW_s| \xrightarrow{\mathbb{P}} 0, \tag{4.4}$$

$$\sup_{t\in[0,T]} |\int_t^T g(s,y_s^n,z_s^n)dB_s - \int_t^T g(s,\underline{y}_s,\underline{z}_s)dB_s| \xrightarrow{\mathbb{P}} 0, \tag{4.5}$$

and for almost all $\omega \in \Omega$, passing to a subsequence if necessary, we have

$$f^n(t,y_t^n,z_t^n) - f(t,\underline{y}_t,\underline{z}_t) \to 0,\ dt - a.e.,\ \text{as } n \to \infty.$$

Combining the above inequalities with the dominated convergence theorem yield

$$\int_0^T f^n(s,y_s^n,z_s^n)ds \to \int_0^T f(s,\underline{y}_s,\underline{z}_s)ds, \tag{4.6}$$

as $n \to \infty$. Consequently, (4.4), (4.5) and (4.6) allow us to pass to the limit on both sides of BDSDE (4.1), passing to a subsequence if necessary, it follows that

$$\underline{y}_t = \xi + \int_t^T f(s,\underline{y}_s,\underline{z}_s)ds + \int_t^T g(s,\underline{y}_s,\underline{z}_s)dB_s - \int_t^T \underline{z}_s dW_s,\ t\in[0,T].$$

Consequently, BDSDE with data (f,g,T,ξ) has a solution $(\underline{y},\underline{z})$.

Let (y,z) be any solution of BDSDE (2.1). From $f_1(t,y,z) \le f(t,y,z)$, for all $(t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$, and Theorem 4.2 it follows that $Y_t^1 \le y_t$, a.s., for all $t\in[0,T]$.

For $n=1$, we consider the following BDSDE:

$$\begin{aligned} y_t^1 = \xi &+ \int_t^T \Big[f(s,Y_s^1,Z_s^1) - C(y_s^1 - Y_s^1) - \phi(|z_s^1 - Z_s^1|)\Big]ds \\ &+ \int_t^T g(s,y_s^1,z_s^1)dB_s - \int_t^T z_s^1 dW_s,\ t\in[0,T]. \end{aligned}$$

From $(H5)$, $(H6)$ and $Y^1 \le y$ it follows that

$$f(t,y_t,z_t) \ge f(t,Y_t^1,Z_t^1) - C(y_t - Y_t^1) - \phi(|z_t - Z_t^1|).$$

Thus, by virtue of Theorem 4.2 we have $y_t^1 \le y_t$, a.s., for all $t\in[0,T]$.

For $n \ge 2$, we assume that $y^n \le y$. Let us consider the following BDSDE:

$$\begin{aligned} y_t^{n+1} = \xi &+ \int_t^T \Big[f(s,y_s^n,z_s^n) - C(y_s^{n+1} - y_s^n) - \phi(|z_s^{n+1} - z_s^n|)\Big]ds \\ &+ \int_t^T g(s,y_s^{n+1},z_s^{n+1})dB_s - \int_t^T z_s^{n+1}dW_s,\quad t\in[0,T]. \end{aligned}$$

By the similar argument as $n = 1$ we get

$$f(t, y_t, z_t) \geq f(t, y_t^n, z_t^n) - C(y_t - y_t^n) - \phi(|z_t - z_t^n|).$$

Therefore, using Theorem 4.2 again we obtain $y_t^{n+1} \leq y_t$, a.s., for all $t \in [0, T]$. Then, by virtue of the first part of the proof and taking the limit we have $\underline{y}_t \leq y_t$ a.s., for all $t \in [0, T]$. The proof is complete. □

Remark 4.5 *The above theorem generalizes the result in Lin [12]. In fact, we can take* $\phi(x) = Cx, x \geq 0$, *where* C *is a positive constant.*

Remark 4.6 *Under the assumptions of Theorem 4.4, if* f_2 *satisfies* $(H4)$ *and* $(H10)$, *and BDSDE (4.1) is replaced by the following BDSDE:*

$$\begin{aligned} y_t^n = \xi &+ \int_t^T [f(s, y_s^{n-1}, z_s^{n-1}) - C(y_s^n - y_s^{n-1}) + \phi(|z_s^n - z_s^{n-1}|)]ds \\ &+ \int_t^T g(s, y_s^n, z_s^n)dB_s - \int_t^T z_s^n dW_s, \quad t \in [0, T], \end{aligned}$$

where $n = 1, 2, \cdots$, *and* $(y^0, z^0) = (Y^2, Z^2)$. *Similar to the proof of Lemma 4.3 and Theorem 4.4, we can prove that BDSDE with data* (f, g, T, ξ) *has the maximal solution.*

Remark 4.7 *Under assumptions* $(H5), (H6)$ *and* $(H9)$, *the solution of BDSDE with data* (f, g, T, ξ) *may be non-unique. Let us consider the following BDSDE:*

$$y_t = \int_t^T \left[4sSgn(y_s)\sqrt{|y_s|} + \sqrt{z_s 1_{z_s \geq 0}}\right]ds + \int_t^T \left[1_{\{y_s < 0\}} y_s + \frac{1}{2} z_s\right]dB_s - \int_t^T z_s dW_s, \ t \in [0, T],$$

where $Sgn(x) = 1, x \geq 0;\ Sgn(x) = -1, x < 0$. *We can check that the above equation satisfies* $(H5), (H6)$ *and* $(H9)$, *where*

$$f_1(t, y, z) = -2t^2 - 2|y| + z \text{ and } f_2 = 2t^2 + 2|y| + z.$$

It's easy to check that, for each $c \in [0, T]$ *and* $t \in [0, T]$, $(y_t, z_t) = (0, 0)$ *and* $(y_t, z_t) = \Big([max\{c^2 - t^2, 0\}]^2, 0\Big)$ *are solutions of the above BDSDE.*

Finally, we give a comparison theorem for BDSDEs with discontinuous coefficients.

Theorem 4.8 *We suppose that* f^1 *and* f^2 *satisfy* $(H5), (H6)$ *and* $(H9)$, *and* f_1 *satisfies* $(H4)$ *and* $(H10)$. *Let the minimal solutions* $(\underline{y}^1, \underline{z}^1)$ *and* $(\underline{y}^2, \underline{z}^2)$ *of BDSDEs (2.1) with data* (f^1, g, T, ξ^1) *and* (f^2, g, T, ξ^2), *respectively. If* $\xi^1 \leq \xi^2$, *a.s., and* $f^1(t, y, z) \leq f^2(t, y, z)$, *a.s., then we have* $\underline{y}_t^1 \leq \underline{y}_t^2$, *a.s., for all* $t \in [0, T]$.

Proof: From $(H9)$ we know that there exists the following BDSDE:

$$Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1)ds + \int_t^T g(s, Y_s^1, Z_s^1)dB_s - \int_t^T Z_s^1 dW_s, \ t \in [0, T],$$

such that $f_1(t, Y_t^1, Z_t^1) \in M^2(0, T; \mathbb{R})$.

We consider a sequence of BDSDEs as follows:

$$y_t^n = \xi + \int_t^T \Big[f^1(s, y_s^{n-1}, z_s^{n-1}) - C(y_s^n - y_s^{n-1}) - \phi(|z_s^n - z_s^{n-1}|) \Big] ds$$
$$+ \int_t^T g(s, y_s^n, z_s^n) dB_s - \int_t^T z_s^n dW_s, \quad t \in [0,T],$$

where $n = 1, 2, \cdots$ and $(y^0, z^0) = (Y^1, Z^1)$.

From $(H9)$ and $f^1(t,y,z) \le f^2(t,y,z)$, *a.s.*, we know that $f_1(t,y,z) \le f^2(t,y,z)$, *a.s.* Then, from Theorem 4.2 it follows that $Y_t^1 \le \underline{y}_t^2$, *a.s.*, for all $t \in [0,T]$.

For $n = 1$, by virtue of $(H5)$, $(H6)$ and $Y^1 \le \underline{y}^2$ we get

$$f^2(t, \underline{y}_t^2, \underline{z}_t^2) - f^1(t, Y_t^1, Z_t^1) \ge f^1(t, \underline{y}_t^2, \underline{z}_t^2) - f^1(t, Y_t^1, Z_t^1) \ge -C(\underline{y}_t^2 - Y_t^1) - \phi(|\underline{z}_t^2 - Z_t^1|).$$

Then, we have

$$f^2(t, \underline{y}_t^2, \underline{z}_t^2) \ge f^1(t, Y_t^1, Z_t^1) - C(\underline{y}_t^2 - Y_t^1) - \phi(|\underline{z}_t^2 - Z_t^1|).$$

Thus, by virtue of Theorem 4.2 we have $y_t^1 \le \underline{y}_t^2$, a.s., for all $t \in [0,T]$.

For $n \ge 2$, we suppose that $y^n \le \underline{y}^2$. Then, let us consider the following BDSDE:

$$y_t^{n+1} = \xi + \int_t^T \Big[f^1(s, y_s^n, z_s^n) - C(y_s^{n+1} - y_s^n) - \phi(|z_s^{n+1} - z_s^n|) \Big] ds$$
$$+ \int_t^T g(s, y_s^{n+1}, z_s^{n+1}) dB_s - \int_t^T z_s^{n+1} dW_s, \ t \in [0,T].$$

By virtue of the similar argument as $n = 1$ we have

$$f^2(t, \underline{y}_t^2, \underline{z}_t^2) \ge f^1(t, y_t^n, z_t^n) - C(\underline{y}_t^2 - y_t^n) - \phi(|\underline{z}_t^2 - z_t^n|).$$

Then, thanks to Theorem 4.2 we get $y_t^{n+1} \le \underline{y}_t^2$, a.s., for all $t \in [0,T]$. From the proof of Theorem 4.4 it follows that $\underline{y}_t^1 \le \underline{y}_t^2$, a.s., for all $t \in [0,T]$. The proof is complete. □

Remark 4.9 *Similar to the proof of Theorem 4.8, we can prove that a comparison theorem for the maximal solution of BDSDE with data (f, g, T, ξ) by using Remark 4.6.*

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