# A Lagrangian Description of the Higher-Order Painlevé Equations 

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#### Abstract

We derive the Lagrangians of the higher-order Painlevé equations using Jacobi's last multiplier technique. Some of these higher-order differential equations display certain remarkable properties like passing the Painlevé test and satisfy the conditions stated by Juráš, (Acta Appl. Math. 66 (2001) 25-39), thus allowing for a Lagrangian description.


## 1 Introduction

The study of higher-order Painlevé equations is interesting from the mathematical point of view because of the possibility of existence of new transcendental functions beyond the six Painlevé transcendents. In addition such higher-order Painlevé often have interesting physical and mathematical applications. For example it is known that special solutions of equations for the Korteweg de Vries hierarchy which are used for describing water waves can be expressed via the higher-order Painlevé equations.

The first Painlevé hierarchy was first introduced in [1]. Thereafter many results were obtained in the analysis of the higher Painlevé equations. Scaling similarity solutions of three integrable PDEs namely the Sawada-Kotera,
fifth order KdV and Kaup-Kupershmidt equations were considered in [2] where it was shown that these fourth-order ordinary differential equations (ODEs) may be written as non-autonomous Hamiltonian equations for time dependent generalizations of integrable cases of the Hénon-Heiles system.

In [3] it was proved that higher-order members for the first and second Painlevé hierarchies do not have polynomial first integrals and that their solutions can determine new transcendental functions. Lax pairs for some equations of these hierarchies are presented in [4] and the Cauchy problem for equations of these hierarchies can be solved by an analogy with the Cauchy problem of the well known Painlevé equations. The Painlevé tests for higherorder Painlevé equations were demonstrated in [5-7].

In [8] two new hierarchies of nonlinear ODEs were introduced which were called the $K_{1}$ and $K_{2}$ hierarchies and which may be considered as new higher Painlevé hierarchies. The equations of these hierarchies have all the properties that are unique to the famous Painlevé equations.

Shimomura in [9] presented an interesting expression for the first Painlevé hierarchy which allows us to consider new properties of equations. Poles and $\alpha$ - points of the meromorphic solutions of the first Painlevé hierarchy was studied by Shimomura in [10], where a lower estimate for the number of poles of meromorphic solution is also given.

In 11 instanton-type solutions and some leading expressions for the second member of the first hierarchy were constructed using multiple-scale analysis. Recently Mo in [12] has applied a twistor description of the similarity reductions to the case of the KdV hierarchy to obtain the twistor spaces of the Painlevé I and Painlevé II hierarchy. Dai and Zhang [13] have extended the results by Boutroux [14,15] for the first Painlevé equation to the case of the first Painlevé hierarchy. The authors have shown that there are solutions characterized by divergent asymptotic expansions near infinity in specified sectors of the complex plane for higher-order analogue of the first Painlevé equation.

Some important results connected with higher-order Painlevé equations were also obtained in the papers [16,17]. In [16] Claeys and Vanlessen proved the universality of the correlation kernel in a double scaling limit near singular edge points in random matrix models that were built out of functions associated with a special solution of the second member for the first Painlevé hierarchy. In [17] the authors established the existence of real solution of the fourth-order analogue of the Painlevé equation and obtained the solvability of an associated Riemann - Hilbert problem through the approach of a vanishing lemma and found additionally the asymptotics of solutions.

The Hamiltonian structure of the second Painlevé hierarchy was considered in [18]. Here the authors introduced new canonical coordinates and
obtained the Hamiltonian for evolutions. They also gave an explicit formulae for these Hamiltonians and demonstrated that these Hamiltonians are polynomials in the canonical coordinates.

The aim of this paper is to obtain the Lagrangians for the four higher Painlevé hierarchies using the same approach. In recent years much attention has been paid to the Lagrangian framework of higher-order differential equations. Although a Lagrangian always exists for any second-order ordinary differential equation its connection with Jacobi's last multiplier (JLM) [19,20 is perhaps not very widely known. The credit for resurrecting the JLM, in recent years, must go to Leach and Nucci, who have shown how it may be used to determine the first integrals and also Lagrangians of a wide variety of nonlinear differential equations [21]. While it appears that the connection of the Jacobi last multiplier to the existence of Lagrangian functions were the subjects of investigation by a few authors in the early 1900's, the precise nature of this interrelation was brought out by Rao, in the 1940's [22]. Thereafter it does not appear to have attracted the attention of most researchers working in the field of differential equations.

According to the classical theory of Darboux [23] every scalar secondorder ordinary differential equation is multiplier variational. The problem of finding a Lagrangian for a given ODE is generally referred to as the inverse variational problem of classical mechanics. The necessary and sufficient conditions for an equation $y^{\prime \prime}=F\left(x, y, y^{\prime}\right)$ to be derivable from the EulerLagrange equation $\frac{\partial L}{\partial y}-\frac{d}{d x}\left(\frac{\partial L}{\partial y^{\prime}}\right)=0$, was enunciated by Helmholtz [24, (25] in the form of certain identities.

The variational multiplier problem for higher-order scalar ordinary differential equations has been studied by Fels [26] and Juráś [27]. The inverse problem for a fourth-order ODE was solved by Fels who investigated scalar fourth-order ordinary differential equations of the form

$$
\frac{d^{4} u}{d x^{4}}=f\left(x, u, \frac{d u}{d x}, \frac{d^{2} u}{d x^{2}}, \frac{d^{3} u}{d x^{3}}\right) .
$$

Fels approach for solving the fourth-order inverse problem was essentially based on a modified version of Douglas's [28] classical solution to the multiplier problem as refined by Anderson and Thompson in [29], who used the variational bicomplex theory [30] to derive the multiplier and showed that the existence of a multiplier was in a direct correspondence with the existence of special cohomology classes arising in the variational bicomplex associated with a differential equation. Fels conditions ensure the existence and uniqueness of the Lagrangian in the case of a fourth-order equation and it has been shown by Nucci and Arthurs [31] and more recently by us 32] that when
these conditions are satisfied, a Lagrangian can be derived from the Jacobi last multiplier.

In fact Fels approached the problem using Cartan's equivalence method, and arrived at two differential invariants whose vanishing completely characterizes the existence of a variational multiplier. Unlike the second-order case, the multiplier is unique up to a constant multiple. The programme was further developed by Juráš [27] who studied the inverse problem for sixth and eighth-order equations. In fact Juráš obtained a similar solution by using, however, a more direct approach in the spirit of the variational bicomplex; the differential invariants becoming increasingly complicated for higher-order systems. By analyzing the structure equations of the horizontal differential he uncovered a two-form $\Pi$ with the property $d \Pi \equiv 0 \bmod \Pi$, if and only if the equation

$$
\frac{d^{2 n} u}{d x^{2 n}}=f\left(x, u, \frac{d u}{d x}, \cdots, \frac{d^{2 n-1} u}{d x^{2 n-1}}\right)
$$

is multiplier variational. He proved that a Lagrangian, if it exists, is unique up to the multiplication by a constant and found functions $I_{1}, I_{2}, \ldots, I_{n}$, whose vanishing provides a necessary condition for the above equation to be variational. These functions are not relative contact invariants, but their simultaneous vanishing is a contact invariant condition.

In [32] the authors made use of the Jacobi Last Multiplier (JLM) to derive Lagrangians for a set of fourth-order ODEs which pass the Painlevé test, i.e., their solutions are free of movable critical points. Recently the conjugate Hamiltonian equations for such fourth-order equations passing the Painlevé test have also been derived in [33].

## 2 Four Painlevé hierarchies

Now the first and the second Painlevé hierarchy are well known and can be written as the following

$$
\begin{align*}
& \sum_{m=1}^{N} t_{m} L_{m}[u]=x  \tag{1}\\
& \left(\frac{d}{d x}+u\right) \sum_{m=1}^{M} t_{m} L_{m}\left[u_{x}-u^{2}\right]-x u-\beta_{N}=0 \tag{2}
\end{align*}
$$

where $N$ and $M$ are integers, $t_{m},(m=1, \ldots, N)$ is the sequence of operators $L_{m}[u]$ that satisfies the Lenard recursion relation

$$
\begin{equation*}
d_{x} L_{m+1}[u]=\left(d_{x}^{3}+4 u d_{x}+2 u_{x}\right) L_{m}[u], \quad L_{0}[u]=\frac{1}{2} . \tag{3}
\end{equation*}
$$

Taking the operator (3) into account we obtain

$$
\begin{align*}
& L_{1}[u]=u,  \tag{4}\\
& L_{2}[u]=u_{x x}+3 u^{2},  \tag{5}\\
& L_{3}[u]=u_{x x x x}+10 u u_{x x}+5 u_{x}^{2}+10 u^{3}  \tag{6}\\
& L_{4}[u]=u_{x x x x x x}+14 u u_{x x x x}+28 u_{x} u_{x x x}+21 u_{x x}^{2}+ \\
& \qquad 70 u^{2} u_{x x}+70 u u_{x}^{2}+35 u^{4} \tag{7}
\end{align*}
$$

Using the values of operators $L_{1}, L_{2}, L_{3}, L_{4}$ and so on we can obtain the equations of the first and the second Painlevé hierarchies.

The sixth-order ordinary differential equations of the first and the second Painlevé hierarchies have the form

$$
\begin{align*}
& t_{4}\left(u_{x x x x x x}+14 u u_{x x x x}+28 u_{x} u_{x x x}+21 u_{x x}^{2}+70 u^{2} u_{x x}+\right. \\
& \left.+70 u u_{x}^{2}+35 u^{4}\right)+t_{3}\left(u_{x x x x}+10 u u_{x x}+5 u_{x}^{2}+10 u^{3}\right)+  \tag{A}\\
& +t_{2}\left(u_{x x}+3 u^{2}\right)+t_{1} u=x, \\
& t_{3}\left(u_{x x x x x x}-14 u^{2} u_{x x x x}-56 u u_{x} u_{x x x}-28 u_{x}^{2} u_{x x}-42 u u_{x x}^{2}+\right. \\
& \left.+70 u^{4} u_{x x}+140 u^{3} u_{x}^{2}-20 u^{7}\right)+t_{2}\left(u_{x x x x}-10 u^{2} u_{x x}-\right.  \tag{B}\\
& \left.-10 u u_{x}^{2}+6 u^{5}\right)+t_{1}\left(u_{x x}-2 u^{3}\right)-x u-\beta_{3}=0,
\end{align*}
$$

We see that equations of the first and the second hierarchy have even integer orders $2 N-2$ and $2 M$ respectively.

Equations (A) and (B) are important and interesting because setting the constants $t_{3}=t_{2}=0$ one recovers the Painlevé equations. When $t_{1}=t_{3}=0$ these yield equations which we have studied recently. In the case $t_{1}=t_{2}=0$ they reduce to sixth-order equations which are the third members of the first and second Painlevé hierarchies. The general case of these equations correspond to the first and second Painlevé hierarchies.

There are two other hierarchies of nonlinear ordinary differential equations that have the properties similar to Painlevé equations. These hierarchies were introduced in [8] and were referred to in [34] as the $K_{1}$ and $K_{2}$ hierarchies. These hierarchies can be presented as the following

$$
\begin{align*}
& \sum_{m=1}^{N} t_{m} H_{m}[u]=x  \tag{8}\\
& \left(\frac{d}{d x}+u\right) \sum_{m=1}^{M} t_{m} H_{m}\left[u_{x}-\frac{1}{2} u^{2}\right]-x u-\beta_{M}=0 \tag{9}
\end{align*}
$$

where $N$ and $M$ are integers, $t_{m}$ are parameters of the equation and the operator $H_{m}$ may be calculated by means of the formulae

$$
\begin{equation*}
H_{n+2}=J[v] \Omega[v] H_{n}, \tag{10}
\end{equation*}
$$

under the conditions

$$
\begin{equation*}
H_{0}[v]=1, \quad H_{1}[v]=v_{x x}+4 v^{2} \tag{11}
\end{equation*}
$$

where the operators $\Omega[v]$ and $J[v]$ are determined by the relations

$$
\begin{align*}
& \Omega=D^{3}+2 v D+v_{x}, \quad D=\frac{d}{d x} \\
& J= \\
& \quad D^{3}+3(v D+D v)+2\left(D^{2} v D^{-1}+D^{-1} v D^{2}\right)+  \tag{13}\\
& \\
& \quad+8\left(v^{2} D^{-1}+D^{-1} v^{2}\right), \quad D^{-1}=\int d x
\end{align*}
$$

Taking conditions (11) and operators (12), (13) into account we have the operators $H_{2}$ and $H_{3}$ as the following

$$
\begin{equation*}
H_{2}[v]=v_{x x x x}+12 v v_{x x}+6 v_{x}^{2}+\frac{32}{3} v^{3}, \tag{14}
\end{equation*}
$$

$$
\begin{gather*}
H_{3}[v]=v_{x x x x x x x x}+20 v v_{x x x x x x}+60 v_{x} v_{x x x x x}+134 v_{x x} v_{x x x x}+ \\
+136 v^{2} v_{x x x x}+84 v_{x x x}^{2}+544 v v_{x} v_{x x x}+408 v v_{x x}^{2}+396 v_{x}^{2} v_{x x}+  \tag{15}\\
+\frac{1120}{3} v^{3} v_{x x}+560 v^{2} v_{x}^{2}+\frac{256}{3} v^{5} .
\end{gather*}
$$

Note that hierarchies (8) and (9) can also be presented using another operator $G_{k}[u]$. In terms of this operator these hierarchies take in the form

$$
\begin{align*}
& \sum_{k=1}^{N} t_{k} G_{k}[u]=x .  \tag{16}\\
& \left(u-\frac{1}{2} \frac{d}{d x}\right) \sum_{k=1}^{M} t_{k} G_{k}\left[-2 u_{x}-2 u^{2}\right]-x u-\beta_{M}=0 . \tag{17}
\end{align*}
$$

Hierarchy (16) can be transformed to (8) but hierarchy (17) coincides with hierarchy (9). The recursion relation $G_{k}$ is determined by the nonlinear operator

$$
\begin{equation*}
G_{k+2}=J_{1}[v] \Omega[v] G_{k}, \tag{18}
\end{equation*}
$$

under the conditions

$$
\begin{equation*}
G_{0}[v]=1, \quad G_{1}[v]=v_{x x}+\frac{1}{4} v^{2} . \tag{19}
\end{equation*}
$$

The operator $J_{1}[v]$ takes the form

$$
\begin{equation*}
J_{1}=D^{3}+\frac{1}{2}\left(D^{2} v D^{-1}+D^{-1} v D^{2}\right)+\frac{1}{8}\left(v^{2} D^{-1}+D^{-1} v^{2}\right) \tag{20}
\end{equation*}
$$

Hierarchies $K_{1}$ and $K_{2}$ though similar to the first and the second Painlevé hierarchies have a fundamental difference in the sense that we cannot transform equations of hierarchies (16) and (17) to hierarchies (1) and (2). Moreover the hierarchy $K_{1}$ has even integer order except $6 k(k=1,2, \ldots)$ and hierarchy $K_{2}$ also has even integer order except $6 k(k=1,2, \ldots)$.

The fourth order equation corresponding to the hierarchy $K_{1}$ takes the form

$$
\begin{equation*}
t_{2}\left(u_{x x x x}+12 u u_{x x}+6 u_{x}^{2}+\frac{32}{3}\right)+t_{1}\left(u_{x x}+4 u^{2}\right)=0 . \tag{C}
\end{equation*}
$$

At $t_{2}=0$ equation (C) is the first Painlevé equation but at $t \neq 0$ the forth order equation differs from the the fourth order equation of the first Painlevé equation and we hope that this one may give a new transcendal function.

On the other hand the sixth-order equation from hierarchy $K_{2}$ can be written as

$$
\begin{gather*}
t_{2}\left(u_{x x x x x x}+7 u_{x} u_{x x x x}-7 u^{2} u_{x x x x}+14 u_{x x} u_{x x x}-28 u u_{x} u_{x x x}-\right. \\
-28 u_{x}^{2} u_{x x}-21 u u_{x x}^{2}-\frac{28}{3} u u_{x}^{3}-14 u^{2} u_{x} u_{x x}+14 u^{4} u_{x x}+ \\
\left.+28 u^{3} u_{x}^{2}-\frac{4}{3} u^{7}\right)+t_{1}\left(u_{x x x x}+5 u_{x} u_{x x}-5 u^{2} u_{x x}-\right.  \tag{D}\\
\left.-5 u u_{x}^{2}+u^{5}\right)-x u-\beta_{2}=0 .
\end{gather*}
$$

Equation (D) is a sixth-order nonlinear ordinary differential equation with properties similar to the Painlevé equations but cannot be transformed to the equation of the second Painlevé hierarchy. This equation does not have a first integral in the polynomial form and it is possible that it determines a new transcendental function.

In the following section we find the Lagrangians for the nonlinear ordinary differential equations (A), (B) and (D).

## 3 Inverse problem for sixth-order equations and their Lagrangians

Consider a sixth-order equation in the normal form, $u_{6}=f\left(x, u, u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right)$. Here we introduce the abridged notation $u_{k}=d^{k} u / d x^{k}$. The following theorem due to Juráš gives the necessary and sufficient conditions for a sixth-order equation to admit a variational multiplier [27].

Theorem. A sixth-order ordinary differential equation admits a variational multiplier and non-degenerate third-order Lagrangian if and only if following two conditions are satisfied

$$
0=-\frac{2}{3} D_{x}^{4}\left(\frac{\partial f}{\partial u_{5}}\right)+\frac{10}{9} \frac{\partial f}{\partial u_{5}} D_{x}^{3}\left(\frac{\partial f}{\partial u_{5}}\right)+D_{x}^{3}\left(\frac{\partial f}{\partial u_{4}}\right)+\frac{20}{9} D_{x}\left(\frac{\partial f}{\partial u_{5}}\right) D_{x}^{2}\left(\frac{\partial f}{\partial u_{5}}\right)
$$

$$
\begin{gathered}
-\frac{20}{27}\left(\frac{\partial f}{\partial u_{5}}\right)^{2} D_{x}^{2}\left(\frac{\partial f}{\partial u_{5}}\right)-\frac{1}{3} \frac{\partial f}{\partial u_{4}} D_{x}^{2}\left(\frac{\partial f}{\partial u_{5}}\right)-\frac{\partial f}{\partial u_{5}} D_{x}^{2}\left(\frac{\partial f}{\partial u_{4}}\right)-D_{x}^{2}\left(\frac{\partial f}{\partial u_{3}}\right) \\
-\frac{10}{9} \frac{\partial f}{\partial u_{5}}\left(D_{x}\left(\frac{\partial f}{\partial u_{5}}\right)\right)^{2}-D_{x}\left(\frac{\partial f}{\partial u_{5}}\right) D_{x}\left(\frac{\partial f}{\partial u_{4}}\right)+\frac{20}{81}\left(\frac{\partial f}{\partial u_{5}}\right)^{3} D_{x}\left(\frac{\partial f}{\partial u_{5}}\right) \\
+\frac{1}{3}\left(\frac{\partial f}{\partial u_{5}}\right)^{2} D_{x}\left(\frac{\partial f}{\partial u_{4}}\right)+\frac{1}{3} \frac{\partial f}{\partial u_{5}} \frac{\partial f}{\partial u_{4}} D_{x}\left(\frac{\partial f}{\partial u_{5}}\right)+\frac{1}{3} \frac{\partial f}{\partial u_{3}} D_{x}\left(\frac{\partial f}{\partial u_{5}}\right)+\frac{2}{3} \frac{\partial f}{\partial u_{5}} D_{x}\left(\frac{\partial f}{\partial u_{3}}\right) \\
+D_{x}\left(\frac{\partial f}{\partial u_{2}}\right)-\frac{2}{243}\left(\frac{\partial f}{\partial u_{5}}\right)^{5}-\frac{1}{27}\left(\frac{\partial f}{\partial u_{5}}\right)^{3} \frac{\partial f}{\partial u_{4}}-\frac{1}{9}\left(\frac{\partial f}{\partial u_{5}}\right)^{2} \frac{\partial f}{\partial u_{3}}-\frac{1}{3} \frac{\partial f}{\partial u_{5}} \frac{\partial f}{\partial u_{2}}-\frac{\partial f}{\partial u_{1}},
\end{gathered}
$$

and

$$
\begin{gathered}
0=\frac{5}{3} D_{x}^{2}\left(\frac{\partial f}{\partial u_{5}}\right)-\frac{5}{3} \frac{\partial f}{\partial u_{5}} D_{x}\left(\frac{\partial f}{\partial u_{5}}\right)-2 D_{x}\left(\frac{\partial f}{\partial u_{4}}\right)+ \\
\frac{5}{27}\left(\frac{\partial f}{\partial u_{5}}\right)^{3}+\frac{2}{3} \frac{\partial f}{\partial u_{5}} \frac{\partial f}{\partial u_{4}}+\frac{\partial f}{\partial u_{3}} .
\end{gathered}
$$

Prove. Suppose the sixth-order equation

$$
u_{6}=f\left(x, u, u_{1}, u_{2}, u_{3}, u_{4}\right)
$$

is independent of $u_{5}$. Then it admits a variational multiplier and a nondegenerate third-order Lagrangian if and only if the following two conditions are satisfied:
$0=D_{x}^{3}\left(\frac{\partial f}{\partial u_{4}}\right)-D_{x}^{2}\left(\frac{\partial f}{\partial u_{3}}\right)+D_{x}\left(\frac{\partial f}{\partial u_{2}}\right)-\frac{\partial f}{\partial u_{1}}$ and $0=-2 D_{x}\left(\frac{\partial f}{\partial u_{4}}\right)+\frac{\partial f}{\partial u_{3}}$.

### 3.1 The Jacobi Last Multiplier and construction of Lagrangians for sixth-order equations

In this section we describe the connection of the Jacobi Last Multiplier with the Lagrangian function for sixth-order ODEs.

Proposition. Let $u_{6}=f\left(x, u, u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right)$ be a sixth-order ordinary differential equation which admits a Lagrangian $L$. Then the function $\mathcal{M}:=$ $\left(\frac{\partial^{2} L}{\partial u_{3}^{2}}\right)^{3}$, is a Jacobi last multiplier, i.e., it satisfies the equation $\frac{d \mathcal{M}}{d x}+\frac{\partial f}{\partial u_{5}} \mathcal{M}=$ 0 , where $u_{5}=u_{x x x x x}$.

Proof : Considering the higher-order Euler operator, E, the EulerLagrange equation of motion for the ODE $u_{6}=f\left(x, u, u_{1}, \ldots, u_{5}\right)$ is given by

$$
\begin{equation*}
E(L)=\frac{\partial L}{\partial u}-D_{x}\left(\frac{\partial L}{\partial u_{1}}\right)+D_{x}^{2}\left(\frac{\partial L}{\partial u_{2}}\right)-D_{x}^{3}\left(\frac{\partial L}{\partial u_{3}}\right)=0, \tag{21}
\end{equation*}
$$

where $L=L\left(x, u, u_{1}, u_{2}, u_{3}\right)$ is a third-order Lagrangian. It is obvious that the partial derivatives of $L$, namely $L_{u}, L_{u_{1}}, \ldots . L_{u_{3}}$ are all functions of $x, u, \ldots, u_{3}$. Upon expanding the Euler-Lagrange equation we find that
$0=E(L)=u_{5} L_{u_{2} u_{3}}-\left[2 u_{5} L_{u_{3} u_{3} x}+u_{5} L_{u_{2} u_{3}}+f\left(x, u, u_{1}, \ldots, u_{5}\right) L_{u_{3} u_{3}}+u_{5} D_{x}\left(L_{u_{3} u_{3}}\right)+\right.$ $\left.+2 u_{4} u_{5} L_{u_{3} u_{3} u_{3}}+2 u_{5} u_{1} L_{u_{3} u_{3} u}+2 u_{2} u_{5} L_{u_{3} u_{3} u_{1}}+2 u_{3} u_{5} L_{u_{3} u_{3} u_{2}}\right]+$ terms independent of $u_{5}$.
Here the subscripts denote partial derivatives with respect to the indicated variables. Since the partial derivative of this equation with respect to $u_{5}$ must also be identically zero, we find that $3 D_{x}\left(L_{u_{3} u_{3}}\right)+\frac{\partial f}{\partial u_{5}}\left(L_{u_{3} u_{3}}\right)=0$.

Let be $M_{(3)}=L_{u_{3} u_{3}}$, then the above equation, $E(L)=0$ is expressible as be $D_{x}\left(\log M_{(3)}^{3}\right)+\frac{\partial f}{\partial u_{5}}=0$, showing thereby that the Jacobi Last multiplier is given by

$$
\mathcal{M}=M_{(3)}^{3} .
$$

Remark: Note that for the fourth-order ODE, $u_{4}=f\left(x, u, \ldots, u_{3}\right)$, admitting a second-order Lagrangian the analog of (3.1) is the following equation:

$$
D_{x}\left(\log M_{(2)}^{2}\right)+\frac{\partial f}{\partial u_{3}}=0,
$$

so that the JLM is $\mathcal{M}=M_{(2)}^{2}$ where $M_{(2)}=L_{u_{2} u_{2}}$. On the other hand for the second-order ODE, $u_{2}=f\left(x, u, u_{1}\right)$, it is the solution of

$$
D_{x}\left(\log M_{(1)}\right)+\frac{\partial f}{\partial u_{1}}=0,
$$

with $\mathcal{M}=M_{(1)}=L_{u_{1} u_{1}}$.
Equation (??) provides us a tool for determining the Lagrangian of a fourth-order equation once a solution of the defining equation for the JLM , $\mathcal{M}$, is obtained from (??). In fact in the event $f$ is independent of $u_{5}$, so that the condition (??) is trivially satisfied one obtains the solution $M_{(3)}=$ constant, which may be set equal to unity, without loss of generality. In such a situation the Jurás conditions are also considerably simplified as evident from the Corollary 1.

### 3.2 Determination of the Lagrangians

We wish to determine a nondegenerate third-order Lagrangian $L=L\left(x, u, u_{1}, u_{2}, u_{3}\right)$ such that $E(L)=0$, where $\frac{\partial^{2} L}{\partial u_{3}{ }^{2}} \neq 0$, where $E$ is the Euler-Lagrange operator $E=\frac{\partial}{\partial u}-D_{x}\left(\frac{\partial}{\partial u_{1}}\right)+D_{x}^{2}\left(\frac{\partial}{\partial u_{2}}\right)-D_{x}^{3}\left(\frac{\partial}{\partial u_{3}}\right)$, and $D_{x}$ denotes the total derivative operator $D_{x}=\frac{\partial}{\partial x}+u_{1} \frac{\partial}{\partial u}+u_{2} \frac{\partial}{\partial u_{1}}+u_{3} \frac{\partial}{\partial u_{2}}$. If there is a thirdorder Lagrangian satisfying the conditions stated in Theorem 3.1, one says
that the ordinary differential equation $u_{6}=f\left(x, u, u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right)$ admits a variational multiplier.

In the new notation equation (A) is given by

$$
\begin{aligned}
& \quad t_{3}\left(u_{6}+14 u u_{4}+28 u_{1} u_{3}+21 u_{2}^{2}+70 u^{2} u_{2}+70 u u_{1}^{2}+35 u^{4}\right)+ \\
& +t_{2}\left(u_{4}+10 u u_{2}+5 u_{1}^{2}+10 u^{3}\right)+t_{1}\left(u_{2}+3 u^{2}\right)=x .
\end{aligned}
$$

Proposition. Equation (A) admits a Lagrangian description with Lagrangian $L=t_{3}\left(-\frac{1}{2} u_{3}^{2}+7 u^{5}-35 u^{2} u_{1}^{2}+7 u u_{2}^{2}\right)+t_{2}\left(\frac{1}{2} u_{2}^{2}-5 u u_{1}^{2}+\frac{5}{2} u^{4}\right)+$ $t_{1}\left(-\frac{1}{2} u_{1}^{2}+u^{3}\right)-x u$, where $u_{k}=u^{(k)}, \quad k=1,2, \ldots$.

Outline of the proof : In order to show this we will adopt the technique used in [32], to derive Lagrangians for a certain class of fourth-order ODEs, namely that of the Jacobi Last Multiplier (JLM). For a sixth-order ODE written in the form

$$
u_{6}=f\left(x, u, u_{1}, \ldots, u_{5}\right)
$$

one can rewrite the equation as a first-order system: $u_{1}=v, v_{1}=w, w_{1}=$ $s, s_{1}=t, t_{1}=r, r_{1}=f(x, u, v, w, s, t, r)$, where the subscript 1 denotes differentiation with respect to the independent variable $x$. Then by definition the JLM, $\mathcal{M}$, for the above system of first-order ODEs is defined as the solution of the following equation $\frac{d \log \mathcal{M}}{d x}+\left(\frac{\partial v}{\partial u}+\frac{\partial w}{\partial v}+\frac{\partial s}{\partial w}+\frac{\partial t}{\partial s}+\frac{\partial r}{\partial t}+\frac{\partial f}{\partial r}\right)=$ 0 . Since in case of eqns (A) and (B) the function $f$ is independent of $u_{5}$ i.e., of $r$ in this notation it follows that $\frac{d \log M_{(3)}}{d x}=0 \Rightarrow M_{(3)}=$ constant . Furthermore since the JLM is connected to the Lagrangian by the following relation $M_{(3)}=\frac{\partial^{2} L}{\partial u_{3}^{2}}$, where $u_{3}=\frac{d^{3} u}{d x^{3}}$, therefore setting the constant in (??) to be $-t_{3}$ we have

$$
\frac{\partial^{2} L}{\partial u_{3}^{2}}=-t_{3} \Rightarrow L=-t_{3} \frac{u_{3}^{2}}{2}+R\left(x, u, u_{1}, u_{2}\right) u_{3}+V\left(x, u, u_{1}, u_{2}\right) .
$$

Consequently the determination of the Lagrangian essentially reduces to finding appropriate functions $R$ and $V$ such that the Euler-Lagrange equation $E(L)=0$, reproduces the desired equation, namely (??). Detailed calculations however show that it is possible to choose $R=0$ so as to simplify the resulting calculations, and therefore the Lagrangian is of the form, $L_{A}=t_{3}\left(-\frac{1}{2} u_{3}^{2}\right)+V\left(x, u, u_{1}, u_{2}\right)$. To deduce the unknown function $V$ we substitute this form of the Lagrangian into (21) and compare the resulting equation with our original sixth-order ODE. One finds that in this case

$$
E\left(L_{A}\right)=\frac{\partial L_{A}}{\partial u}-D_{x}\left(\frac{\partial L_{A}}{\partial u_{1}}\right)+D_{x}^{2}\left(\frac{\partial L_{A}}{\partial u_{2}}\right)-D_{x}^{3}\left(\frac{\partial L_{A}}{\partial u_{3}}\right)=0
$$

gives
$-t_{3} u_{6}=V_{u}-V_{x u_{1}}-u_{1} V_{u u_{1}}-u_{2} V_{u_{1} u_{1}}+u_{4} V_{u_{2} u_{2}}+V_{x x u_{2}}+2 u_{1} V_{x u u_{2}}+2 u_{2} V_{x u_{1} u_{2}}+2 u_{3} V_{x u_{2} u_{2}}$
$+u_{1}^{2} V_{u u u_{2}}+u_{2}^{2} V_{u_{1} u_{1} u_{2}}+u_{3}^{2} V_{u_{2} u_{2} u_{2}}+2 u_{1} u_{2} V_{u u_{1} u_{2}}+2 u_{1} u_{3} V_{u u_{2} u_{2}}+2 u_{2} u_{3} V_{u_{1} u_{2} u_{2}}$.
Inserting the value of $u_{6}$ from the original equation and equating the coefficient of $u_{4}$ we find that

$$
V_{u_{2} u_{2}}=t_{3} 14 u+t_{2}
$$

which leads to the following solution, namely $V=t_{3} 7 u u_{2}^{2}+t_{2} \frac{u_{2}^{2}}{2}+T u_{2}+S$ where $T$ and $S$ are functions of $x, u, u_{1}$. Once again we may set $T=0$ so that $L_{A}=t_{3}\left(-\frac{1}{2} u_{3}^{2}+7 u u_{2}^{2}\right)+t_{2}\left(\frac{1}{2} u_{2}^{2}\right)+S$, and it remains therefore to determine the unknown function $S$. From the remaining terms we find that one must have $S_{u}-S_{x u_{1}}-u_{1} S_{u u_{1}}-u_{2} S_{u_{1} u_{1}}=t_{3}\left(70 u^{2} u_{2}+70 u u_{1}^{2}+35 u^{4}\right)+t_{2}\left(10 u u_{2}+\right.$ $\left.5 u_{1}^{2}+10 u^{3}\right)+t_{1}\left(u_{2}+3 u^{2}\right)-x$. Next equating the coefficients of $u_{2}$ it is seen that $-S=t_{3} 35 u^{2} u_{1}^{2}+t_{2} 5 u u_{1}^{2}+t_{1} \frac{u_{1}^{2}}{2}+K u_{1}+N(x, u)$. Again choosing $K=0$, we have ultimately from (3.2),

$$
-N_{u}=t_{3} 35 u^{4}+t_{2} 10 u^{3}+t_{1} 3 u^{2}-x .
$$

This yields

$$
-N=t_{3} 7 u^{5}+t_{2} \frac{5}{2} u^{4}+t_{1} u^{3}-x u
$$

and finally the following expression for the unknown function $S$

$$
S=-t_{3} 35 u^{2} u_{1}^{2}-t_{2} 5 u u_{1}^{2}+t_{1}\left(-\frac{1}{2} u_{1}^{2}\right)+t_{3} 7 u^{5}+t_{2} \frac{5}{2} u^{4}+t_{1} u^{3}-x u .
$$

We find finally that the expression for the Lagrangian of eqn. (A) is, $L_{A}=t_{3}\left(-\frac{1}{2} u_{3}^{2}+7 u^{5}-35 u^{2} u_{1}^{2}+7 u u_{2}^{2}\right)+t_{2}\left(\frac{1}{2} u_{2}^{2}-5 u u_{1}^{2}+\frac{5}{2} u^{4}\right)+t_{1}\left(-\frac{1}{2} u_{1}^{2}+u^{3}\right)-x u$.

In a similar manner we find that the sixth-order equations (B) and (D) also admit a Lagrangian description, which are stated below.

Proposition. The Lagrangians associated with the equations (B) and (D) are given by

$$
\begin{align*}
& L_{B}=t_{1}\left(-\frac{1}{2} u_{1}^{2}-\frac{1}{2} u^{4}\right)+t_{2}\left(\frac{1}{2} u_{2}^{2}+5 u^{2} u_{1}^{2}+u^{6}\right)+ \\
& +t_{3}\left(-\frac{1}{2} u_{3}^{2}-\frac{5}{2} u^{8}-35 u^{4} u_{1}^{2}-7 u^{2} u_{2}^{2}\right)-\frac{1}{2} x u^{2}-\beta_{3} u,  \tag{22}\\
& L_{D}=t_{2}\left(-\frac{1}{2} u_{3}^{2}+\frac{7}{2} u_{1} u_{2}^{2}-\frac{7}{2} u^{2} u_{2}^{2}+\frac{7}{6} u_{1}^{4}+\frac{7}{3} u^{2} u_{1}^{3}-7 u^{4} u_{1}^{2}-\frac{u^{8}}{6}\right)+ \\
& +t_{1}\left(\frac{1}{2} u_{2}^{2}+\frac{5}{2} u^{2} u_{1}^{2}-\frac{5}{6} u_{1}^{3}+\frac{u^{6}}{6}\right)-\frac{1}{2} x u^{2}-\beta_{2} u . \tag{23}
\end{align*}
$$

## 4 Conclusion

Let us briefly discuss the results of this paper. We have found the Lagrangians $L_{A}, L_{B}$ and $L_{D}$ for the sixth-order nonlinear differential equations from the Painlevé hierarchies. These Lagrangians are generalizations of the well known Lagrangians of the Painlevé equations. It is interesting to look at the properties of the Lagrangians $L_{A}, L_{B}$ and $L_{D}$. For example we know that there exists the following symmetry of equation (B) when $u(x) \rightarrow-u(x)$ and $\beta_{3} \rightarrow-\beta_{3}$. This symmetry is preserved for the Lagrangian $L_{B}$ as well. Equations (A), (B) and (D) posses the Painlevé property. The Lagrangians of several mechanical systems usually involve a difference of their kinetic and potential energies respectively. Here this property holds for the Lagrangians $L_{A}$ and $L_{B}$ at $t_{2}=t_{3}=x=0$. However this is not true in the general case and as such the Lagrangians derived here are really examples of nonstandard ones. The question of irreducibility of the higher Painlevé equations is an open problem.

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