Application of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations

Pavel N. Ryabov, Dmitry I. Sinelshchikov, Mark B. Kochanov

Department of Applied Mathematics, National Research Nuclear University MEPHI, 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

Abstract

The application of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations is considered. Some classes of solitary wave solutions for the families of nonlinear evolution equations of fifth, sixth and seventh order are obtained. The efficiency of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations is demonstrated.

Keywords: Kudryashov method; Nonlinear evolution equations; Nonlinear differential equations; Ordinary differential equations; Exact solutions.

PACS 02.30.Hq - Ordinary differential equations

1 Introduction

The powerful and effective method for finding exact solutions of nonlinear ordinary differential equations was proposed in work [1]. In works [2–4] author applied this method to construct the exact solutions of the nonlinear nonintegrable equations. The first modification of this method was presented in work [5]. The final and the most successful modification of this method allow us in the straightforward manner to construct solitary wave solutions for a wide class of nonlinear ordinary differential equations. The main idea of the Kudryashov method is to use special form of the singularity manifold in the truncation method [1–5]. This approach allows us to reduce the problem of constructing exact solutions to solving the overdetermined system of algebraic equations. The main advantage of the Kudryashov method is that we can more effectively construct exact solutions of high order nonlinear evolution equations in comparison with other methods for finding exact solutions [7–13].

Using this method in works [1–3] exact solutions of the generalized Kuramoto–Sivashinsky equation, the Burgers–Korteweg–de Vries equation, the Bretherton equation and the Kawahara equation were obtained. Traveling wave solutions for class of third order nonlinear evolution equations were constructed in [14] with help of the Kudryashov method. The Kudryashov method was used for constructing traveling wave solutions of several nonlinear evolution equation in works [15, 16] as well.

The aim of this work is to demonstrate efficiency of the Kudryashov method for finding exact solitary wave solutions of high order nonlinear evolution equations. For this purpose we consider three

^{*}E-mail: pnryabov@mephi.ru

families of nonlinear evolution equations of fifth, six and seven order. This families of equations have some applications in physics and other fields of science. We present classes of solitary wave solutions for considered families of nonlinear evolution equations.

This work is organized as follows. In the next section we give brief description of the Kudryashov method algorithm. In the sections 3,4 and 5 we construct solitary wave solutions for the families of nonlinear evolution equations of fifth, six and seven order respectively. In the last section we summarize and discuss our results.

2 Method applied

The aim of this section is to present the algorithm of the Kudryashov method for finding exact solutions of the nonlinear evolution equations. To reach this purpose we will follow the works [5, 6, 14].

Let us consider the nonlinear partial differential equation in the form

$$E_1[u_t, u_x, \dots, x, t] = 0 \tag{1}$$

Using the following ansatz

$$u(x,t) = y(z), \ z = kx - \omega t.$$
⁽²⁾

from Eq. (1) we obtain the ordinary nonlinear differential equation

$$E_2[-\omega y_z, ky_z, k^2 y_{zz}, k^3 y_{zzz}, \ldots] = 0.$$
(3)

Now we show how one could obtain the exact solution of the Eq. (3) using the approach by Kudryashov. This method consist of the following steps [5, 6, 14].

The first step. Determination of the dominant terms.

To find dominant terms we substitute

$$y = z^p, \tag{4}$$

into all terms of Eq. (3). Then we compare degrees of all terms in Eq. (3) and choose two or more with the smallest degree. The minimum value of p define the pole of Eq. (3) solution and we denote it as N. We have to point out that method can be applied when N is integer. If the value N is noninteger one can transform the equation studied and repeat the procedure.

The second step. The solution structure.

We look for exact solution of Eq. (3) in the form

$$y = a_0 + a_1 Q(z) + a_2(z)Q(z)^2 + \dots + a_N Q(z)^N,$$
(5)

where a_i - unknown constants, Q(z) is the following function

$$Q(z) = \frac{1}{1 + e^z}.$$
 (6)

This function satisfies to the first order ordinary differential equation

$$Q_z = Q^2 - Q. \tag{7}$$

The Eq. (7) is necessary to calculate the derivatives of function y(z). The third step. Derivatives calculation. We should calculate all derivatives of function y. One can do it using the computer algebra systems Maple or Mathematica. As an example we consider following case:

The derivatives of function y(z) in the case of N = 2 can be written in the form

$$y = a_0 + a_1 Q + a_2 Q^2,$$

$$y_z = -a_1 Q + (a_1 - 2a_2)Q^2 + 2a_2 Q^3,$$

$$y_{zz} = a_1 Q + (4a_2 - 3a_1)Q^2 + (2a_1 - 10a_2)Q^3 + 6a_2 Q^4.$$
(8)

The relations (8) can be generalized for any value of N. Differentiating the expression (5) with respect to z and taking into account (7) we have

$$y_{z} = \sum_{i=1}^{N} a_{i} i (Q-1)Q^{i},$$

$$y_{zz} = \sum_{i=1}^{N} a_{i} i ((i+1)Q^{2} - (2i+1)Q + i)Q^{i}.$$
(9)

The high order derivatives of function y(z) can be found in works [5,6].

The fourth step. Defining the values of unknown parameters.

We substitute expressions (9) in Eq. (1). After it we have take into account (5). Thus Eq. (1) takes the form

$$P[Q(z)] = 0, (10)$$

where P[Q(z)] - is a polynomial of function Q(z). Then we collect all items with the same powers of function Q(z) and equate this expressions equal to zero. As a result we obtain algebraic system of equations. Solving this system we get the values of unknown parameters.

The Kudryashov method is a very powerful method for finding exact solutions of the nonlinear differential equations. It has a set of advantages. They are:

- 1. The first step is not necessary, because all redundant terms in (5) becomes equal to zero when we start to define unknown parameters from the system of algebraic equations;
- 2. We construct the solution as a set of Q-blocks. The function Q does not contain any parameters. It is very comfortable, because all of them is in equation;
- 3. Given method can be easily programmed in Maple or Mathematica, because we use the substitutions (5) and (9) in Eq. (3) it takes the polynomial form;
- 4. This method is powerful and effective even if we construct the exact solutions of the high order nonlinear evolution equations.
- 5. It easy to show that tanh, coth, (G'/G) methods and Kudryashov method can be reduced to each other [6]. Moreover the Kudryashov method gives the same results as an Exp-function method. However it is well known that Exp-function method can not be applied for the equations of hight order;

Let us give several examples to demonstrate it's efficiency.

3 Exact solitary wave solutions of the fifth order evolution equation

As an example let us consider the fifth order nonlinear evolution equation in the form

$$u_t + uu_x + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x + \alpha u_{xx} + \beta u_{xxx} + + \gamma u_{xxxx} + u_{xxxxx} = 0.$$
(11)

Eq. (11) is new and does not present in the periodic literature. However it is an interesting equation because this equation consist of the fifth order Korteweg-de Vries equation with additional dispersive and dissipative terms. The equation (11) can arise in the physical applications when we consider the wave processes in active dispersive-dissipative media.

Using the traveling waves (2) we have

$$-wy_{z} + kyy_{z} + 10k^{3}yy_{zzz} + 20k^{3}y_{z}y_{zz} + 30ky^{2}y_{z} + \alpha k^{2}y_{zz} + \beta k^{3}y_{zzz} + + \gamma k^{4}y_{zzzz} + k^{5}y_{zzzzz} = 0.$$
(12)

The pole of the Eq. (12) is equal to N = 2, thus we look for exact solution in the form

$$y = a_0 + a_1 Q + a_2 Q^2, (13)$$

where a_0, a_1 and a_2 – are unknown constants.

Substituting (9) in Eq. (12) and taking into account (13) we obtain the polynomial of function Q(z). Collecting all terms with the same power of function Q(z) and equate this expressions to zero we obtain the system of algebraic equations. Solving this system we find that solution of Eq. (12) exists only in eight cases. They are

$$k_{(1,2)} = \pm \frac{\gamma}{21}, \quad k_{(3,4)} = \pm \frac{3\gamma}{7}, \quad k_{(5,6)} = \pm \frac{1}{84}\sqrt{988\gamma^2 + 2646\beta - 441},$$

$$k_{(7,8)} = \pm \frac{1}{35\gamma}\sqrt{5\gamma(972\gamma^3 + 343\alpha)}.$$
(14)

However in the case of $k = k_{(5,6)}$ and $k = k_{(7,8)}$ the parameters and solution presentation is very cumbersome and we do not give them in the present manuscript.

In the case of $k = k_{(1,2)}$ and $k = k_{(3,4)}$ the values of parameters $\alpha, \beta, w, a_0, a_1, a_2$ are

$$k = \frac{\gamma}{21}, \quad \alpha = \frac{5}{84}\gamma - \frac{5}{14}\gamma\beta - \frac{44}{441}\gamma^{3},$$

$$a_{0} = \frac{\beta}{20} + \frac{16\gamma^{2}}{2205} - \frac{1}{40}, \quad a_{1} = \frac{4}{147}\gamma^{2}, \quad a_{2} = -\frac{2}{147}\gamma^{2},$$

$$\omega = \frac{\gamma \left(9472\gamma^{4} - 10416\gamma^{2} - 9261 - 37044\beta + 62496\beta\gamma^{2} + 111132\beta^{2}\right)}{31116960},$$

$$k = -\frac{\gamma}{21}, \quad \alpha = \frac{5}{84}\gamma - \frac{5}{14}\gamma\beta - \frac{44}{441}\gamma^{3},$$

$$a_{0} = \frac{\beta}{20} + \frac{46\gamma^{2}}{2205} - \frac{1}{40}, \quad a_{1} = 0, \quad a_{2} = -\frac{2}{147}\gamma^{2},$$

$$(16)$$

$$\omega = -\frac{\gamma \left(9472\gamma^{4} - 10416\gamma^{2} - 9261 - 37044\beta + 62496\beta\gamma^{2} + 111132\beta^{2}\right)}{31116960},$$

$$k = \frac{3\gamma}{7}, \quad \beta = \frac{1}{6} - \frac{18}{7}\gamma^{2},$$

$$a_{0} = \frac{7\alpha}{150\gamma} - \frac{1}{60} + \frac{222}{1225}\gamma^{2}, \quad a_{1} = 0, \quad a_{2} = -\frac{18}{49}\gamma^{2},$$

$$\omega = \frac{67228\alpha^{2} - 7056\alpha\gamma^{3} + 347328\gamma^{6} - 8575\gamma^{2}}{2401000\gamma},$$

$$k = -\frac{3\gamma}{7}, \quad \beta = \frac{1}{6} - \frac{18}{7}\gamma^{2},$$

$$a_{0} = \frac{7\alpha}{150\gamma} - \frac{1}{60} - \frac{228}{1225}\gamma^{2}, \quad a_{1} = \frac{36}{49}\gamma^{2}, \quad a_{2} = -\frac{18}{49}\gamma^{2},$$

$$\omega = -\frac{67228\alpha^{2} - 7056\gamma^{3}\alpha + 347328\gamma^{6} - 8575\gamma^{2}}{2401000\gamma},$$
(18)

The solutions of the Eq. (12) which corresponds to the relations (15)-(18) are

$$y(z) = a_0 + \frac{2}{147} \gamma^2 \left[2 - Q(z)\right] Q(z), \tag{19}$$

$$y(z) = a_0 - \frac{2\gamma^2}{147}Q(z)^2,$$
(20)

$$y(z) = a_0 - \frac{18\gamma^2}{49}Q(z)^2,$$
(21)

$$y(z) = a_0 + \frac{18\gamma^2}{49} \left[2 - Q(z)\right] Q(z).$$
(22)

Graphical presentation of solutions (19), (20) is shown on Fig. 1.



Figure 1: Exact solutions (19) and (20) – 1, 2 respectively at $\beta = \gamma = 1$.

The solutions (21), (22) have the same structure so we have decided not to picture them.

4 Exact solitary waves of the sixth order evolution equation

Let us consider the following equation

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxxx} + \gamma u_{xxxxxx} = 0.$$
⁽²³⁾

This equation was proposed in work [17] for describing the longitudinal seismic waves in a viscoelastic medium. Moreover this equation is used for modeling the "soft" – type of the turbulence [18]. Also this equation describes the chemical reactions in reaction-diffusion systems [19]. The numerical modeling of the wave processes describing by Eq. (23) was performed in work [20]. Exact solutions of this equation was obtained in work [21].

Taking into account the traveling waves (2) in Eq. (23) we obtain

$$-\omega y_z + kyy_z + \alpha k^2 y_{zz} + \beta k^4 y_{zzzz} + \gamma k^6 y_{zzzzz} = 0.$$
⁽²⁴⁾

Integrating the Eq. (24) we have

$$C_1 - \omega y + k \frac{y^2}{2} + \alpha k^2 y_z + \beta k^4 y_{zzz} + \gamma k^6 y_{zzzzz} = 0.$$
(25)

Dominant terms of the Eq. (25) are $k^6 y_{zzzz}, ky^2/2$. Thus, the pole order of the Eq. (25) solution is N = 5. So we look for solution in the form

$$y(z) = a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3 + a_4 Q^4 + a_5 Q^5,$$
(26)

where $a_0 - a_5$ – constants to be determined.

Using the (9) in Eq. (25) and taking into account ansatz (26) we obtain a system of algebraic equations. Solving this system we find four real families of unknown parameters. However we give only three because one of them is cumbersome. The following families are

$$\begin{aligned} \alpha &= -\frac{114}{5}k^2\beta, \quad \gamma = -\frac{\beta}{55k^2}, \quad C_1 = \frac{\omega^2}{2k} - \frac{127008k^7\beta^2}{121}, \\ a_0 &= \frac{\omega}{k} - \frac{504k^3\beta}{11}, \quad a_1 = a_2 = 0, \quad a_3 = \frac{10080k^3\beta}{11}, \\ a_4 &= -\frac{15120k^3\beta}{11}, \quad a_5 = \frac{6048k^3\beta}{11} \\ \alpha &= \frac{219}{44}k^2\beta, \quad \gamma = -\frac{5}{44}\frac{\beta}{k^2}, \quad C_1 = \frac{\omega^2}{2k} - \frac{99225k^7\beta^2}{968}, \\ a_0 &= \frac{\omega}{k} + \frac{315k^3\beta}{22}, \quad a_1 = 0, \quad a_2 = -\frac{19845k^3\beta}{11}, \quad a_3 = 6930k^3\beta, \\ a_4 &= -\frac{94500k^3\beta}{11}, \quad a_5 = \frac{37800k^3\beta}{11} \\ \alpha &= -\frac{3259}{110}\beta k^2, \quad \gamma = -\frac{1}{100}\frac{\beta}{k^2}, \quad C_1 = \frac{\omega^2}{2k} - \frac{321489k^7\beta^2}{242}, \\ a_0 &= \frac{\omega}{k} - \frac{567k^3\beta}{11}, \quad a_1 = 0, \quad a_2 = \frac{1890k^3\beta}{11}, \quad a_3 = \frac{3780k^3\beta}{11}, \\ a_4 &= -\frac{7560k^3\beta}{11}, \quad a_5 = \frac{3024k^3\beta}{11} \end{aligned}$$

From relations (27), (28) and (29) we see that the real solution of Eq. (25) exists in the case of

$$k^{2} = -\frac{5}{114} \frac{\alpha}{\beta}, \quad \beta^{2} = \frac{275}{114} \alpha \gamma,$$

$$k^{2} = \frac{44}{219} \frac{\alpha}{\beta}, \quad \beta^{2} = -\frac{1095}{1936} \alpha \gamma,$$

$$k^{2} = -\frac{110}{3259} \frac{\alpha}{\beta}, \quad \beta^{2} = \frac{3259}{12100} \alpha \gamma.$$
(30)

The solutions of Eq. (25) which corresponds to the relations (27), (28) and (29) takes the form

$$y(z) = \frac{\omega}{k} + \frac{\beta k^3}{11} \left(-504 + 10080Q(z)^3 - 15120Q(z)^4 + 6048Q(z)^5 \right), \tag{31}$$

$$y(z) = \frac{\omega}{k} + \frac{\beta k^3}{11} \left(\frac{315}{2} - 19845Q(z)^2 + 76230Q(z)^3 - 94500Q(z)^4 + 37800Q(z)^5 \right),$$
(32)

$$y(z) = \frac{\omega}{k} + \frac{\beta k^3}{11} \left(-567 + 1890Q(z)^2 + 3780Q(z)^3 - 7560Q(z)^4 + 3024Q(z)^5 \right).$$
(33)

The solutions (31), (32) of the Eq. (23) is presented on Fig. 2. The third solution has the form of the kink as well as (31).



Figure 2: Exact solutions (31) and (32) – 1, 2 respectively at $\beta = \omega = k = 1$.

5 Exact solitary waves of the seventh order evolution equation

Let us consider the nonlinear evolution equation of seventh order

$$u_t + u^n u_x + \alpha u_{xxx} + \beta u_{xxxxx} + \gamma u_{xxxxxx} = 0, \tag{34}$$

where n = 1, 2, 3.

For the first time this equation was considered in work [6] in the case n = 2. We generalizes the results of this work.

Taking the traveling wave ansatz u(x,t) = y(z), z = kx - wt into account from Eq.(34) we have

$$-\omega y_z + ky^n y_z + \alpha k^3 y_{zzz} + \beta k^5 y_{zzzzz} + \gamma k^7 y_{zzzzzz} = 0.$$
(35)

Integrating the Eq. (35) with respect to variable z we obtain

$$C_1 - \omega y + k \frac{y^{n+1}}{n+1} + \alpha k^3 y_{zz} + \beta k^5 y_{zzzz} + \gamma k^7 y_{zzzzz} = 0.$$
(36)

In the case of n = 1 we have to look exact solution of Eq. (36) in the form

$$y(z) = a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3 + a_4 Q^4 + a_5 Q^5 + a_6 Q^6,$$
(37)

because the pole order of Eq. (36) solution is N = 6.

Substituting (9) in Eq. (36) with n = 1 and taking (37) into account we found the following solutions

$$y(z) = \frac{\omega}{k} + \gamma k^6 \left(-7100 + 166320Q(z)^2 + 332640Q(z)^3 - 1829520Q(z)^4 + 1995840Q(z)^5 - 665280Q(z)^6 \right),$$
(38)

$$y(z) = \frac{\omega}{k} + \gamma k^{6} \left(-3600 + 665280Q(z)^{3} - 1995840Q(z)^{4} + 1995840Q(z)^{5} - 665280Q(z)^{6} \right),$$
(39)

The corresponding parameters of Eq. (36) and (37) are

$$\beta = -100k^{2}\gamma, \quad \alpha = 2159k^{4}\gamma, \quad C_{1} = \frac{\omega^{2}}{2k} - 25205000k^{13}\gamma^{2},$$

$$a_{0} = \frac{\omega}{k} - 7100k^{6}\gamma, \quad a_{1} = 0, \quad a_{2} = 166320k^{6}\gamma, \quad a_{3} = 332640k^{6}\gamma,$$

$$a_{4} = -1829520k^{6}\gamma, \quad a_{5} = 1995840k^{6}\gamma, \quad a_{6} = -665280k^{6}\gamma,$$
(40)

$$\beta = -50k^2\gamma, \quad \alpha = 769k^4\gamma, \quad C_1 = \frac{\omega^2}{2k} - 6480000k^{13}\gamma^2,$$

$$a_0 = \frac{\omega}{k} - 3600k^6\gamma, \quad a_1 = a_2 = 0, \quad a_3 = 665280k^6\gamma,$$

$$a_4 = -1995840k^6\gamma, \quad a_5 = -a_4, \quad a_6 = -a_3,$$
(41)

where $\beta/\gamma < 0$. Thus, the solutions (38) and (39) exist when

$$k^{2} = -\frac{\beta}{100\gamma}, \quad \beta^{2} = \frac{1000}{2159}\alpha\gamma, k^{2} = -\frac{\beta}{50\gamma}, \quad \beta^{2} = \frac{2500}{769}\alpha\gamma.$$
(42)

The illustration of (38) is given on Fig. 3.



Figure 3: Exact solution (38) at $\omega = 10, \gamma = 0.001, k = 1$.

The solution (39) has the same form.

Let us consider the case of n = 2. The pole order of Eq. (36) solution in that case is equal to N = 3. Thus, we look for exact solution of Eq. (36) in the form

$$y(z) = a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3, (43)$$

where a_0, a_1, a_2, a_3 the constants to be determined.

Taking into account (9) and relations (43) in Eq. (36) we have the solution in the form

$$y(z) = \pm 6\sqrt{-105\gamma}k^3 \left(1 - 6Q(z)^2 + 4Q(z)^3\right), \tag{44}$$

where $\gamma < 0$.

Values of β, α, w, C_1 are defined by relations

$$\beta = -83k^2\gamma, \quad \alpha = 946k^4\gamma, \qquad w = -1260\gamma k^7, \quad C_1 = 0,$$

$$a_0 = \pm 6\sqrt{-105\gamma}k^3, \quad a_1 = 0, \quad a_2 = \mp 36\sqrt{-105\gamma}k^3,$$

$$a_3 = \pm 24\sqrt{-105\gamma}k^3, \qquad (45)$$

So, the solution (44) exist when the wave number satisfies to equivalence

$$k^2 = -\frac{\beta}{83\gamma}, \quad \beta^2 = \frac{6889}{946}\alpha\gamma.$$
 (46)

The solution (44) is a kink, see Fig. 4.



Figure 4: Exact solution (44) at $\gamma = -1, k = 1$.

In the case of n = 3 the pole order of the Eq. (36) solution is N = 2. So, we use the following ansatz

$$y(z) = a_0 + a_1 Q + a_2 Q^2, (47)$$

where a_0, a_1, a_2 – unknown constants.

Then we substitute (9) and (47) in Eq. (5.3) with n = 3. Collecting all terms with the same power of function Q(z) and equate them to zero we obtain the system of algebraic equations on unknown

parameters. Solving this system we have

$$\alpha = 14k^{4}\gamma + \frac{5\beta^{2}}{28\gamma}, \quad w = \frac{k}{1176\gamma^{2}} \left(3920k^{6}\gamma^{3} - 1764\beta\gamma^{2}k^{4} - 5\beta^{3} \right),$$

$$C_{1} = \frac{\sqrt[3]{315}}{65856} \frac{k}{\gamma^{8/3}} \left(5\beta^{4} - 137200k^{8}\gamma^{4} + 9408k^{6}\gamma^{3}\beta + 392k^{4}\gamma^{2}\beta^{2} \right),$$

$$a_{0} = -\frac{\sqrt[3]{315}}{42} \frac{\beta}{\gamma^{2/3}} - \frac{\sqrt[3]{315}\gamma k^{2}}{3}, \quad a_{1} = 4\sqrt[3]{315\gamma}k^{2}, \quad a_{2} = -a_{1}.$$
(48)

The solution of Eq. (36) with n = 3 which corresponds to (48) takes the form

$$y(z) = -\frac{\sqrt[3]{315}}{42}\frac{\beta}{\gamma^{2/3}} - \frac{\sqrt[3]{315\gamma}k^2}{3}\left(1 - 12Q(z) + 12Q(z)^2\right).$$
(49)

This (49) has the form of the traveling wave (Fig. 5).



Figure 5: Exact solution (49) at $\beta = \gamma = k = 1$.

6 Conclusion

In this work we have demonstrated efficiency of the Kudryashov method for finding exact solutions of high order nonlinear evolution equations. We have obtained solitary wave solutions for the three families of nonlinear evolution equations of fifth, six and seven orders. Graphical representation of these exact solution is presented. We believe that some of exact solutions obtained in this work are new.

7 Acknowledgements

This work was supported by the federal target programm "Research and scientific-pedagogical personnel of innovation in Russia" on 2009-2011.

References

 Kudryashov N.A. Exact soliton solutions of the generalized evolution equation of wave dynemics. J. Appl. Math. Mech. 52 (1988) 361–365.

- [2] Kudryashov N.A. Exact solutions of the generalized Kuramoto-Sivashinsky equation. Phys. Lett. A. 147 (1990) 287–291.
- [3] Kudryashov N.A. On types of nonlinear nonintegrable equations with exact solutions Phys. Lett. A. 155 (1991) 269–275.
- [4] Kudryashov N.A. Singular manifold equations and exact solutions for some nonlinear partial differential equations. Phys. Lett. A. 182 (1993) 356–362.
- [5] Kudryashov N.A. Analitical theory of nonlinear differential equations. Moskow Igevsk. Institute of computer investigations: 2004 [in Russian]
- Kudryashov N.A. On one of methods for finding exact solutions of nonlinear differential equations. arXiv:1108.3288v1 [nlin.SI] 16 Aug 2011
- [7] Parkes E.J. Exact solutions to the two-dimensional Korteweg-de Vries-Burgers equation. J. Phys. A: Math. Gen. 27 (1994) 27 L497–L501.
- [8] Parkes E.J., Duffy B.R. An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations. Comput. Phys. Commun. 98 (1996) 288–300.
- [9] Malfliet W., Hereman W. The tanh method: I. Exact solutions of nonlinear evolution and wave equations. Phys. Scr. 54 (1996) 563–568.
- [10] Fan E. Extended tanh-function method and its applications to nonlinear equations. Phys. Lett. A. 277 (2000) 212–218.
- [11] Biswas A. Solitary wave solution for the generalized Kawahara equation. Appl. Math. Lett. 22 (2009) 208–210.
- [12] Vitanov N.K., Dimitrova Z.I., Kantz H. Modified method of simplest equation and its application to nonlinear PDEs. Applied Mathematics and Computation 216 (2010) 2587–2595
- [13] Kudryashov N.A., Loginova N.B. Extended simplest equation method for nonlinear differential equations. Applied Mathematics and Computation 205 (2008) 396–402
- [14] Ryabov P.N. Exact solutions of the Kudryashov–Sinelshchikov equation. Applied Mathematics and Computation 217 (2010) 3585–3590.
- [15] Kabir M. M., Khajeh A., Abdi Aghdam E., Yousefi Koma A. Modified Kudryashov method for finding exact solitary wave solutions of higher-order nonlinear equations. Mathematical Methods in the Applied Science. doi: 10.1002/mma.1349
- [16] Gepreel K.A., Omran S., Elagan S.K. The Traveling Wave Solutions for Some Nonlinear PDEs in Mathematical Physics. Applied Mathematics 2 (2011) 343–347.
- [17] Nikolaevskii V.N. Dynamics of Viscoelastic Media with Internal Oscillators, in Recent Advances in Engineering Sciences, Ed. by S.L. Koh and C.G. Speciale (Springer, Berlin, 1989), 210-221.
- [18] Tribelsky M.I., Tsuboi K. New Scenario for Transition to Turbulence? Phys. Rev. Lett. 76 (1996) 1631-1634.
- [19] Tanaka D. Chemical Turbulence Equivalent to Nikolaevskii Turbulence, Phys. Rev. E 70 (2004) 015202-1-015202-4

- [20] Kudryashov N.A., Migita A.V. Periodic structures developing with account for dispersion in a turbulence model. Fluid Dyn. 42 (2007) 463–471
- [21] Kudryashov N. A., Nonlinear differential equations with exact solutions expressed via the Weierstrass function. Zeitschrift fur Naturforschung 59 (2004) 443-454.