

Interpretation of percolation in terms of infinity computations

D.I. Iudin*, Ya.D. Sergeyev[†], M. Hayakawa[‡]

Abstract

In this paper, a number of traditional models related to the percolation theory has been considered by means of new computational methodology that does not use Cantor's ideas and describes infinite and infinitesimal numbers in accordance with the principle 'The part is less than the whole'. It gives a possibility to work with finite, infinite, and infinitesimal quantities *numerically* by using a new kind of a computer – the Infinity Computer – introduced recently in [18]. The new approach does not contradict Cantor. In contrast, it can be viewed as an evolution of his deep ideas regarding the existence of different infinite numbers in a more applied way. Site percolation and gradient percolation have been studied by applying the new computational tools. It has been established that in an infinite system the phase transition point is not really a point as with respect of traditional approach. In light of new arithmetic it appears as a critical interval, rather than a critical point. Depending on "microscope" we use this interval could be regarded as finite, infinite and infinitesimal short interval. Using new approach we observed that in vicinity of percolation threshold we have *many* different *infinite clusters* instead of *one infinite cluster* that appears in traditional consideration.

*Dmitry I. Iudin, Ph.D., D.Sc., holds a Full Professorship at the N.I. Lobachevsky State University, Nizhni Novgorod, Russia. He is also Full Professor at the Architecture State University, Nizhni Novgorod, Russia and Senior Researcher at the Institute of Applied physics of Russian Academy of Science, Nizhni Novgorod, Russia. di-iudin@nirfi.sci-nnov.ru

[†]Yaroslav D. Sergeyev, Ph.D., D.Sc., holds a Full Professorship reserved for distinguished scientists at the University of Calabria, Rende, Italy. He is also Full Professor at the N.I. Lobachevsky State University, Nizhni Novgorod, Russia and Affiliated Researcher at the Institute of High Performance Computing and Networking of the National Research Council of Italy. yaro@si.deis.unical.it

[‡]Masaschi Hayakawa, Ph.D., D.Sc., holds a Full Professorship at the Department of Electronic Engineering, University of Electro-Communications, Chofu, Tokyo 182 8585, Japan. hayakawa@whistler.ee.uec.ac.jp

1 Introduction

Numerous trials have been done during the centuries in order to evolve existing numeral systems¹ in such a way that infinite and infinitesimal numbers could be included in them (see [3, 5, 7, 12, 14, 16, 32]). Particularly, in the early history of the calculus, arguments involving infinitesimals played a pivotal role in the derivation developed by Leibnitz and Newton (see [12, 14]). The notion of an infinitesimal, however, lacked a precise mathematical definition and in order to provide a more rigorous foundation for the calculus infinitesimals were gradually replaced by the d'Alembert-Cauchy concept of a limit (see [6, 8]).

The creation of a mathematical theory of infinitesimals to base on the calculus remained an open problem until the end of 1950s when Robinson (see [16]) introduced his famous non-standard analysis approach. He showed that non-archimedean ordered field extensions of the reals contained numbers that could serve the role of infinitesimals and their reciprocals could serve as infinitely large numbers. Robinson then has derived the theory of limits, and more generally of Calculus, and has found a number of important applications of his ideas in many other fields of Mathematics (see [16]).

In his approach, Robinson used mathematical tools and terminology (cardinal numbers, countable sets, continuum, one-to-one correspondence, etc.) taking their origins from the famous ideas of Cantor (see [5]) who has shown that there existed infinite sets having different number of elements. It is well known nowadays that while dealing with infinite sets, Cantor's approach leads to some counterintuitive situations that often are called by non-mathematicians 'paradoxes'. For example, the set of even numbers, \mathbb{E} , can be put in a one-to-one correspondence with the set of all natural numbers, \mathbb{N} , in spite of the fact that \mathbb{E} is a part of \mathbb{N} :

$$\begin{array}{rcccccccc} \text{even numbers:} & 2, & 4, & 6, & 8, & 10, & 12, & \dots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \text{natural numbers:} & 1, & 2, & 3, & 4, & 5, & 6, & \dots \end{array} \quad (1)$$

The philosophical principle of Ancient Greeks '*The part is less than the whole*' observed in the world around us does not hold true for infinite numbers introduced by Cantor, e.g., it follows $x + 1 = x$, if x is an infinite cardinal, although for any finite x we have $x + 1 > x$. As a consequence, the same effects necessary have reflections in the non-standard Analysis of Robinson (this is not the case of the interesting non-standard approach introduced recently in [3]).

Due to the enormous importance of the concepts of infinite and infinitesimal in science, people try to introduce them in their work with computers, too (see, e.g. the IEEE Standard for Binary Floating-Point Arithmetic). However, non-standard Analysis remains a very theoretical field because various arithmetics (see [3, 5, 7, 16]) developed for infinite and infinitesimal numbers are quite different with respect to the finite arithmetic we are used to deal with. Very often they leave undetermined many operations where infinite numbers take part (for example, $\infty - \infty$, $\frac{\infty}{\infty}$, sum of infinitely many items, etc.) or use representation of infinite numbers based on infinite sequences of finite numbers. These crucial difficulties did not allow people to construct computers that would be able to work with infinite and infinitesimal numbers *in the same*

¹ We are reminded that a *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '8', 'eight', and 'VIII' are different numerals, but they all represent the same number.

manner as we are used to do with finite numbers and to study infinite and infinitesimal objects numerically.

Recently a new applied point of view on infinite and infinitesimal numbers has been introduced in [17, 23, 27]. The new approach does not use Cantor's ideas and describes infinite and infinitesimal numbers that are in accordance with the principle 'The part is less than the whole'. It gives a possibility to work with finite, infinite, and infinitesimal quantities *numerically* by using a new kind of computers – the Infinity Computer – introduced in [18, 19, 28, 29]. It is worthwhile noticing that the new approach does not contradict Cantor. In contrast, it can be viewed as an evolution of his deep ideas regarding the existence of different infinite numbers in a more applied way. For instance, Cantor showed that there exist infinite sets having different cardinalities \aleph_0 and \aleph_1 . In its turn, the new approach specifies this result showing that in certain cases within each of these classes it is possible to distinguish sets with the number of elements being different infinite numbers. We emphasize that the new approach has been introduced as an evolution of standard and non-standard Analysis and not as a contraposition to them. One or another version of Analysis can be chosen by the working mathematician in dependence on the problem he deals with.

In this paper, we consider a number of applications related to the theory of percolation and study them using the new approach. On the one hand, percolation represents the simplest model of a disordered system. Disordered structures and random processes that are self-similar on certain length and time scales are very common in nature. They can be found on the largest and the smallest scales: in galaxies and landscapes, in earthquakes and fractures, in aggregates and colloids, in rough surfaces and interfaces, in glasses and polymers, in proteins and other large molecules. Disorder plays a fundamental role in many processes of industrial and scientific interest. On the other hand, percolation reveals a concept of self-similarity and demonstrate numerous fractal features. Owing to the wide occurrence of self-similarity in nature, the scientific community interested in this phenomenon is very broad, ranging from astronomers and geoscientists to material scientists and life scientists. From mathematic point of view, self-similarity implies a recursive process and, consequently, is tightly connected with concept of infinity. This turn us to an idea that percolation is very suitable to demonstrate advantages of the new computational approach proposed recently in [17, 23].

The outline of the paper is as follows. In Sec. 2 we introduce the new approach that allows one to write down different finite, infinite, and infinitesimal numbers by a finite number of symbols as particular cases of a unique framework and to execute numerical computations with all of them. Than in Sec. 3 we apply the new methodology to the percolation phase transition. Generalized percolation problem known as gradient percolation analyzed in terms of infinity computations in Sec. 4 In the final section, the applications are summarized and discussed.

2 Methodology

In this section, we give a brief introduction to the new methodology that can be found in a rather comprehensive form in [23, 27] downloadable from [19] (see also the monograph [17] written in a popular manner). A number of applications of the new approach can be found in [13, 20–22, 24, 28, 29]. We start by introducing three postulates that will fix our methodological positions (having a strong applied character) with respect to infinite and infinitesimal quantities

and Mathematics, in general.

Usually, when mathematicians deal with infinite objects (sets or processes) it is supposed that human beings are able to execute certain operations infinitely many times (e.g., see (1)). Since we live in a finite world and all human beings and/or computers finish operations they have started, this supposition is not adopted.

Postulate 1. *There exist infinite and infinitesimal objects but human beings and machines are able to execute only a finite number of operations.*

Due to this Postulate, we accept a priori that we shall never be able to give a complete description of infinite processes and sets due to our finite capabilities.

The second postulate is adopted following the way of reasoning used in natural sciences where researchers use tools to describe the object of their study and the instrument used influences the results of observations. When physicists see a black dot in their microscope they cannot say: The object of observation *is* the black dot. They are obliged to say: the lens used in the microscope allows us to see the black dot and it is not possible to say anything more about the nature of the object of observation until we change the instrument - the lens or the microscope itself - by a more precise one.

Due to Postulate 1, the same happens in Mathematics studying natural phenomena, numbers, and objects that can be constructed by using numbers. Numeral systems used to express numbers are among the instruments of observations used by mathematicians. Usage of powerful numeral systems gives the possibility to obtain more precise results in mathematics and in the same way usage of a good microscope gives the possibility of obtaining more precise results in Physics. However, the capabilities of the tools will be always so limited due to Postulate 1 and due to Postulate 2 that we shall never tell, **what is**, for example, a number but shall just observe it through numerals expressible in a chosen numeral system.

Postulate 2. *We shall not tell **what are** the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.*

Particularly, this means that from our point of view, axiomatic systems do not define mathematical objects but just determine formal rules for operating with certain numerals reflecting some properties of the studied mathematical objects. Throughout the paper, we shall always emphasize this philosophical triad – researcher, object of investigation, and tools used to observe the object – in various mathematical and computational contexts.

Finally, we adopt the principle of Ancient Greeks mentioned above as the third postulate.

Postulate 3. *The principle ‘The part is less than the whole’ is applied to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).*

Due to this declared applied statement, it becomes clear that the subject of this paper is out of Cantor’s approach and, as a consequence, out of non-standard analysis of Robinson. Such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper because they belong to the theory working with different assumptions. However, the approach used here does not contradict Cantor and Robinson. It can be viewed just as a more strong lens of a mathematical microscope that allows one to distinguish more objects and to work with them.

In [17,23], a new numeral system has been developed in accordance with Postulates 1–3. It gives one a possibility to execute numerical computations not only with finite numbers but also

with infinite and infinitesimal ones. The main idea consists of the possibility to measure infinite and infinitesimal quantities by different (infinite, finite, and infinitesimal) units of measure.

A new infinite unit of measure has been introduced for this purpose as the number of elements of the set \mathbb{N} of natural numbers. It is expressed by the numeral $\textcircled{1}$ called *grossone*. It is necessary to note immediately that $\textcircled{1}$ is neither Cantor's \aleph_0 nor ω . Particularly, it has both cardinal and ordinal properties as usual finite natural numbers (see [23]).

Formally, grossone is introduced as a new number by describing its properties postulated by the *Infinite Unit Axiom* (IUA) (see [17, 23]). This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integer numbers are introduced. It is important to emphasize that we speak about axioms of real numbers in sense of Postulate 2, i.e., axioms define formal rules of operations with numerals in a given numeral system.

Inasmuch as it has been postulated that grossone is a number, all other axioms for numbers hold for it, too. Particularly, associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers. This means that the following relations hold for grossone, as for any other number

$$0 \cdot \textcircled{1} = \textcircled{1} \cdot 0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \frac{\textcircled{1}}{\textcircled{1}} = 1, \quad \textcircled{1}^0 = 1, \quad 1^{\textcircled{1}} = 1, \quad 0^{\textcircled{1}} = 0. \quad (2)$$

Let us comment upon the nature of grossone by some illustrative examples.

Infinite numbers constructed using grossone can be interpreted in terms of the number of elements of infinite sets. For example, $\textcircled{1} - 1$ is the number of elements of a set $B = \mathbb{N} \setminus \{b\}$, $b \in \mathbb{N}$, and $\textcircled{1} + 1$ is the number of elements of a set $A = \mathbb{N} \cup \{a\}$, where $a \notin \mathbb{N}$. Due to Postulate 3, integer positive numbers that are larger than grossone do not belong to \mathbb{N} but also can be easily interpreted. For instance, $\textcircled{1}^2$ is the number of elements of the set V , where $V = \{(a_1, a_2) : a_1 \in \mathbb{N}, a_2 \in \mathbb{N}\}$. \square

Grossone has been introduced as the quantity of natural numbers. As a consequence, similarly to the set

$$A = \{1, 2, 3, 4, 5\} \quad (3)$$

consisting of 5 natural numbers where 5 is the largest number in A , $\textcircled{1}$ is the largest number² in \mathbb{N} and $\textcircled{1} \in \mathbb{N}$ analogously to the fact that 5 belongs to A . Thus, the set, \mathbb{N} , of natural numbers can be written in the form

$$\mathbb{N} = \{1, 2, \dots, \frac{\textcircled{1}}{2} - 2, \frac{\textcircled{1}}{2} - 1, \frac{\textcircled{1}}{2}, \frac{\textcircled{1}}{2} + 1, \frac{\textcircled{1}}{2} + 2, \dots, \textcircled{1} - 2, \textcircled{1} - 1, \textcircled{1}\}. \quad (4)$$

Note that traditional numeral systems did not allow us to see infinite natural numbers

$$\dots, \frac{\textcircled{1}}{2} - 2, \frac{\textcircled{1}}{2} - 1, \frac{\textcircled{1}}{2}, \frac{\textcircled{1}}{2} + 1, \frac{\textcircled{1}}{2} + 2, \dots, \textcircled{1} - 2, \textcircled{1} - 1, \textcircled{1}. \quad (5)$$

²This fact is one of the important methodological differences with respect to non-standard analysis theories where it is supposed that infinite numbers do not belong to \mathbb{N} .

Similarly, Pirahã³ are not able to see finite numbers larger than 2 using their weak numeral system but these numbers are visible if one uses a more powerful numeral system. Due to Postulate 2, the same object of observation – the set \mathbb{N} – can be observed by different instruments – numeral systems – with different accuracies allowing one to express more or less natural numbers. \square

This example illustrates also the fact that when we speak about sets (finite or infinite) it is necessary to take care about tools used to describe a set (remember Postulate 2). In order to introduce a set, it is necessary to have a language (e.g., a numeral system) allowing us to describe its elements and the number of the elements in the set. For instance, the set A from (3) cannot be defined using the mathematical language of Pirahã.

Analogously, the words ‘the set of all finite numbers’ do not define a set completely from our point of view, as well. It is always necessary to specify which instruments are used to describe (and to observe) the required set and, as a consequence, to speak about ‘the set of all finite numbers expressible in a fixed numeral system’. For instance, for Pirahã ‘the set of all finite numbers’ is the set $\{1, 2\}$ and for another Amazonian tribe – Mundurukú⁴ – ‘the set of all finite numbers’ is the set A from (3). As it happens in Physics, the instrument used for an observation bounds the possibility of observation. It is not possible to say how we shall see the object of our observation if we have not clarified which instruments will be used to execute the observation.

Introduction of grossone gives us a possibility to compose new (in comparison with traditional numeral systems) numerals and to see through them not only numbers (3) but also certain numbers larger than ①. We can speak about the set of *extended natural numbers* (including \mathbb{N} as a proper subset) indicated as $\hat{\mathbb{N}}$ where

$$\hat{\mathbb{N}} = \{1, 2, \dots, \textcircled{1} - 1, \textcircled{1}, \textcircled{1} + 1, \textcircled{1} + 2, \textcircled{1} + 3, \dots, \textcircled{1}^2 - 1, \textcircled{1}^2, \textcircled{1}^2 + 1, \dots\} \quad (6)$$

However, analogously to the situation with ‘the set of all finite numbers’, the number of elements of the set $\hat{\mathbb{N}}$ cannot be expressed within a numeral system using only ①. It is necessary to introduce in a reasonable way a more powerful numeral system and to define new numerals (for instance, ②, ③, etc.) of this system that would allow one to fix the set (or sets) somehow. In general, due to Postulate 1 and 2, for any fixed numeral \mathcal{A} system there always be sets that cannot be described using \mathcal{A} .

Analogously to (4), the set, \mathbb{E} , of even natural numbers can be written now in the form

$$\mathbb{E} = \{2, 4, 6 \quad \dots \quad \textcircled{1} - 4, \textcircled{1} - 2, \textcircled{1}\}. \quad (7)$$

Due to Postulate 3 and the IUA (see [17, 23]), it follows that the number of elements of the set of even numbers is equal to $\frac{\textcircled{1}}{2}$ and ① is even. Note that the next even number is ① + 2 but it

³Pirahã is a primitive tribe living in Amazonia that uses a very simple numeral system for counting: one, two, ‘many’(see [10]). For Pirahã, all quantities larger than two are just ‘many’ and such operations as 2+2 and 2+1 give the same result, i.e., ‘many’. Using their weak numeral system Pirahã are not able to distinguish numbers larger than 2 and, as a result, to execute arithmetical operations with them. Another peculiarity of this numeral system is that ‘many’+ 1= ‘many’. It can be immediately seen that this result is very similar to our traditional record $\infty + 1 = \infty$.

⁴Mundurukú (see [15]) fail in exact arithmetic with numbers larger than 5 but are able to compare and add large approximate numbers that are far beyond their naming range. Particularly, they use the words ‘some, not many’ and ‘many, really many’ to distinguish two types of large numbers (in this connection think about Cantor’s \aleph_0 and \aleph_1).

is not natural because $\textcircled{1} + 2 > \textcircled{1}$, it is extended natural (see (6)). Thus, we can write down not only initial (as it is done traditionally) but also the final part of (1)

$$\begin{array}{cccccccccccc} 2, & 4, & 6, & 8, & 10, & 12, & \dots & \textcircled{1} - 4, & \textcircled{1} - 2, & \textcircled{1} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4 & 5, & 6, & \dots & \frac{\textcircled{1}}{2} - 2, & \frac{\textcircled{1}}{2} - 1, & \frac{\textcircled{1}}{2} \end{array}$$

concluding so (1) in a complete accordance with Postulate 3. It is worth noticing that the new numeral system allows us to solve many other ‘paradoxes’ related to infinite and infinitesimal quantities (see [17, 23, 24]). \square

In order to express numbers having finite, infinite, and infinitesimal parts, records similar to traditional positional numeral systems can be used (see [17, 23]). To construct a number C in the new numeral positional system with base $\textcircled{1}$, we subdivide C into groups corresponding to powers of $\textcircled{1}$:

$$C = c_{p_m} \textcircled{1}^{p_m} + \dots + c_{p_1} \textcircled{1}^{p_1} + c_{p_0} \textcircled{1}^{p_0} + c_{p_{-1}} \textcircled{1}^{p_{-1}} + \dots + c_{p_{-k}} \textcircled{1}^{p_{-k}}. \quad (8)$$

Then, the record

$$C = c_{p_m} \textcircled{1}^{p_m} \dots c_{p_1} \textcircled{1}^{p_1} c_{p_0} \textcircled{1}^{p_0} c_{p_{-1}} \textcircled{1}^{p_{-1}} \dots c_{p_{-k}} \textcircled{1}^{p_{-k}} \quad (9)$$

represents the number C , where all numerals $c_i \neq 0$, they belong to a traditional numeral system and are called *grossdigits*. They express finite positive or negative numbers and show how many corresponding units $\textcircled{1}^{p_i}$ should be added or subtracted in order to form the number C .

Numbers p_i in (9) are sorted in the decreasing order with $p_0 = 0$

$$p_m > p_{m-1} > \dots > p_1 > p_0 > p_{-1} > \dots > p_{-(k-1)} > p_{-k}.$$

They are called *grosspowers* and they themselves can be written in the form (9). In the record (9), we write $\textcircled{1}^{p_i}$ explicitly because in the new numeral positional system the number i in general is not equal to the grosspower p_i . This gives the possibility to write down numerals without indicating grossdigits equal to zero.

The term having $p_0 = 0$ represents the finite part of C because, due to (2), we have $c_0 \textcircled{1}^0 = c_0$. The terms having finite positive grosspowers represent the simplest infinite parts of C . Analogously, terms having negative finite grosspowers represent the simplest infinitesimal parts of C . For instance, the number $\textcircled{1}^{-1} = \frac{1}{\textcircled{1}}$ is infinitesimal. It is the inverse element with respect to multiplication for $\textcircled{1}$:

$$\textcircled{1}^{-1} \cdot \textcircled{1} = \textcircled{1} \cdot \textcircled{1}^{-1} = 1. \quad (10)$$

Note that all infinitesimals are not equal to zero. Particularly, $\frac{1}{\textcircled{1}} > 0$ because it is a result of division of two positive numbers. All of the numbers introduced above can be grosspowers, as well, giving thus a possibility to have various combinations of quantities and to construct terms having a more complex structure.

3 Geometric phase transition

In 1957, two mathematicians, S.R. Broadbent and J.M. Hammersley, have published an article [4] where they have shared with readers an idea of probabilistic formalizations of water

infiltration in electric coffee maker. Their description, named later *percolation theory*, represents one of the simplest models of a disordered system.

Consider a square lattice, where each site is occupied randomly with probability p or empty with probability $1 - p$. Occupied and empty sites may stand for very different physical properties [1, 2, 9, 31]. For simplicity, let us assume that the occupied sites are electrical conductors (represented by gray pixels in figure 1), the empty sites (shown by black pixels in figure 1) represent insulators, and that electrical current can flow only between nearest neighbor conductor sites.

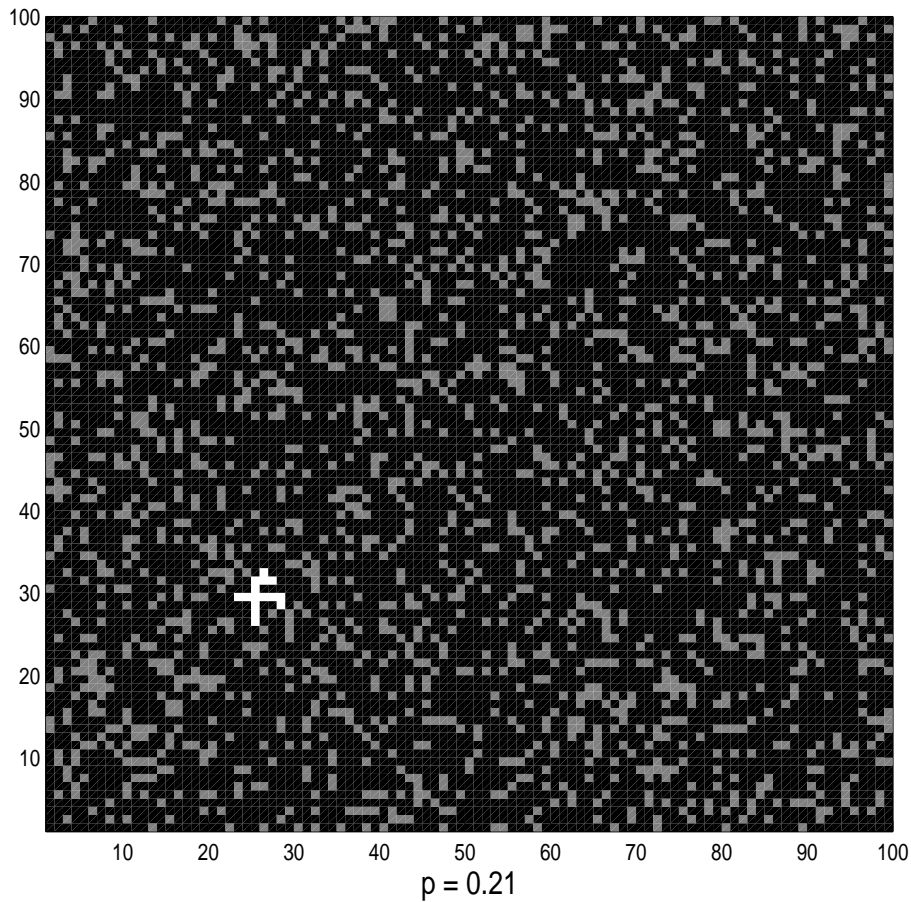


Figure 1: Site percolation on a square lattice. Grey cells of a square lattice correspond to conducting pixels, black stand for non-conducting, white cells belong to maximal conducting cluster. Concentration of conducting pixels equals to $p = 0,21$

At a low concentration p , the conductor sites are either isolated or form small clusters of nearest neighbor sites (see figure 1). We suppose that two conductor sites belong to the same cluster if they are connected by a path of nearest neighbor conductor sites, and a current can flow between them. At low p values, the mixture is an insulators, since a conducting path connecting opposite edges of our lattice does not exist. At large p values, on the other hand,

many conducting paths between opposite edges exist, where electrical current can flow, and the mixture is a conductor (see figure 2).

At some concentration in between, therefore, a threshold concentration p_c must exist where for the first time electrical current can percolate from one edge to the other (see figure 3). Thus, for the values $p < p_c$ we have an insulator, and for $p \geq p_c$ we have a conductor. The threshold concentration is called the *percolation threshold*, or, since it separates two different phases, the *critical concentration*. For a site problem on a square lattice the percolation threshold is approximately equal to 0.59, i.e., $p \approx 0.59$ [1, 2, 9, 31]. A situation for a value p close to the threshold is displayed in Figure 3.

If the occupied sites are superconductors and the empty sites are conductors, then p_c separates a normal-conducting phase for values $p < p_c$ transition from a superconducting phase where $p \geq p_c$. Another example is a mixture of magnets and paramagnets, where the system changes at p_c from a paramagnet to a magnet.

In contrast to the more common thermal phase transitions, where the transition between two phases occurs at a critical temperature, the percolation transition described here is a geometrical phase transition, which is characterized by the geometric features of large clusters in the neighborhood of p_c . At low values of p only small clusters of occupied sites exist. When the concentration p increases, the average size of the clusters increases, as well. At the critical concentration p_c , a large cluster appears which connects opposite edges of the lattice. This cluster commonly named *spanning cluster* or *percolating cluster* [1, 2, 9, 31]. In the thermodynamic limit, i.e. in the infinite system limit spanning cluster named *infinite cluster*, since its size diverges when the size of the lattice increases to infinity. It should be emphasized here that from traditional standpoint there exist unique *infinite cluster* and this *infinite cluster* always coincides with *spanning cluster* [1, 2, 9, 31].

When p increases further, the density of the infinite cluster also increases, since more and more sites start to be a part of the infinite cluster. Simultaneously, the average size of the finite clusters, which do not belong to the infinite cluster, decreases. At $p = 1$, trivially, all sites belong to the infinite cluster. In percolation, the concentration p of occupied sites plays the same role as the temperature in thermal phase transitions. Similar to thermal transitions, long range correlations control the percolation transition and the relevant quantities near p_c are described by power laws and critical exponents [1, 2, 9, 31].

The percolation transition is characterized by the geometrical properties of clusters for values of p that are close to p_c . One of important characteristics describing these properties is the probability, P_∞ , that a site belongs to the infinite cluster. For $p < p_c$, only finite clusters exist, and, therefore, it follows $P_\infty = 0$. For values $p > p_c$, P_∞ behaves similarly to the magnetization below critical temperature, and increases with p by a power law

$$P_\infty \sim (p - p_c)^\beta, \quad (11)$$

where $\beta = 5/36$ is critical exponent in 2D case [1, 2, 9, 31].

The linear size of the finite clusters, below and above percolation transition, is characterized by the *correlation length* ξ . The correlation length is defined as the mean distance between two sites on the same finite cluster. When p approaches p_c , ξ increases as

$$\xi \simeq a \cdot |p - p_c|^{-\nu}, \quad (12)$$



Figure 2: Site percolation on a square lattice. Concentration of conducting pixels is equal to $p = 0,63$. Grey cells of a square lattice correspond to the conducting pixels isolated from maximal (white) cluster

with the same exponent $\nu = 4/3$ below and above the threshold [1, 2, 9, 31].

To obtain ξ averages over all finite clusters in the lattice are required.

That is whelsy to note that all quantities described above are defined in the thermodynamic limit of large systems. In a finite system, P_∞ , for example, is not strictly zero below p_c .

The structure of percolation cluster can be well described in the framework of the fractal

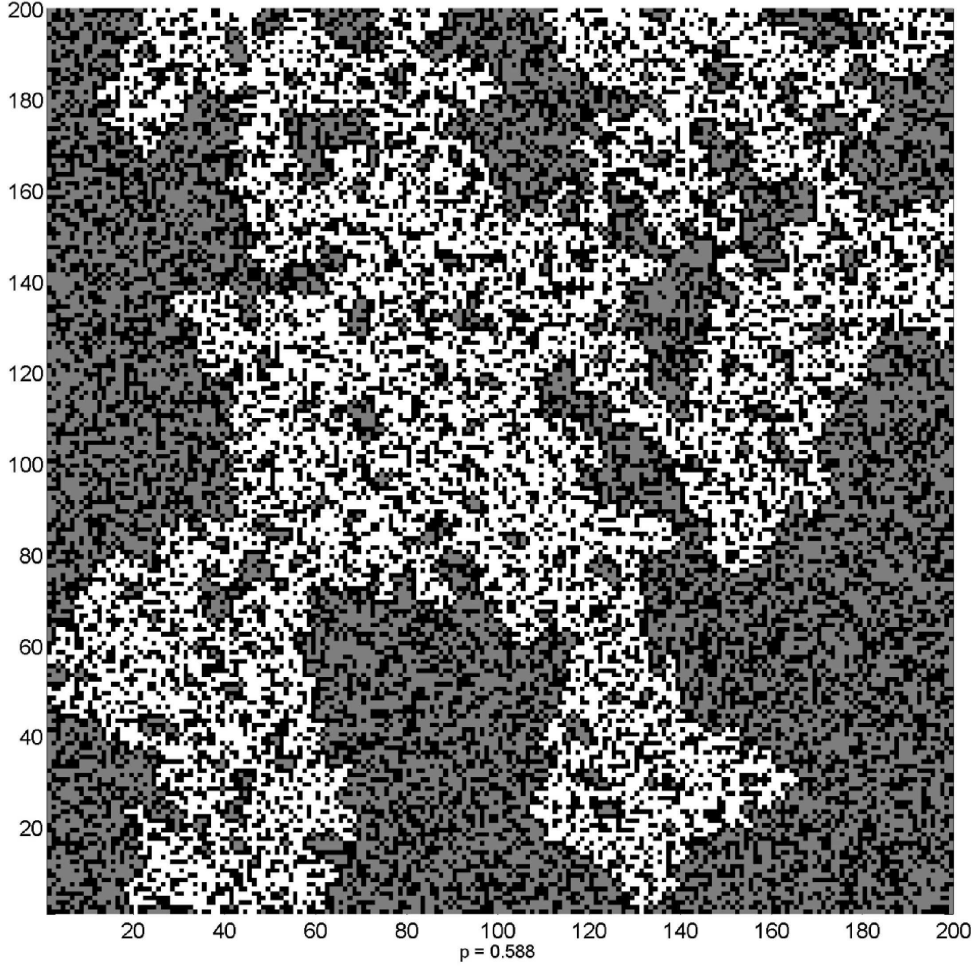


Figure 3: Site percolation on a square lattice. Concentration of conducting pixels is equal to $p = 0,588$

theory. We begin by considering the percolation cluster at the critical concentration p_c . A representative example of the *spanning* clusters shown in Fig. 3. As seen in the figure, the *infinite* cluster contains holes of all sizes. The cluster is self-similar on all length scales (larger than the unit size and smaller than the lattice size), and can be regarded as a fractal. The *fractal dimension*, d_f , describes how, on the average, the mass, M , of the cluster within a sphere of radius r scales with the r ,

$$M(r) \sim r^{d_f}. \quad (13)$$

In random fractals, $M(r)$ represents an average over many different cluster configurations or, equivalently, over many different centers of spheres on the same *infinite* cluster. Below and above p_c , the mean size of the finite clusters in the system is described by the correlation length ξ . At p_c , ξ diverges and holes occur in the *infinite* cluster on all length scales. Above p_c , ξ also represents the linear size of the holes in the *infinite* cluster. Since ξ is finite above p_c , the *infinite* cluster can be self-similar only on length scales smaller than ξ . We can interpret $\xi(p)$

as a typical length up to which the cluster is self-similar and can be regarded as a fractal. For length scales larger than ξ , the structure is not self-similar and can be regarded as homogeneous. If our length scales is smaller than ξ , we see a fractal structure. On length scales larger than ξ , we see a homogeneous system which is composed of many unit cells of size ξ . Mathematically, this can be summarized as

$$M(r) \sim \begin{cases} r^{d_f}, & r \ll \xi, \\ r^d, & r \gg \xi. \end{cases} \quad (14)$$

One can relate the fractal dimension d_f of percolation cluster to the exponents β and ν [1, 2, 9, 31]. The probability that an arbitrary site within a circle of radius r smaller than ξ belongs to the *infinite* cluster, is the ratio between the number of sites on the *infinite* cluster and the total number of sites,

$$P_\infty \sim \frac{r^{d_f}}{r^2}, \quad r < \xi. \quad (15)$$

This equation is certainly correct for $r = \lambda \xi$, where λ is an arbitrary constant smaller than 1. Substituting $r = \lambda \xi$ in (15) yields

$$P_\infty \sim \lambda^{d_f-2} \cdot \frac{\xi^{d_f}}{\xi^2} \sim \frac{\xi^{d_f}}{\xi^2}. \quad (16)$$

Both sides are powers of $p - p_c$. By substituting (11) and (12) into (16) we obtain,

$$d_f = 2 - \frac{\beta}{\nu}. \quad (17)$$

Thus the fractal dimension of the *infinite* cluster at p_c is not a new independent exponent but depends on β and ν . Since β and ν are universal exponents, d_f is also universal. It can be shown [31] that (17) also represents the fractal dimension of the finite clusters at p_c and below p_c , as long as their linear size is smaller than ξ .

The exponents β , ν , and γ describe the critical behavior of typical quantities associated with the percolation transition, and are called the *critical exponents*. The exponents are universal and depend neither on the structural details of the lattice (e.g., square or triangular) nor on the type of percolation (site, bond, or continuum), but only on the dimension d of the lattice ($d = 2$ in our present consideration).

This universality is a general feature of phase transitions, where the order parameter vanishes continuously at the critical point (second order phase transition). In Table 1, the values of the critical exponents β , ν , and γ in percolation are listed for 2D case [1].

Table 1.

Percolation	$d = 2$
Order parameter $P_\infty : \beta$	5/36
Correlation length $\xi : \nu$	4/3
Mean cluster size $S : \gamma$	43/18
Fractal dimension	91/48

The fractal dimension, however, is not sufficient to fully characterize a percolation cluster. For a further intrinsic characterization of a fractal we consider the shortest path between two sites on the cluster. We denote the length of this path, which is called the ‘chemical distance’, by l [1, 2, 9, 31]. The *graph* dimension d_l , which is also called the ‘chemical’ or ‘topological’ dimension, describes how the cluster mass M within the chemical distance l from a given site scales with l ,

$$M(l) \sim l^{d_l}. \quad (18)$$

While the fractal dimension d_f characterizes how the mass of the cluster scales with the Euclidean distance r , the graph dimension d_l characterizes how the mass scales with the chemical distance l .

The concept of the chemical distance also plays an important role in the description of spreading phenomena such as epidemics and forest fires, which propagate along the shortest path from the seed.

Let us investigate the percolation problem from positions of the new arithmetics of infinite and infinitesimal numbers (see [17, 20, 23]). Consider a 2D square lattice with period a and the linear size $L = a \cdot \mathbb{1}$. The full number of cells of such a lattice is, therefore, infinite and is equal to $V = \mathbb{1}^2$. Since the critical parameter is defined as the attitude of the occupied sites number N to their full number $p = N/V = N/\mathbb{1}^2$ then the smallest change in concentration $\delta p = \mathbb{1}^{-2}$ is equivalent to adding or subtracting only one occupied site. The infinitesimal small value δp is the maximum precision level we can distinguish by considering the critical parameter p on the $\mathbb{1} \times \mathbb{1}$ lattice. In order to obtain a higher precision level we should increase our lattice linear size. For example, if we use a lattice with period a and linear size $L = a \cdot \mathbb{1}^{1+\vartheta/2}$, where $\vartheta > 0$, the maximum precision level we can distinguish by considering the critical parameter p is $\delta p = 1/V = \mathbb{1}^{-(2+\vartheta)}$.

When we investigate the percolation problem we increase or decrease the critical parameter p using an appropriate precision level δp starting from an arbitrary point in between $p = 0$ and $p = 1$. According to the **Postulate 1** we are able to execute only a finite number of steps with length δp . Therefore, the length of critical parameter p interval that we can investigate is determined by the precision level we chose.

Consider the behavior of correlation radius. In the vicinity of percolation threshold the correlation radius diverges according to (12). On the other hand, the radius of correlation cannot exceed the system linear size $\xi \lesssim \xi_{max} = L = a \cdot \mathbb{1}$, where $\xi_{max} = a \cdot \mathbb{1}$ is the maximal correlation length. The situation is depicted in Figure 4. We see that in the range $[p_c - \mathbb{1}^{-1/\nu}, p_c + \mathbb{1}^{-1/\nu}]$ the radius of correlation in our $\mathbb{1} \times \mathbb{1}$ lattice does not change and keeps the value $\xi_{max} = a \cdot \mathbb{1}$.

Now we should decide which step we shall use to express different points on p axis. Infinitely many variants can be chosen dependent on the precision level we want to obtain. All these variants form three groups. The first group appears when in order to change p we use a small but still finite step $\delta p \ll 1$. In the case the phase transition is infinitely sharp because $\delta p \gg \mathbb{1}^{-1/\nu}$. The second group appears when $\delta p = c \cdot \mathbb{1}^{-1/\nu}$, where c is a finite grossdigit that is less than one. In the case the phase transition occupies the finite interval $[p_c - \mathbb{1}^{-1/\nu}, p_c + \mathbb{1}^{-1/\nu}]$. The third group appears when $\delta p = \mathbb{1}^{-\varsigma}$, where $\frac{1+\nu}{\nu} \leq \varsigma \leq 2$.⁵

⁵ For example, if we add only one occupied site in our grid, then p increases by $\delta p = \mathbb{1}^{-2}$, and that is the smallest step along p we can distinguish in our $\mathbb{1} \times \mathbb{1}$ lattice.

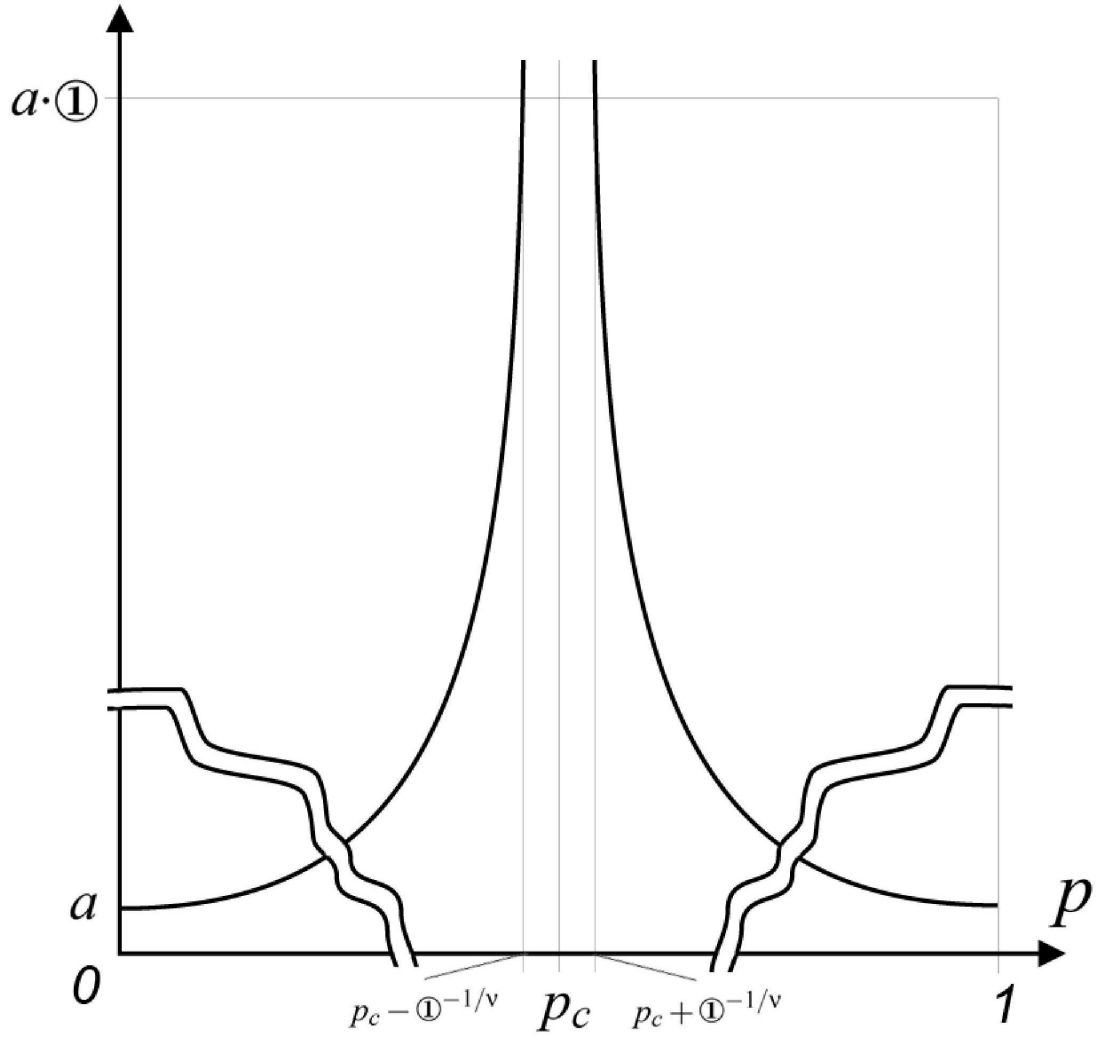


Figure 4: Correlation length versus p

In the case phase transition interval contains more than ① different points and if we execute a finite number of steps with length δp along this infinite transition area there exist three possibilities: 1) system contains a lot of finite and *infinite* clusters that coagulate but *spanning* cluster is still absent; 2) *spanning* cluster already exists and absorbs finite and infinite clusters; 3) at the beginning of our execution *spanning* cluster is absent but it appears after finite number of steps. This appearance is due to adding only one occupied site in our grid that produce confluence of either two infinite clusters or one finite and one infinite clusters.

Figure 5 shows that *spanning* cluster could envelop a set of embedded infinite clusters of different scales when we choose the critical parameter infinitesimally close to percolation threshold value. One infinite cluster is embedded into another also infinite but already spanning cluster. This situation is similar to that with finite clusters, when in finite system one finite cluster is embedded into another also finite but already spanning cluster. Some of the embedded

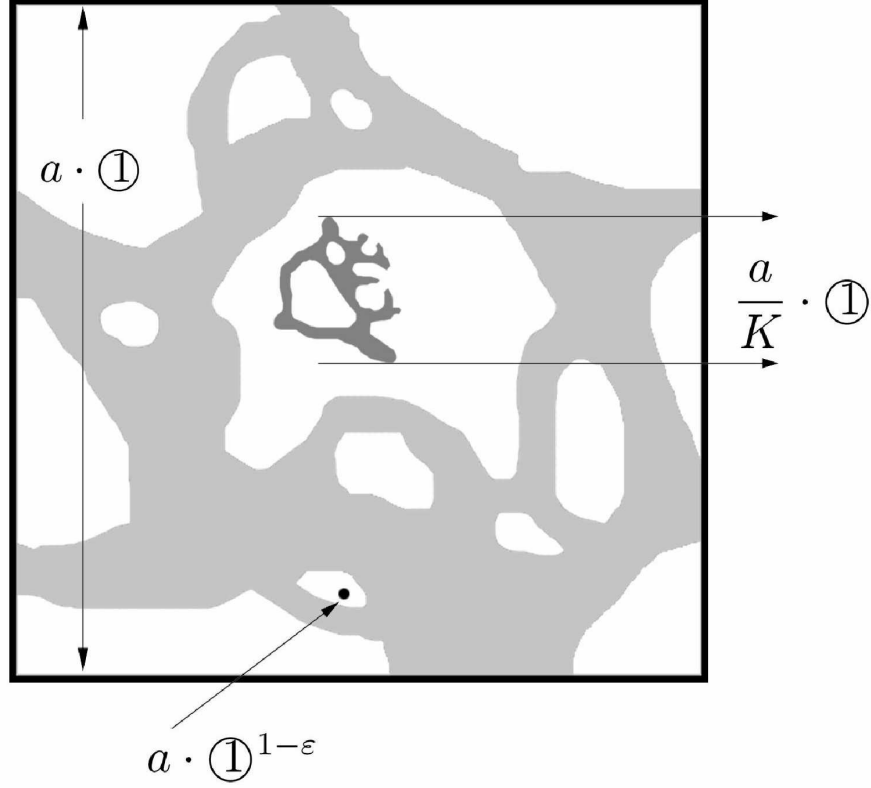


Figure 5: Embedded infinite clusters of different scales

infinite clusters are comparable with the *spanning* cluster and have linear sizes R that could be expressed by following:

$$R = \frac{a}{K} \textcircled{1}, \quad (19)$$

where $K > 1$ is a finite number. Remainder of the embedded infinite clusters has linear sizes that are indefinitely small as compared with $\textcircled{1}$ and could be expressed as $R = \varepsilon \cdot \textcircled{1}$, where ε is infinitesimal number.

On the step when *spanning* cluster appears the order parameter jumps from zero value up to the infinitesimal value as seen in Fig. 6

$$P_{\infty min} \sim C_1 \textcircled{1}^{d_f-2} = C_1 \textcircled{1}^{-\beta/\nu} = C_1 \textcircled{1}^{-\frac{5}{48}}, \quad (20)$$

where C_1 is a finite number. The first equality in (20) defines $P_{\infty min}$ as a measure of the relative size of *spanning* cluster expressible as a proportion of elements number of spanning cluster $C_1 \textcircled{1}^{d_f}$ to total number of grid elements $C_1 \textcircled{1}^2$. The second equality in (20) appears as a consequence of Eq. (17). Actually, spanning clusters could come in different size and shapes, so

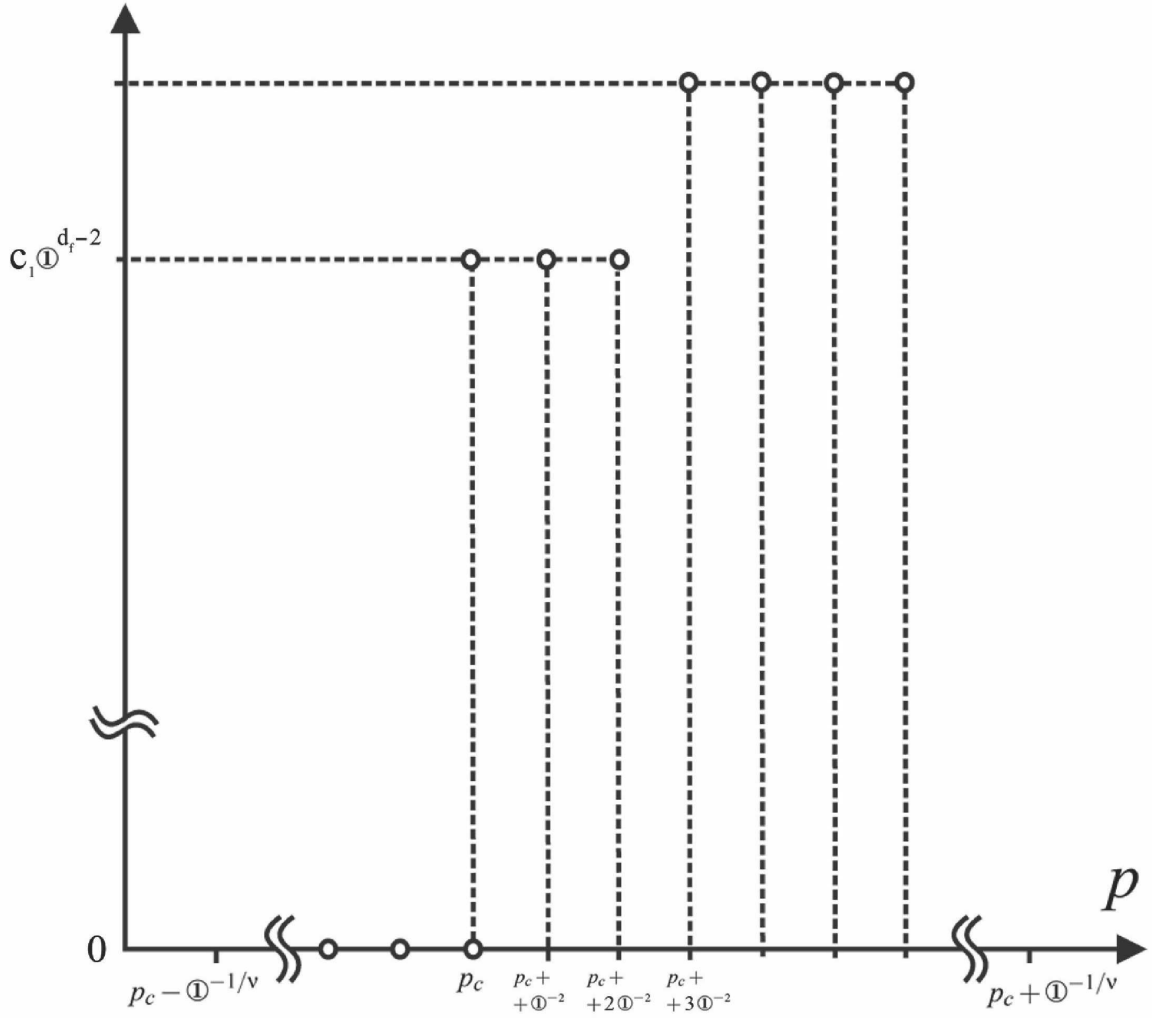


Figure 6: P_∞ vs p

the constant C_1 depicted in Fig. 6 could vary distinctly. In addition, when spanning cluster already exists and we increase p by adding new occupied sites the spanning cluster could expand because it engrosses other finite and infinite clusters.

We see that application of the new arithmetic of infinite and infinitesimal numbers gives us a unique opportunity to consider a point of phase transition in more detail (viewed just like a point with respect of traditional approach).

4 Gradient percolation

An important site-percolation problem generalization appears when the concentration p of occupied sites varies with the vertical distance z in our square grid. In literature (see [11]), this generalization is commonly named as the *gradient percolation*. It can be conveniently pictured in a geographical description in which the set of sites connected to the area $p \lesssim 1$ is called the ‘land’. In Fig 7, it is shown by white pixels. In this geographical language the set of connected empty sites not surrounded by land is called the ‘sea’, in Fig 7, it is shown by black pixels. Then, there naturally exist groups of occupied sites that are not connected with the land called ‘islands’. They are shown by grey pixels in Fig 7. Analogously, there exist also connected empty sites surrounded by the land. They are called ‘lakes’, which are shown also in black in Fig 7. In this geographical description, the part of the land in contact with the sea is called the ‘seashore’. In [11] this line is attributed as the *diffusion front*.

The diffusion front is conveniently described (see [11]) by its average width h_f , that can be related easily to the concentration gradient dp/dz at the position of the front. We see in Fig. 7 that, far from the front, islands or lakes are very small, whereas, near the front, their size becomes comparable to the width of the front. The islands correspond to the finite clusters in a percolation system, and the lakes correspond to the finite holes. The typical linear size of both quantities scales as ξ . Relation (12) tells that the size of the islands or lakes should increase when approaching the mean position of the front. But this size, even at z_f , is bounded due to the finite gradient of $p(z)$. The maximum typical size of islands and lakes is then given by the width of the front, which represents the only characteristic length scale in the problem, and we can assume

$$h_f \simeq \xi(z_c \pm h_f). \quad (21)$$

This assumption expresses our observation that islands or lakes near the front have the size comparable to the width of the front. Using (21) and expanding $p(z)$ around $z = z_c$ we obtain

$$h_f \simeq a|p(z_c \pm h_f) - p_c|^{-\nu} \simeq a \left| h_f \frac{dp}{dz}(z_c) \right|^{-\nu},$$

which gives

$$h_f \simeq a^{\frac{\beta_f}{\nu}} \left| \frac{dp}{dz}(z_c) \right|^{-\beta_f}, \quad (22)$$

where

$$\beta_f = \frac{\nu}{1 + \nu}. \quad (23)$$

As percolation is a critical phenomenon, the exponent β_f depends only on the dimensionality of the system (for $d = 2$ it follows $\beta_f = 4/7$), and not on the particular lattice structure (square, triangular, etc.).

Let us assume now, that we examine the gradient percolation phenomenon on a square lattice N^2 where $N = \mathbb{1}$, and the critical parameter p changes linearly, accepting infinitesimal value $p(z = a) = \frac{1}{a} \cdot \mathbb{1}^{-1}$ (value equal to zero) in the first line of lattice cells and value equal to unit $p(z = a \cdot \mathbb{1}) = 1$ in the last, $\mathbb{1}$ -s. In other words,

$$p(z) = A \cdot z,$$

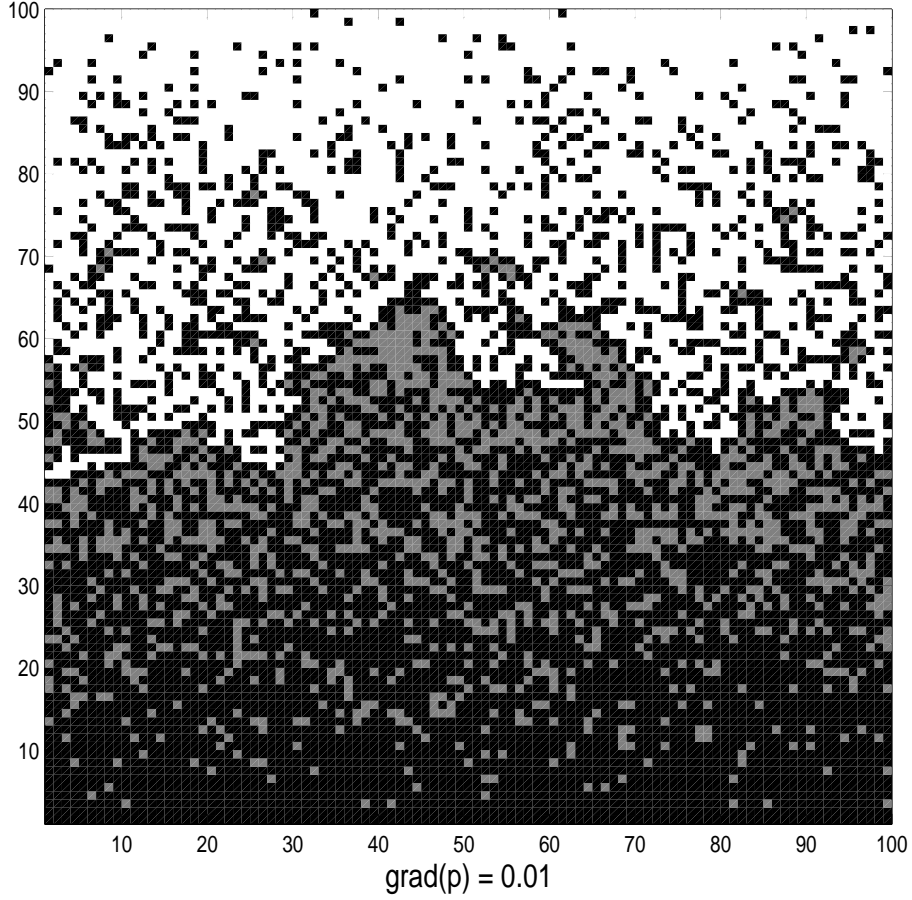


Figure 7: Gradient percolation in two dimensions

where $A = \frac{1}{a} \cdot \mathbb{1}^{-1}$ and z changes discretely. Then the value of the derivative in (22) is $\frac{1}{a} \cdot \mathbb{1}^{-1}$, and the diffusion front width makes

$$h_f \simeq a \cdot \mathbb{1}^{\beta_f} = a \cdot \mathbb{1}^{4/7}, \quad (24)$$

Thus, on scales of the observation commensurable with the size of the entire system, the diffusion front width is viewed as infinitesimally small and it is represented by the sharp border of two contrast phases – ‘sea’ and ‘land’. On the contrary, length scales commensurable with the finite number of the lattice periods a are completely absorbed by huge fluctuations of the front. At last, on scales proportional with h_f the width of front appears to the observer as a finite value.

5 A brief conclusion

In this paper, it has been shown that infinite and infinitesimal numbers introduced in [17, 23, 27] allow us to obtain exact numerical results instead of traditional asymptotic forms at different

points at infinity. We consider a number of traditional models related to the percolation theory using the new computational methodology. It has been shown that the new computational tools allow one to create new, more precise models of percolation and to study the existing models more in detail. The introduction in these models new, computationally manageable notions of the infinity and infinitesimals gives a possibility to pass from the traditional qualitative analysis of the situations related to these values to the quantitative one. Naturally, such a transition is very important from both theoretical and practical viewpoints.

The point of view presented in this paper uses strongly two methodological ideas borrowed from Physics: relativity and interrelations holding between the object of an observation and the tool used for this observation. The latter is directly related to connections between Analysis and Numerical Analysis because the numeral systems we use to write down numbers, functions, etc. are among our tools of investigation and, as a result, they strongly influence our capabilities to study mathematical objects.

Note that foundations of Analysis have been developed more than 200 years ago with the goal to develop mathematical tools allowing one to solve problems arising in the real world, as a result, they reflect ideas that people had about Physics in that time. Thus, Analysis that we use now does not include numerous achievements of Physics of the XX-th century. The brilliant efforts of Robinson made in the middle of the XX-th century have been also directed to a reformulation of the classical Analysis in terms of infinitesimals and not to the creation of a new kind of Analysis that would incorporate new achievements of Physics. In fact, he wrote in paragraph 1.1 of his famous book [16]: ‘It is shown in this book that Leibnitz’ ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics’.

Site percolation and gradient percolation have been studied by applying the new computational tools. It has been established that in infinite system phase transition point is not really a point as with respect of traditional approach. In light of new arithmetic it appears as a critical interval, rather than a critical point. Depending on “microscope” we use this interval could be regarded as finite, infinite and infinitesimal short interval. Using new approach we observed that in vicinity of percolation threshold we have *many* different *infinite clusters* instead of *one infinite cluster* that appears in traditional consideration. Moreover, we have now a tool to distinguish those infinite clusters. In particular, we can distinguish *spanning infinite* clusters from *embedded* infinite clusters.

Then we consider gradient percolation phenomenon on infinite square lattice with infinitesimal gradient of critical parameter p that changes linearly, accepting infinitesimal value $p(z = a) = \frac{1}{a} \cdot \textcircled{1}^{-1}$ (value equal to zero) in the first line of lattice cells and value equal to unit $p(z = a \cdot \textcircled{1}) = 1$ in the last, $\textcircled{1}$ -s line of lattice cells. We observe that diffusion front width in this case stretches for an infinite number of lattice spacing: $h_f \simeq a \cdot \textcircled{1}^{\beta_f} = a \cdot \textcircled{1}^{4/7}$. And again this value could be regarded as finite, infinite and infinitesimal short depending on “microscope” we use.

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