# Dependence on parameters for a discrete Emden-Fowler equation 

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November 20, 2018


#### Abstract

We investigate the dependence on parameters for the discrete boundary value problem connected with the Emden-Fowler equation. A variational method is used in order to obtain a general scheme allowing for investigation the dependence on paramaters of discrete boundary value problems.


MSC Subject Classification: 34B16, 39M10
Keywords: variational method; discrete Emden-Fowler equation; coercivity; dependence on parameters

## 1 Introduction

The discrete version of the Emden-Fowler equation received some considerable interest lately by the use of critical point theory, see for example [7], [8], [9], [11]. Various variational approaches towards the existence of solutions for this problem can be applied as being the finite dimensional counterparts of the methods used in the continuous variational case. However, due to the finite dimensionality of the space in which solutions are obtained, we have
much more tools at our disposal. In finite dimensional space the weak solution - which is a key notion in the variational approach - is always a strong one. Also weak convergence, and thus weak lower semicontinuity coincides with strong convergence and classical lower semicontinuity which can in turn be obtained with no additional convexity assumption. Moreover, the coercivity of the action functional can be very often investigated together with its anti-coercivity which of course involves different growth conditions on the nonlinear term. Thus we have a lot of tools at our disposal as far as the existence is concerned, compare with [1], [6]. Although the application of variational methods to the discrete problems is rather a new topic, started apparently by [3], [6] the list of our references is by no means complete since research in this area has been very active.

In the boundary value problems for differential equations it is also important to know whether the solution, once its existence is proved, depends continuously on a functional parameter. This question has a great impact on future applications of any model. As it is well known difference equations serve as mathematical models in diverse areas, such as economy, biology, computer science, finance, see for example [2], [4], [10]. Thus it is desirable to know whether the solution to the small deviation from the model would return, in a continuous way, to the solution of the original model. This is known in differential equation as stability or continuous dependence on parameter, see [12], but it has not been investigated in the area of boundary value problems for difference equations, apart from some work done in [5]. In this paper we are going to get some general scheme for investigating the dependence on parameter in difference equations which we illustrate with the Emden-Fowler equation as a model example. Thus our interest lies not in the existence of solutions, which in fact has been vastly researched, but in their dependence on parameters. We mention that the approach of 5 ] required that each boundary value problem should be treated separately, while in this submission we provide some general result which could be further applied for any discrete boundary value problem for which the action functional is either coercive or anti-coercive.

## 2 Problem formulation and main results

In what follows $T$ is a fixed natural number, $T \geq 3 ;[a, b]$, where $a \leq b$ are integers, stands for the discrete interval $\{a, a+1, \ldots, b-1, b\} ; M>0$ is fixed, $L_{M}=\left\{v \in C([1, T], R):\|v\|_{C} \leq M\right\} ;\|v\|_{C}=\max _{k \in[1, T]}|v(k)|$. We consider the discrete equation

$$
\begin{equation*}
\Delta(p(k-1) \Delta x(k-1))+q(k) x(k)+f(k, x(k), u(k))=g(k) \tag{1}
\end{equation*}
$$

subject to a parameter $u \in L_{M}$ and with boundary conditions

$$
\begin{equation*}
x(0)=x(T), p(0) \Delta x(0)=p(T) \Delta x(T) \tag{2}
\end{equation*}
$$

known as the discrete version of the Emden-Fowler equation. We assume that

A1 $f \in C([1, T] \times R \times[-M, M], R), p \in C([0, T+1], R), q, g \in C([1, T], R) ;$ $g\left(k_{1}\right) \neq 0$ for certain $k_{1} \in[1, T]$.

The growth conditions on $f$ will be given later on. Solutions to (11)-(2) are such functions $v:[0, T+1] \rightarrow R$ that satisfy (1) as identity and further $v(0)=v(T), p(0) \Delta v(0)=p(T) \Delta v(T)$. Hence solutions to (1)-(2) are investigated on a finite dimensional space

$$
E=\{v:[0, T+1] \rightarrow R: v(0)=v(T), p(0) \Delta v(0)=p(T) \Delta v(T)\}
$$

Any function from $E$ can be identified with a vector from $R^{T}$ and therefore solutions to (11)-(2) can be investigated in $R^{T}$ with classical Euclidean norm. By $|\cdot|$ we denote the Euclidean norm, and by $\langle\cdot, \cdot\rangle$ the scalar product. We note that with A1 any solution to (1)-(2) is in fact nontrivial in the sense that no function $v:[0, T+1] \rightarrow R$ such that $v(k)=0$ for all $k \in[1, T]$ would satisfy (11)-(2). To reach this conclusion suppose that $v:[0, T+1] \rightarrow R$ such that $v(k)=0$ for all $k \in[1, T]$ satisfies (11)-(2). We see that at least for $k_{1}$ it follows $0=g\left(k_{1}\right) \neq 0$.

Variational approach towards (11)-(2) relays on investigation of critical points to a suitable action functional. Thus problem (11)-(21), but without parameter $u$ and a forcing term $g$, can be considered either as it stands, as
it is done in [13], or else one may write it in a matrix form in which form we will further investigate it. Let us denote as in 9 ]

$$
M=\left[\begin{array}{cccccc}
p(0)+p(1) & -p(1) & 0 & \ldots & 0 & -p(0) \\
-p(1) & p(1)+p(2) & -p(2) & \cdots & 0 & 0 \\
0 & -p(2) & p(2)+p(3) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p(T-2)+p(T-1) & -p(T-1) \\
-p(0) & 0 & 0 & \cdots & -p(T-1) & p(T-1)+p(0)
\end{array}\right]
$$

and

$$
Q=\left[\begin{array}{cccccc}
-q(1) & 0 & 0 & \cdots & 0 & 0 \\
0 & -q(2) & 0 & \cdots & 0 & 0 \\
0 & 0 & -q(3) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -q(T-1) & 0 \\
0 & 0 & 0 & \cdots & 0 & -q(T)
\end{array}\right] .
$$

For a fixed $u \in L_{M}$ we introduce the action functional $J: R^{T} \rightarrow R$ for (1)-(2) by the formula

$$
\begin{equation*}
J(x, u)=\frac{1}{2}\langle(M+Q) x, x\rangle-\sum_{k=1}^{T} F(k, x(k), u(k))+\sum_{k=1}^{T} g(k) x(k) . \tag{3}
\end{equation*}
$$

Calculating the Gâteaux derivative of $J(x, u)$ with respect to $x$ we relate critical points to (3) with solutions to (11)-(2) as in [8]. In fact any critical point to the action functional (3) is a solution to (1)-(2) and any solution to (1)-(2) provides a critical point to $J$. Hence, in order to find at least one solution (1)-(22) it suffice to find at least one critical point to (3). We denote

$$
V_{u}=\left\{x \in R^{T}: J(x, u)=\inf _{v \in R^{T}} J(v, u), \frac{d}{d x} J(x, u)=0\right\}
$$

for any fixed $u \in L_{M}$. We will employ the following assumptions concerning the growth of the nonlinear term $f$.

A2 there exist constants $\varepsilon_{1}>0, \varepsilon_{2} \in R$ and $r \in(1,2)$ such that

$$
\begin{equation*}
f(k, y, u) \leq \varepsilon_{1}|y|^{r-1}+\varepsilon_{2} \tag{4}
\end{equation*}
$$

uniformly for $u \in[-M, M], k \in[1, T]$ and $|y| \geq B$, where $B>0$ is certain (possibly large) constant.

A3 there exist constants $\varepsilon_{1}>0, \varepsilon_{2} \in R$ and $r>2$ such that

$$
\begin{equation*}
f(k, y, u) \geq \varepsilon_{1}|y|^{r-1}+\varepsilon_{2} \tag{5}
\end{equation*}
$$

uniformly for $u \in[-M, M], k \in[1, T]$ and $|y| \geq B$, where $B>0$ is certain (possibly large) constant.

Our main results are as follows.
Theorem 1 (sublinear case) Assume A1, A2 and further that $M+Q$ is either positive or negative definite matrix. For any fixed $u \in L_{M}$ there exists at least one non trivial solution $x \in V_{u}$ to problem (1)-(目). Let $\left\{u_{n}\right\}_{n=1}^{\infty} \subset$ $L_{M}$ be a sequence of parameters. For any sequence $\left\{x_{n}\right\}_{n=1}^{\infty}$ of solutions $x_{n} \in V_{u_{n}}$ to the problem (11)-(2) corresponding to $u_{n}$, there exist subsequences $\left\{x_{n_{i}}\right\}_{i=1}^{\infty} \subset R^{T},\left\{u_{n_{i}}\right\}_{i=1}^{\infty} \subset L_{M}$ and elements $\bar{x} \in R^{T}, \bar{u} \in L_{M}$ such that $\lim _{i \rightarrow \infty} x_{n_{i}}=\bar{x}, \lim _{n \rightarrow \infty} u_{n_{i}}=\bar{u}$. Moreover, $\bar{x} \in V_{\bar{u}}$ (which means that $\bar{x}$ satisfies (1)-(2) with $\bar{u}$ ), i.e.

$$
\begin{gathered}
\Delta(p(k-1) \Delta \bar{x}(k-1))+q(k) \bar{x}(k)+f(k, \bar{x}(k), \bar{u}(k))=g(k), \\
\bar{x}(0)=\bar{x}(T), p(0) \Delta \bar{x}(0)=p(T) \Delta \bar{x}(T) .
\end{gathered}
$$

Theorem 2 (superlinear case) Assume A1, A3. Then the assertion of Theorem 11 is valid.

We note that in Theorem 2 matrix $M+Q$ could be singular and its definiteness is not important.

In order to consider the case when $r=2$ we first introduce some necessary notation. In case when $M+Q$ is positive definite there exists a number $a_{M+Q}>0$ such that for all $y \in R^{T}$

$$
\begin{equation*}
\langle(M+Q) y, y\rangle \geq a_{M+Q}|y|^{2} \tag{6}
\end{equation*}
$$

while in case when $M+Q$ is negative definite there exists a number $b_{M+Q}>0$ such that for all $y \in R^{T}$

$$
\begin{equation*}
\langle(M+Q) y, y\rangle \leq-b_{M+Q}|y|^{2} \tag{7}
\end{equation*}
$$

Now we may formulate the assumptions in case $r=2$.
A4 let $M+Q$ be positive definite and let there exist constants $\varepsilon_{1} \in$ $\left(0,2 a_{M+Q}\right), \varepsilon_{2} \in R$ such that

$$
\begin{equation*}
f(k, y, u) \leq \varepsilon_{1}|y|+\varepsilon_{2} \tag{8}
\end{equation*}
$$

uniformly for $u \in[-M, M], k \in[1, T]$ and $|y| \geq B$, where $B>0$ is certain (possibly large) constant.

A5 let $M+Q$ be negative definite and let there exist constants $\varepsilon_{1} \in$ $\left(0,2 b_{M+Q}\right), \varepsilon_{2} \in R$ such that (8) holds uniformly for $u \in[-M, M], k \in[1, T]$ and $|y| \geq B$, where $B>0$ is certain (possibly large) constant.

Theorem 3 ( $r=2$ ) Assume either A1, A4 or A1, A5. Then the assertion of Theorem 1 is valid.

## 3 Auxiliary results

We will prove the following lemmas concerning the existence (11)-(2) with fixed $u \in L_{M}$. These lemmas improve certain existence results from [9]. In contrast to the boundary values for ODE, compare with [14], it is the superlinear case which is easier to be dealt with, while the sublinear one involves more restrictive assumptions.

Lemma 4 Assume A1, A2 and that $M+Q$ is either positive or negative definite matrix. Then for any fixed $u \in L_{M}$ there exists at least one non trivial solution $x \in V_{u}$ to problem (1)-(2).

Proof. Let us fix $u \in L_{M}$. We see that $x \rightarrow J(x, u)$ is continuous and differentiable the sense of Gâteaux on $R^{T}$ in either case. Thus we would have the assertion provided that $x \rightarrow J(x, u)$ is coercive or anti-coercive since functional $x \rightarrow J(x, u)$ would have either an argument of a minimum or an argument of a maximum which must be its critical point in turn.

Let $M+Q$ be a positive definite matrix. Let us take sufficiently large $B>0$ as in A2. We denote $|\bar{g}|=\sqrt{\sum_{k=1}^{T} g^{2}(k)}$ and observe that for
$c_{1}=T^{1 /\left(1-\frac{2}{r}\right)}$ it follows by Hölder's inequality

$$
\begin{align*}
& \sum_{k=1}^{T} y(k) \leq \sqrt{\sum_{k=1}^{T}|y(k)|^{2}} \sqrt{\sum_{k=1}^{T} 1}=\sqrt{T}|y| \\
& \sum_{k=1}^{T} y(k) g(k) \leq|\bar{g}||y|  \tag{9}\\
& \sum_{k=1}^{T}|y(k)|^{r} \leq \sqrt[2 / r]{\sum_{k=1}^{T}|y(k)|^{r \cdot \frac{2}{r}}}\left(1-\frac{2}{r}\right) \sqrt{\sum_{k=1}^{T} 1}=c_{1}|y|^{r} .
\end{align*}
$$

Now by (6), (4), (9) we have for all $|y| \geq B$

$$
\begin{align*}
& J(x, u) \geq a_{M+Q}|y|^{2}-\frac{\varepsilon_{1}}{r} \sum_{k=1}^{T}|y(k)|^{r}-\varepsilon_{2} \sum_{k=1}^{T}|y(k)|-|y||\bar{g}| \geq \\
& a_{M+Q}|y|^{2}-\frac{\varepsilon_{1}}{r} c_{1}|y|^{r}-\varepsilon_{2} \sqrt{T}|y|-|y||\bar{g}| \underset{|y| \rightarrow \infty}{\rightarrow} \infty . \tag{10}
\end{align*}
$$

Now let $M+Q$ be a negative definite matrix. Again, let us take sufficiently large $B>0$ as in A2. By (7), (4), (9) we have for all $|y| \geq B$

$$
\begin{equation*}
J(x, u) \leq-b_{M+Q}|y|^{2}+\frac{\varepsilon_{1}}{r} c_{1}|y|^{r}+\varepsilon_{2} \sqrt{T}|y|+|y||\bar{g}| \underset{|y| \rightarrow \infty}{\rightarrow}-\infty . \tag{11}
\end{equation*}
$$

Lemma 5 Assume A1, A3. Then for any fixed $u \in L_{M}$ there exists at least one non trivial solution $x \in V_{u}$ to problem (1)-(园).

Proof. Fix $u \in L_{M}$. We see that for all $y \in R^{T}$

$$
\begin{equation*}
\langle(M+Q) y, y\rangle \leq\|M+Q\||y|^{2} \tag{12}
\end{equation*}
$$

where $\|M+Q\|$ denotes the norm of a matrix $M+Q$. Further by (5) we have for all $|y| \geq B$, where $B$ is as in A3

$$
J(x, u) \leq\|M+Q\||y|^{2}-\frac{\varepsilon_{1}}{r} c_{1}|y|^{r}-\varepsilon_{2} \sqrt{T}|y|-|y||\bar{g}| \underset{|y| \rightarrow \infty}{\rightarrow}-\infty .
$$

Hence the assertion follows with the same arguments as in Lemma 4.
Lemma 6 Assume $\boldsymbol{A} 1$ and either $\boldsymbol{A} 4$ or A5. Then for any fixed $u \in L_{M}$ there exists at least one non trivial solution $x \in V_{u}$ to problem (1)-(2).

Proof. Fix $u \in L_{M}$. With A4 we see by (10) that $x \rightarrow J(x, u)$ is coercive while with A5 we see by (11) it is anti-coercive. Hence the assertion follows with the same arguments as in the above lemmas.

## 4 Dependence on parameters

In order to derive the results concerning the dependence on parameters for problem (1)-(21), we employ the following general principle which we could further apply in a finite dimensional setting.

Let $E$ be finite dimensional Euclidean space with inner product $\langle\cdot, \cdot\rangle$ and with norm $\|\cdot\|$. Let $C$ be a finite dimensional complete normed space with norm $\|\cdot\|_{C}$. Let us consider a family of action functionals $x \rightarrow J(x, u)$, where $x \in E$ and where $u \in C$ is a parameter.

Theorem 7 Assume that there exist constants $\mu, a, \beta>0, b \in R$ such that

$$
\begin{equation*}
J(x, u) \geq a\|x\|^{\mu}+b \text { for all } x \in E \text { with }\|x\| \geq \beta \text { and all } u \in C . \tag{13}
\end{equation*}
$$

Assume also that $x \rightarrow J(x, u)$ continuous and differentiable in the sense of Gâteaux in the first variable for any $u \in C$. Then for any $u \in C$ there exists at least one solution $x_{u}$ to problem

$$
\begin{equation*}
\frac{d}{d x} J(x, u)=0 . \tag{14}
\end{equation*}
$$

Assume further that there exists a constant $\alpha>0$ such that

$$
\begin{equation*}
J(0, u) \leq \alpha \text { for all } u \in E \tag{15}
\end{equation*}
$$

Let $\left\{u_{n}\right\}_{n=1}^{\infty} \subset C$ be a convergent sequence of parameters, where $\lim _{n \rightarrow \infty} u_{n}=$ $\bar{u} \in C$ and let us assume that either $J$ is continuous on $E \times C$ or the Gâteaux derivative of $J$ with respect to $x, \frac{d}{d x} J(x, u)$, is bounded on bounded sets in $E \times C$ and $J$ is continuous with respect to $u$ on $C$ for any $x \in E$. Then for any sequence $\left\{x_{n}\right\}_{n=1}^{\infty}$ of solutions $x_{n} \in E$ to the problem (14) corresponding to $u_{n}$ for $n=1,2, \ldots$ there exist a subsequence $\left\{x_{n_{i}}\right\}_{i=1}^{\infty} \subset E$ and an element $\bar{x} \in E$ such that $\lim _{i \rightarrow \infty} x_{n_{i}}=\bar{x}$ and

$$
J(\bar{x}, \bar{u})=\inf _{y \in E} J(y, \bar{u})
$$

Moreover, $\bar{x}$ satisfies (14) with $u=\bar{u}$, i.e.

$$
\begin{equation*}
\frac{d}{d x} J(\bar{x}, \bar{u})=0 \tag{16}
\end{equation*}
$$

Proof. Let us fix $u \in C$. Assumption (13) leads to the coercivity of $x \rightarrow J(x, u)$. Since $x \rightarrow J(x, u)$ is also continuous and since $E$ is finite dimensional it follows that $J$ has an argument of a minimum $x_{u}$ which satisfies (14). Next, let us take a sequence $\left\{u_{n}\right\}_{n=1}^{\infty} \subset X$ converging to some $\bar{u} \in X$ and let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be a sequence of solutions to (14) corresponding to the relevant elements of the sequence $\left\{u_{n}\right\}_{n=1}^{\infty}$. By (13) and by (15) we see that for all $n \in N$ we have

$$
a\left\|x_{n}\right\|^{\mu}+b \leq J\left(x_{n}, u_{n}\right) \leq \alpha .
$$

Hence a sequence $\left\{x_{n}\right\}_{n=1}^{\infty}$ is norm bounded, say by some $c>0$, and again since $E$ is finite dimensional, it contains the convergent subsequence, $\left\{x_{n_{i}}\right\}_{i=1}^{\infty}$, such that $\lim _{i \rightarrow \infty} x_{n_{i}}=\bar{x}$, where $\bar{x} \in E$.

We will prove that (16) holds. We observe that there exists $x_{0} \in E$ such that $\frac{d}{d x} J\left(x_{0}, \bar{u}\right)=0$ and $J\left(x_{0}, \bar{u}\right)=\inf _{y \in E} J(y, \bar{u})$. We see that there are two possibilities: either $J\left(x_{0}, \bar{u}\right)<J(\bar{x}, \bar{u})$ or $J\left(x_{0}, \bar{u}\right)=J(\bar{x}, \bar{u})$. If we have $J\left(x_{0}, \bar{u}\right)=J(\bar{x}, \bar{u})$, then by the Fermat's rule we have (16). Let us suppose that $J\left(x_{0}, \bar{u}\right)<J(\bar{x}, \bar{u})$, so there exists $\delta>0$ such that

$$
\begin{equation*}
J(\bar{x}, \bar{u})-J\left(x_{0}, \bar{u}\right)>\delta>0 . \tag{17}
\end{equation*}
$$

We investigate the inequality

$$
\begin{align*}
& \delta<\left(J\left(x_{n_{i}}, u_{n_{i}}\right)-J\left(x_{0}, \bar{u}\right)\right)-\left(J\left(x_{n_{i}}, u_{n_{i}}\right)-J\left(\bar{x}, u_{n_{i}}\right)\right)  \tag{18}\\
&-\left(J\left(\bar{x}, u_{n_{i}}\right)-J(\bar{x}, \bar{u})\right)
\end{align*}
$$

which is equivalent to (17). In case $J$ is jointly continuous we have

$$
\begin{align*}
& -\left(J\left(x_{n_{i}}, u_{n_{i}}\right)-J\left(\bar{x}, u_{n_{i}}\right)\right)-\left(J\left(\bar{x}, u_{n_{i}}\right)-J(\bar{x}, \bar{u})\right)= \\
& -J\left(x_{n_{i}}, u_{n_{i}}\right)+J(\bar{x}, \bar{u}) \underset{i \rightarrow \infty}{\rightarrow} 0 . \tag{19}
\end{align*}
$$

In case the Gâteaux derivative is bounded on bounded sets recalling that $\left\|x_{n_{i}}\right\| \leq c$ for all $i \in N$ we have

$$
\begin{equation*}
\left|J\left(x_{n_{i}}, u_{n_{i}}\right)-J\left(\bar{x}, u_{n_{i}}\right)\right| \leq \sup _{\|v\| \leq c}\left\|\frac{d}{d x} J\left(v, u_{n_{i}}\right)\right\|\left\|x_{n_{i}}-\bar{x}\right\| \underset{i \rightarrow \infty}{\rightarrow} 0 . \tag{20}
\end{equation*}
$$

Indeed, we fix $n_{i}$ and introduce the auxiliary function $g:[0,1] \rightarrow R$

$$
g_{n_{i}}(t)=J\left(t x_{n_{i}}+(1-t) \bar{x}, u_{n_{i}}\right)=J\left(\bar{x}+t\left(x_{n_{i}}-\bar{x}\right), u_{n_{i}}\right)
$$

(see that $\left.g_{n_{i}}(0)=J\left(\bar{x}, u_{n_{i}}\right), g_{n_{i}}(1)=J\left(x_{n_{i}} \bar{x}, u_{n_{i}}\right)\right)$ and we have by the Mean Value Theorem that there exists some $\xi \in(0,1)$ such that

$$
\left|g_{n_{i}}(0)-g_{n_{i}}(1)\right|=\left|\frac{d}{d t} g_{n_{i}}(\xi)\right|=\left|\left\langle\frac{d}{d x} J\left(\xi x_{n_{i}}+(1-\xi) \bar{x}, u_{n_{i}}\right), x_{n_{i}}-\bar{x}\right\rangle\right|
$$

since $\left\|t x_{n_{i}}+(1-t) \bar{x}\right\| \leq c$ and thus (20) holds. By (20) and by continuity of $J$ with respect to $u$ for any $x \in E$ we see that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left(J\left(\bar{x}, u_{n_{i}}\right)-J(\bar{x}, \bar{u})\right)=0 \text { and } \lim _{i \rightarrow \infty}\left(J\left(x_{n_{i}}, u_{n_{i}}\right)-J\left(\bar{x}, u_{n_{i}}\right)\right)=0 \tag{21}
\end{equation*}
$$

Finally, since $x_{n_{i}}$ minimizes $x \rightarrow J\left(x, u_{n_{i}}\right)$ over $E$ we get $J\left(x_{n_{i}}, u_{n_{i}}\right) \leq$ $J\left(x_{0}, u_{n_{i}}\right)$ and next

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left(J\left(x_{n_{i}}, u_{n_{i}}\right)-J\left(x_{0}, \bar{u}\right)\right) \leq \lim _{i \rightarrow \infty}\left(J\left(x_{0}, u_{n_{i}}\right)-J\left(x_{0}, \bar{u}\right)\right)=0 . \tag{22}
\end{equation*}
$$

So putting (19), (22) into (18) in case $J$ is jointly continuous and (21), (22) in the other case, we see that $\delta \leq 0$, which is a contradiction. Thus $J(\bar{x}, \bar{u})=\inf _{y \in E} J(y, \bar{u})$ and thus (16) holds.

## 5 Proofs of main results and some corollaries

Proof of Theorem 1. In order to prove Theorem 1 we need to demonstrate that all assumptions of Theorem 7 are satisfied. Firstly, let $M+Q$ be positive definite. We see that (13) is satisfied by (10). We see that $F(k, 0, u(y))=0$ so that $J(0, u)=\sum_{k=1}^{T} \int_{0}^{0} f(k, t, u(k)) d t=0$ and (15) holds. By A1 we see that $J$ is jointly continuous in $(x, u)$. Next, we chose a subsequence $\left\{u_{n_{i}}\right\}_{i=1}^{\infty} \subset L_{M}$ from a sequence $\left\{u_{n}\right\}_{n=1}^{\infty} \subset L_{M}$ such that $\lim _{i \rightarrow \infty} u_{n_{i}}=\bar{u}$. Such a subsequence necessarily exists since $C([1, T], R)$ is a finite dimensional space. Next, we chose a corresponding sequence $\left\{x_{n_{i}}\right\}_{i=1}^{\infty} \subset E$ and rename both sequences as $\left\{u_{n}\right\}_{n=1}^{\infty}$ and $\left\{x_{n}\right\}_{n=1}^{\infty}$. Thus all assumptions of Theorem 7 are satisfied.

Secondly, when $M+Q$ is negative definite we multiply functional (3) by -1 and apply the above reasoning.

In order to prove Theorem 2 we proceed as in the second part of the proof of Theorem 1. Exactly in the same manner as in the proof of Theorem 1, we prove Theorem 3.

Theorem 7 suggests that $f$ need not be jointly continuous in its all variables. While in case of a discrete variable $k$ it is equivalent to assume that either $f \in C([1, T] \times R \times[-M, M], R)$ or $f \in C(R \times[-M, M], R)$ for all $k \in[1, T]$ this is not the case with respect to other variables. Hence in order to get results concerning the dependence on a parameter, we may assume that $\mathbf{A 1}$ is replaced with the following one:

A1a $f:[1, T] \times R \times[-M, M] \rightarrow R$; for all $k \in[1, T]$ and all $u \in$ $[-M, M]$ function $x \rightarrow f(k, x, u)$ is continuous; for all $k \in[1, T]$ and all $x \in R$ function $u \rightarrow f(k, x, u)$ is continuous; $p \in C([0, T+1], R), q, g \in$ $C([1, T], R) ; g\left(k_{1}\right) \neq 0$ for certain $k_{1} \in[1, T]$; for any $d>0$ there exists a function $h \in C([1, T], R)$ such that

$$
\begin{equation*}
|f(k, x, u)| \leq h(k) \text { for all } k \in[1, T], x \in[-d, d], u \in[-M, M] . \tag{23}
\end{equation*}
$$

All Theorems 1, 2, 3, are valid with A1 replaced by A1a. What must be shown is the boundedness of the Gâteaux derivative of the action functional on bounded subsets of $R^{T}$. Such a property follows by (23). Indeed, we have the following

Lemma 8 Assume A1a. The Gâteaux derivative of a functional $x \rightarrow J(x, u)$ given by (3) is bounded on a set $[1, T+1] \times[-d, d] \times[-M, M]$, where $d>0$ is a arbitrarily fixed constant.

## 6 Further applications

The discrete Emden-Fowler equation can also be considered with other type of general boundary conditions, compare with [7]. In this section we consider the discrete equation

$$
\begin{equation*}
\Delta(p(k-1) \Delta x(k-1))+q(k) x(k)=f(k, x(k), u(k))+g(k) \tag{24}
\end{equation*}
$$

subject to a parameter $u \in L_{M}$ and with boundary conditions

$$
\begin{equation*}
x(0)+\alpha_{1} x(1)=A_{1}, x(T+1)+\beta_{1} x(T)=B_{1}, \tag{25}
\end{equation*}
$$

where $\alpha_{1}, \beta_{1}, A_{1}, B_{1}$ are fixed constants. We assume A1. Solutions to (24)-(25) being elements of a finite dimensional space

$$
E_{1}=\left\{v:[0, T+1] \rightarrow R: v(0)+\alpha_{1} v(1)=A_{1}, v(T+1)+\beta_{1} v(T)=B_{1}\right\}
$$

are identified with vectors from $R^{T}$. The action functional $J_{1}: R^{T} \rightarrow R$ for problem (24)-(25) for a fixed $u \in L_{M}$ reads

$$
J_{1}(x, u)=\frac{1}{2}\langle M x, x\rangle+\langle q, x\rangle-\sum_{k=1}^{T} F(k, x(k), u(k))+\sum_{k=1}^{T} g(k) x(k),
$$

where $c(k)=q(k)-p(k)-p(k+1)$ and

$$
\left.\begin{array}{c}
P=\left[\begin{array}{cccccc}
c(1)-\alpha_{1} p(1) & p(1) & 0 & \ldots & 0 & 0 \\
p(2) & c(2) & p(3) & \ldots & 0 & 0 \\
0 & p(3) & c(3) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & c(T-1) & p(T) \\
0 & 0 & 0 & \ldots & p(T) & c(T)-\beta_{1} x(T+1)
\end{array}\right] \\
\\
\\
\\
\\
\end{array} \begin{array}{cc}
p(1) A_{1} \\
0 \\
\vdots \\
0 \\
p(T+1) B_{1}
\end{array}\right] .
$$

Theorems 1, 2 and 3 remain valid with the understanding that now matrix $M+Q$ is replaced by $P$. In what follows $a_{P}, b_{P}$ have the same meaning as $a_{Q+M}, b_{Q+M}$. The term $\langle q, x\rangle$ has no impact on the coercivity or anticoercivity of $J_{2}$. As an example, we formulate the version of Theorem 3 assuming what follows.

A6 let $M+Q$ be positive definite and let there exist constants $\varepsilon_{1} \in$ $\left(0,2 a_{P}\right), \varepsilon_{2} \in R$ such that

$$
f(k, y, u) \leq \varepsilon_{1}|y|+\varepsilon_{2}
$$

uniformly for $u \in[-M, M], k \in[1, T]$ and $|y| \geq B$, where $B>0$ is certain (possibly large) constant.

A7 let $M+Q$ be negative definite and let there exist constants $\varepsilon_{1} \in$ $\left(0,2 b_{P}\right), \varepsilon_{2} \in R$ such that

$$
f(k, y, u) \leq \varepsilon_{1}|y|+\varepsilon_{2}
$$

uniformly for $u \in[-M, M], k \in[1, T]$ and $|y| \geq B$, where $B>0$ is certain (possibly large) constant.

Theorem $9(r=2)$ Assume either $\boldsymbol{A} 1$ and $\boldsymbol{A} \boldsymbol{6}$ or $\boldsymbol{A} 1$ and $\boldsymbol{A} 7$. For any fixed $u \in L_{M}$ there exists at least one non trivial solution $x \in V_{u}$ to problem (24)-(25). Let $\left\{u_{n}\right\}_{n=1}^{\infty} \subset L_{M}$ a sequence of parameters. For any sequence $\left\{x_{n}\right\}_{n=1}^{\infty}$ of solutions $x_{n}$ to the problem (24)-(25) corresponding to $u_{n}$ and such that

$$
x_{n} \in\left\{x \in R^{T}: J_{1}(x, u)=\inf _{v \in R^{T}} J_{1}(v, u), \frac{d}{d x} J_{1}(x, u)=0\right\}
$$

there exist subsequences $\left\{x_{n_{i}}\right\}_{i=1}^{\infty} \subset R^{T},\left\{u_{n_{i}}\right\}_{i=1}^{\infty} \subset L_{M}$ and elements $\bar{x} \in R^{T}$, $\bar{u} \in L_{M}$ such that $\lim _{i \rightarrow \infty} x_{n_{i}}=\bar{x}, \lim _{n \rightarrow \infty} u_{n_{i}}=\bar{u}$. Moreover, $\bar{x}$ satisfies (24)-(25) with $\bar{u}$, i.e.

$$
\begin{gathered}
\Delta(p(k-1) \Delta \bar{x}(k-1))+q(k) \bar{x}(k)=f(k, \bar{x}(k), \bar{u}(k))+g(k), \\
\bar{x}(0)+\alpha \bar{x}(1)=A, \bar{x}(T+1)+\beta \bar{x}(T)=B
\end{gathered}
$$

and $J_{1}(\bar{x}, \bar{u})=\inf _{v \in R^{T}} J_{1}(v, \bar{u}), \frac{d}{d x} J_{1}(\bar{x}, \bar{u})=0$.

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