Another proof of Pell identities by using the determinant of tridiagonal matrix

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Abstract

In this paper, another proof of Pell identities is presented by using the determinant of tridiagonal matrices. It is calculated via the Laplace expansion.

Key words: Pell numbers, Pell identities, tridiagonal matrix, Laplace expansion, determinant.

1 Introduction

Pell numbers are defined as

$$P_n = 2P_{n-1} + P_{n-2}$$

with the initial conditions $P_0 = 0, P_1 = 1$ for $n \ge 2$.

In [1], a complex factorization formula for (n+1)th Pell number is obtained to provided the tridiagonal matrix

$$N(n) = \begin{bmatrix} 2i & 1 & & & \\ 1 & 2i & 1 & & \\ & 1 & 2i & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2i & 1 \\ & & & & 1 & 2i \end{bmatrix}$$

as in the following:

$$P_{n+1} = m |N(n)|, \quad m = \begin{cases} 1, & n \equiv 0 \pmod{4} \\ -i, & n \equiv 1 \pmod{4} \\ -1, & n \equiv 2 \pmod{4} \\ i, & n \equiv 3 \pmod{4} \end{cases}$$
(1)

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In [2], an identity of Fibonacci numbers is proved via the determinant of the tridiagonal matrix. In [3], the authors showed the connection between Fibonacci numbers and Chebyshev polynomials and obtained a complex factorization for Fibonacci numbers by using a sequence of the tridiagonal matrices. Then with a small difference in the tridiagonal matrix, it is showed that how Lucas numbers and Chebyshev polynomials are connected to each other. Two complex factorization are obtained by using the $n \times n$ tridiagonal and anti-tridiagonal matrix for n is even in [4].

In this paper, we give another proof of Pell identities

$$P_{2n} = P_n(P_{n+1} + P_{n-1}) \tag{2}$$

$$P_n = P_k P_{n-k+1} + P_{k-1} P_{n-k} \tag{3}$$

where k is a positive integer. For $1 \le k \le n$

$$P_{n} = P_{1}P_{n} + P_{0}P_{n-1}$$

$$P_{n} = P_{2}P_{n-1} + P_{1}P_{n-2}$$

$$P_{n} = P_{3}P_{n-2} + P_{2}P_{n-3}$$

$$P_{n} = P_{4}P_{n-3} + P_{3}P_{n-4}$$

$$\vdots$$

$$P_{n} = P_{n}P_{1} + P_{n-1}P_{0}.$$
(4)

2 Main Result

Let A be an $n \times n$ matrix, $A([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k])$ be the $k \times k$ submatrix of A and $M([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k])$ be the $(n - k) \times (n - k)$ minor of the matrix A. The cofactor of A is defined by

$$\dot{A}([i_1, i_2, \dots, i_k], [j_1, j_2, \dots, j_k]) = (-1)^m M([i_1, i_2, \dots, i_k], [j_1, j_2, \dots, j_k])$$

where $1 \le i_1, i_2, \cdots, i_k \le n$ and $m = \sum_{r=1}^k (i_r + j_r)$. The determinant of the matrix A is

$$\det(A) = \sum_{\substack{1 \le i_1, i_2, \cdots, i_k \le n \\ \times \det(\mathring{A}([i_1, i_2, \dots, i_k], [j_1, j_2, \dots, j_k]))} \\ \times \det(\mathring{A}([i_1, i_2, \dots, i_k], [j_1, j_2, \dots, j_k])).$$

If $A(i,j) = a_{ij}$, then $\mathring{A}(i,j) = (-1)^{i+j} M(i,j) = \mathring{A}_{ij}$ and the determinant is

$$\det(A) = \sum_{i=1}^{n} a_{ij} \mathring{A}_{ij}.$$

This is the famous Laplace expansion formula [2]. We will use this formula to proof of Pell identities in (3).

The cofactors of the first row of the matrix N(n) are

$$\mathring{A}_{11} = \left\{ \begin{array}{ll} -P_{n-1} &, n \equiv 0 (\bmod \, 4) \\ -iP_{n-1} &, n \equiv 1 (\bmod \, 4) \\ P_{n-1} &, n \equiv 2 (\bmod \, 4) \\ iP_{n-1} &, n \equiv 3 (\bmod \, 4) \end{array} \right. \\ \\ \mathring{A}_{12} = \left\{ \begin{array}{ll} -iP_{n-2} &, n \equiv 0 (\bmod \, 4) \\ P_{n-2} &, n \equiv 1 (\bmod \, 4) \\ iP_{n-2} &, n \equiv 2 (\bmod \, 4) \\ -P_{n-2} &, n \equiv 3 (\bmod \, 4) \end{array} \right.$$

By using the Laplace expansion formula the determinant of the matrix ${\cal N}(n-1)$ is

$$\det(N(n-1)) = 2P_{n-1} + P_{n-2}.$$
(5)

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From (1), the initial value $P_0 = 0, P_1 = 1, P_2 = 2$ and the fact $P_2 = 2P_1$ are used in (5), then we have

$$P_n = 2P_1P_{n-1} + P_1P_{n-2} = P_2P_{n-1} + P_1P_{n-2}.$$
 (6)

If the first two rows of the matrix N(n-1) are chosen, there are only three 2×2 submatrices of the matrix N(n-1) whose determinants are nonzero. i.e.

$$A([1,2],[1,2]) = \begin{bmatrix} 2i & 1\\ 1 & 2i \end{bmatrix} = -P_3$$
$$A([1,2],[1,3]) = \begin{bmatrix} 2i & 0\\ 1 & 1 \end{bmatrix} = iP_2$$
$$A([1,2],[2,3]) = \begin{bmatrix} 1 & 0\\ 2i & 1 \end{bmatrix} = P_1$$

and their cofactors are

$$\mathring{A}([1,2],[1,2]) = \begin{cases} iP_{n-2} &, n \equiv 0 \pmod{4} \\ -P_{n-2} &, n \equiv 1 \pmod{4} \\ -iP_{n-2} &, n \equiv 2 \pmod{4} \\ P_{n-2} &, n \equiv 3 \pmod{4} \end{cases}$$
$$\mathring{A}([1,2],[1,3]) = \begin{cases} -P_{n-3} &, n \equiv 0 \pmod{4} \\ -iP_{n-3} &, n \equiv 1 \pmod{4} \\ P_{n-3} &, n \equiv 2 \pmod{4} \\ iP_{n-3} &, n \equiv 3 \pmod{4} \end{cases}$$

$\dot{A}([1,2],[2,3]) = 0.$

By using the Laplace expansion the determinant of the matrix N(n-1) is

$$\det(N(n-1)) = \begin{cases} -iP_3P_{n-2} - iP_2P_{n-3} & , n \equiv 0 \pmod{4} \\ P_3P_{n-2} + P_2P_{n-3} & , n \equiv 1 \pmod{4} \\ iP_3P_{n-2} + iP_2P_{n-3} & , n \equiv 2 \pmod{4} \\ -P_3P_{n-2} - P_2P_{n-3} & , n \equiv 3 \pmod{4} \end{cases}$$
(7)

From (1) and (7), we obtain

$$P_n = P_3 P_{n-2} + P_2 P_{n-3}.$$
 (8)

If the first three rows of the matrix N(n-1) are chosen, there are only four 3×3 submatrices of the matrix N(n-1) whose determinants are nonzero:

$$\begin{aligned} A([1,2,3],[1,2,3]) &= \begin{bmatrix} 2i & 1 & 0 \\ 1 & 2i & 1 \\ 0 & 1 & 2i \end{bmatrix} = -iP_4 \\ A([1,2,3],[1,2,4]) &= \begin{bmatrix} 2i & 1 & 0 \\ 1 & 2i & 0 \\ 0 & 1 & 1 \end{bmatrix} = -P_3 \\ A([1,2,3],[1,3,4]) &= \begin{bmatrix} 2i & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2i & 1 \end{bmatrix} = iP_2 \\ A([1,2,3],[2,3,4]) &= \begin{bmatrix} 1 & 0 & 0 \\ 2i & 1 & 0 \\ 1 & 2i & 1 \end{bmatrix} = P_1 \end{aligned}$$

and their cofactors are

$$\begin{split} \mathring{A}([1,2,3],[1,2,3]) &= \begin{cases} P_{n-3} &, n \equiv 0 \pmod{4} \\ iP_{n-3} &, n \equiv 1 \pmod{4} \\ -P_{n-3} &, n \equiv 2 \pmod{4} \\ -iP_{n-3} &, n \equiv 2 \pmod{4} \\ -iP_{n-3} &, n \equiv 3 \pmod{4} \end{cases} \\ \mathring{A}([1,2,3],[1,2,4]) &= \begin{cases} iP_{n-4} &, n \equiv 0 \pmod{4} \\ -P_{n-4} &, n \equiv 1 \pmod{4} \\ -iP_{n-4} &, n \equiv 2 \pmod{4} \\ P_{n-4} &, n \equiv 3 \pmod{4} \\ \end{cases} \\ \mathring{A}([1,2,3],[1,3,4]) = 0 \\ \mathring{A}([1,2,3],[2,3,4]) = 0 \end{cases}$$

By using the Laplace expansion the determinant of the matrix N(n-1) is evaluated as follows:

$$\det(N(n-1)) = \begin{bmatrix} -iP_4P_{n-3} - iP_3P_{n-4} & ,n \equiv 0(\mod 4) \\ P_4P_{n-3} + P_3P_{n-4} & ,n \equiv 1(\mod 4) \\ iP_4P_{n-3} + iP_3P_{n-4} & ,n \equiv 2(\mod 4) \\ -P_4P_{n-3} - P_3P_{n-4} & ,n \equiv 3(\mod 4) \end{bmatrix}$$
(9)

From (1) and (9), we have

$$P_n = P_4 P_{n-3} + P_3 P_{n-4}.$$
 (10)

The remaining identities in (4) can be shown similarly.

Now, we give another proof of following Pell identity:

$$P_{2n} = P_n (P_{n+1} + P_{n-1}). \tag{11}$$

If we choose the first (n-1) rows of the matrix N(2n-1), there are only n the $(n-1) \times (n-1)$ submatrices of the matrix N(2n-1) whose determinants are nonzero but only the cofactors of two of them are nonzero. i.e.

$$A([1,2,\ldots,n-1],[1,2,\ldots,n-2,n-1]) = \begin{cases} -iP_n &, n \equiv 0 \pmod{4} \\ P_n &, n \equiv 1 \pmod{4} \\ iP_n &, n \equiv 2 \pmod{4} \\ -P_n &, n \equiv 3 \pmod{4} \end{cases}$$
$$A([1,2,\ldots,n-1],[1,2,\ldots,n-2,n]) = \begin{cases} -P_{n-1} &, n \equiv 0 \pmod{4} \\ -iP_{n-1} &, n \equiv 1 \pmod{4} \\ P_{n-1} &, n \equiv 2 \pmod{4} \\ iP_{n-1} &, n \equiv 3 \pmod{4} \end{cases}$$

and their cofactors are

$$\mathring{A}([1,2,\ldots,n-1],[1,2,\ldots,n-2,n-1]) = \begin{cases} P_{n+1} , n \equiv 0 \pmod{4} \\ iP_{n+1} , n \equiv 1 \pmod{4} \\ -P_{n+1} , n \equiv 2 \pmod{4} \\ -iP_{n+1} , n \equiv 3 \pmod{4} \end{cases}$$
$$\mathring{A}([1,2,\ldots,n-1],[1,2,\ldots,n-2,n]) = \begin{cases} iP_n , n \equiv 0 \pmod{4} \\ -P_n , n \equiv 1 \pmod{4} \\ -iP_n , n \equiv 2 \pmod{4} \\ -P_n , n \equiv 1 \pmod{4} \\ -iP_n , n \equiv 3 \pmod{4} \end{cases}$$

From Laplace expansion the determinant of the matrix N(2n-1) is

$$\det(N(2n-1)) = \begin{cases} -iP_n(P_{n+1} + P_{n-1}) & , n \equiv 0, 2 \pmod{4} \\ iP_n(P_{n+1} + P_{n-1}) & , n \equiv 1, 3 \pmod{4} \end{cases} .$$
(12)

From (1) and (12) we have

$$P_{2n} = P_n(P_{n+1} + P_{n-1}).$$

Thus the proof is completed.

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