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ABSTRACT

The aim of this comment is to extend the study of the dynamics of a finite extensibility nonlinear oscillator conducted by Febbo [M. Febbo, A finite extensibility nonlinear oscillator, Applied Mathematics and Computation 217 (2011) 6464-6475]. We show that the linearized harmonic balance method is not sufficiently adequate for this oscillator and that the harmonic balance method without linearization provides better results. We also discuss what happens when the oscillation amplitude approaches 1 and why the harmonic balance method does not give optimum results. For these values of the oscillation amplitude the periodic solution becomes markedly anharmonic and is almost straight between the turning points. Finally, a 'heuristic' solution is proposed which is adequate for the whole amplitude range $0 < A < 1$.

KEYWORDS: Finite extensibility; Nonlinear oscillator; Approximate solutions; Harmonic balance method.

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1. Introduction

In a recent paper [1], Febbo studied analytically the dynamics of a finite extensibility nonlinear oscillator (FENO) using two different approaches. One involved a linearized harmonic balance (LHB) procedure, which allowed him to obtain analytical approximations to the frequency of oscillations and periodic solution. In Febbo’s paper the approximate period obtained using a LHB method is compared with the exact one (numerically integrated) and very good agreement is obtained for amplitudes (A) between 0 and 0.9 with a relative error of less than 3.53%. However, for the rest of the amplitude range ($0.9 < A < 1$), the relative error for the approximate period increases exponentially and the author mentioned that higher order perturbation solutions are needed in such cases.

In this note, we would first like to take the opportunity to congratulate the author of Ref. [1] for his interesting, comprehensive study of a finite extensibility nonlinear oscillator. We will then add some interesting results about application of the harmonic balance method to a finite extensibility nonlinear oscillator –whether or not it is better to use a linearized version of this procedure– which are not included in Ref. [1] and compare the approximate and exact periodic solutions. Our results may provide information about why approximate methods fail when the oscillation amplitude approaches 1. In particular, we show that linearized harmonic balance procedures, even though they simplify the harmonic balancing, do not always provide optimal results. For instance, if the second order harmonic balance method without linearization is applied to a finite extensibility nonlinear oscillator, the relative error is as low as 0.60% for A = 0.9, whereas this error is 3.53% when a linearized harmonic balance method is used. This means that great care must be exercised when linearized harmonic balance procedures are applied to nonlinear oscillators with very strong nonlinearities. Finally, we present some comments on the behaviour of the periodic solutions for amplitudes approaching 1, which may not only enable us to understand this type of nonlinear oscillator better, but also provide the basis for a more detailed study of the dynamics of a finite extensibility nonlinear oscillator and an extension of Febbo’s paper.

2. Application of the harmonic balance method

The non-dimensional equation of motion governing a finite extensibility nonlinear oscillator is [1]

$$\frac{d^2x}{dt^2} + \frac{x}{1-x^2} = 0 \tag{1}$$

with initial conditions

$$x(0) = A \quad (\text{with } 0 < A < 1) \qquad \text{and} \qquad \frac{dx}{dt}(0) = 0 \tag{2}$$

where A is the oscillation amplitude.

The harmonic balance (HB) method provides a technique for calculating analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series [2-4]. As we mentioned in the introduction, the HB method without linearization will be applied to a finite extensibility nonlinear oscillator.

Before applying the HB method to Eq. (1), this equation is rewritten in a form that does not contain the fractional expression

$$(1-x^2)\frac{d^2x}{dt^2} + x = 0 \tag{3}$$

Introducing a new independent variable $\tau = \omega t$, where ω is the frequency of the oscillations, Eqs. (2) and (3) can be rewritten as

$$\omega^2(1-x^2)\frac{d^2x}{d\tau^2} + x = 0, \quad x(0) = A \ \ (0 < A < 1), \quad \frac{dx}{d\tau}(0) = 0 \tag{4}$$

The new independent variable τ is chosen in such a way that the solution of Eq. (4) is a periodic function of τ of period 2π [3] Applying the lowest harmonic balance method it is easy to obtain the following first-order analytical approximate frequency (Eq. (32) in Febbo’s paper [1])

$$\omega_1(A) = \frac{2}{\sqrt{4-3A^2}} \tag{5}$$

and the corresponding approximate periodic solution is $x_1(t) = A\cos\omega_1 t$. The second-order approximate solution to Eq. (4) can be expressed as

$$x_2(t) = A\cos\tau + c_1(\cos 3\tau - \cos\tau) \tag{6}$$

which satisfies the initial conditions in Eq. (4) and where c_1 depends on the initial amplitude A. In Febbo's paper [1], Eq. (6) is substituted in Eq. (4) and higher-order corrections in c_1 are discarded. This is the linearized harmonic balance (LHB) method that Febbo mentioned. This approximation is usually sufficient due to the low values of c_1. However, as we can see in this note, when A approaches 1, the higher-order corrections in c_1 are important and can not be discarded. Then, the LHB method does not give accurate approximations to the frequency and the periodic solution when A increases (for A = 0.9, Febbo obtained a relative error of 3.53% for the approximate period using the LHB method, see Table 1 in [1]).

Substituting Eq. (6) into Eq. (4), expanding the expression in a trigonometric series, simplifying and setting the coefficient of the resulting items $\cos\tau$ and $\cos 3\tau$ equal to zero yields

$$(4-3A^2-5c_1A-30c_1^2)\omega^2-4=0 \tag{7}$$

and

$$[A^3-(36-19A^2)c_1-41c_1^2A+48c_1^3]\omega^2+4c_1=0 \tag{8}$$

From Eq. (7) the second approximate frequency ω_2 can be obtained as follows

$$\omega_2(A) = \frac{2}{\sqrt{4-3A^2-5Ac_1(A)-30c_1^2(A)}} \tag{9}$$

Substituting Eq. (9) into Eq. (7) and simplifying, the following cubic equation is obtained

$$c_1^3 - \frac{23}{9}Ac_1^2 - \frac{8}{9}(2-A^2)c_1 + \frac{1}{18}A^3 = 0 \tag{10}$$

which allows us to obtain c_1. In Figure 1 we plot the discriminant Δ of the cubic equation (10) as a function of A. For $0 < A < 0.421673$ this discriminant is $\Delta > 0$ and there will be one real root and two complex conjugate roots; for $A = 0.421673$ the discriminant is $\Delta = 0$ and there will three real roots, of which at least two are equal; and for $0.421673 < A < 1$ the discriminant of the equation is $\Delta < 0$ and there will be three unequal real roots. Instead of solving Eq. (10) for these intervals of A to find the corresponding values of c_1, it would be better to have only one expression for c_1 as a function of A. In order to obtain this expression, we proceed as follows. A first approximate value of c_{10} can be determined by disregarding the first two terms in this equation

$$-\frac{8}{9}(2-A^2)c_{10} + \frac{1}{18}A^3 = 0 \tag{11}$$

Solving this equation we obtain

$$c_{10} = \frac{A^3}{16(2-A^2)} \tag{12}$$

To further improve this result, we assume that c_1 can be written as follows

$$c_1 = c_{10} + \delta \tag{13}$$

where δ is a correction term and $|\delta| << |c_{10}|$. Substituting Eq. (13) into Eq. (10) and linearizing with respect to the correction term δ gives

$$c_{10}^3\left(1+\frac{3\delta}{c_{10}}\right) - \frac{23}{9}Ac_{10}^2\left(1+\frac{2\delta}{c_{10}}\right) - \frac{8}{9}(2-A^2)c_{10}\left(1+\frac{\delta}{c_{10}}\right) + \frac{1}{18}A^3 = 0 \tag{14}$$

which is a linear instead of a cubic equation. Solving Eq. (14) and taking into account Eq. (12) and (13) we obtain

$$c_1(A) = \frac{8192A^3 - 12288A^5 + 6512A^7 - 1217A^9}{8(32768 - 65536A^2 + 52096A^4 - 19383A^6 + 2811A^8)} \tag{15}$$

For example, for A = 0.9 we obtain c_1 = 0.0355911 using Eq. (15) and c_1 = 0.0355781 solving numerically the exact equation in (10). As we can see, for A = 0.9 the percentage error is less than 0.037% when Eq. (15) is used. In Figure 2 we plot c_1 as a function of the oscillation amplitude A. As can be seen, c_1 tends to 1199/22056 = 0.0543616 when A approaches 1.

For A = 0.9 and using Eqs. (9) and (15) we obtain for the period $T_2 = 2\pi/\omega_2$ = 3.67972 while the exact value is T_{ex} = 3.65767, which means that the relative error for the approximate period is −0.60%. As can be seen in Table 1 in Ref. [1], the relative error is −3.53% for A = 0.9 when the LHB method is used, almost six times more than the relative error obtained using the HB method without linearization (Eqs. (9) and (15)). Now, to reach the same relative error (−3.53%) it is necessary to consider A = 0.954 in Eqs. (9) and (15). In order to produce a global estimator of the accuracy of the solution, the L_2 norm over one period [1] has been obtained to compare the approximate solution in Eq. (6) and we have obtained that increases from 0.03 to 0.13 when A increases from o.9 to 0.954. In Figure 3 we plot the relative errors for the approximate frequencies ω_2 (obtained in this paper, Eqs. (9)) and ω_{Lhb2} (obtained using the linearized HBM, Eq. (33) in Ref. [1]) for $0 < A \leq 0.9$. As can be seen in this figure, for this range of values of A the LHB method provides poorer results than the HB method without any linearization.

The exact periodic solutions achieved by numerically integrating Eq. (1), and the proposed normalized second-order approximate periodic solutions in Eq. (6), for one complete cycle are plotted in Figures 4, 5 and 6 for A = 0.4, 0.8 and 0.9, respectively. In these figures parameter h is defined as follows

$$h = 2\pi T_{ex} t \tag{16}$$

These figures show that the HB method provides a good approximation to the exact periodic solution and is adequate for obtaining the approximate analytical expression of $x(t)$ for $0 < A \leq 0.954$

3. What happen when A approaches 1?

To better understand what happens when the oscillation amplitude, A, approaches 1, it is necessary to plot the exact periodic solution (achieved numerically) for one complete cycle. Figure 7 shows this exact solution for $A = 0.999$. As we can see in Figures 4, 5 and 6, the periodic solution is very close to the cosine function (or it can be approximate with only two harmonics, Eq. (6)) for $0 < A < 0.9$. However, when A approaches 1, the curvature becomes more concentrated at the turning points ($x = \pm A$). For these values of A, $x(t)$ becomes markedly anharmonic and is almost straight between the turning points. As the velocity is the derivative of the displacement with respect to time, we can conclude that the velocity is practically constant between the turning points, and its value is given by the slope of the straight lines between $x = +A$ and $-A$, and between $x = -A$ and $+A$. Figure 8 shows the exact velocity for $A = 0.999$. Only in the vicinity of the turning points, where the magnitude of the restoring force is maximum and the velocity becomes zero, is the force effective in changing the velocity. We investigated what type of functions could verify the behaviour of the periodic solutions of Eq. (1) when A approaches 1 and found that the following solution satisfies this behaviour not only when A approaches 1, but also over the whole range $0 < A < 1$,

$$x(t) = A\frac{\sin^{-1}(r\cos\omega t)}{\sin^{-1} r} \tag{17}$$

where r is a parameter which depends on A and which tends to 0 when A tends to 0. In Figures 9, 10 and 11 we plot the exact solutions and the approximate solutions obtained using Eq. (17) for $A = 0.99$, 0.999 and 0.9999, respectively, for which the values of parameter r used were $r = 0.985$, 0.9968 and 0.99895, respectively. These values were obtained by means of a least squares adjustment of Eq (17) to the exact numerical values. Obviously, r has to function a function of A. We have proposed the solution in

Eq. (17) because the behaviour we can see in Figures 9-11 is similar to that we can see for the relativistic nonlinear oscillator [5, 6], and for this oscillator an analytical approximate solution is similar to this equation. For small values of A, parameter r will also be small and it is possible to do the Taylor series expansion of Eq. (17) giving

$$x(t) \approx \left(A - \frac{A}{24} r^2 \right) \cos \omega t + \frac{A}{24} r^2 \cos 3\omega t \tag{18}$$

which coincides with Eq. (6) and for small values of A we can write

$$c_1 \approx \frac{A}{24} r^2 \quad \text{and} \quad r \approx \sqrt{\frac{24 c_1}{A}} \tag{19}$$

Obviously, Eq. (17) is a 'heuristic' solution to Eq. (1) and more studies are necessary to better understand the behaviour of a finite extensibility nonlinear oscillator for oscillation amplitudes approaching 1.

4. Conclusions

In summary, this paper shows that the LHB method is not adequate for applying to a finite extensibility nonlinear oscillator and that the HB method provides better results for the approximate frequency and periodic solutions, reducing the relative error from 3.53% to 0.60% for A = 0.9. We discussed the reason why the accuracy of the approximate solutions obtained using the harmonic balance method is not good when A approaches 1. This is due to the fact that for these values of A the solution $x(t)$ becomes markedly anharmonic and is almost straight between the turning points. The behaviour of these periodic solutions was also investigated when the oscillation amplitude approaches 1 and a 'heuristic' solution was proposed. As Febbo pointed out at the end of his paper, a more detailed study is necessary to better understand the behaviour of this oscillator when A approaches 1 and to obtain analytical approximate solutions for these values of the oscillation amplitude.

Acknowledgements

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FIGURE CAPTIONS

Figure 1.- The discriminant Δ of the cubic equation (10) as a function of de oscillation amplitude.

Figure 2.- c_1 (Eq. (15)) as a function of the oscillation amplitude A

Figure 3.- Relative errors for the approximate frequencies ω_2 (obtained in this paper, Eqs. (9), continuous line) and ω_{Lhb2} (obtained using the linearized HBM, Eq. (33) in Ref. [1], dashed line).

Figure 4.- Comparison of the normalized approximate solution in Eq. (6) (△ and dashed line) with the exact solution (○ and continuous line) for A = 0.4.

Figure 5.- Comparison of the normalized approximate solution in Eq. (6) (△ and dashed line) with the exact solution (○ and continuous line) for A = 0.8.

Figure 6.- Comparison of the normalized approximate solution in Eq. (6) (△ and dashed line) with the exact solution (○ and continuous line) for A = 0.9.

Figure 7.- Normalized exact periodic solution obtained by numerically integrating Eq. (1) for A = 0.999.

Figure 8.- Normalized exact velocity obtained using from Eq. (1) for A = 0.999.

Figure 9.- Comparison of the normalized approximate solution in Eq. (17) with r = 0.985 (△ and dashed line) with the exact solution (○ and continuous line) for A = 0.99.

Figure 10.- Comparison of the normalized approximate solution in Eq. (17) with r = 0.99968 (△ and dashed line) with the exact solution (○ and continuous line) for A = 0.999.

Figure 11.- Comparison of the normalized approximate solution in Eq. (17) with r = 0.99895 (△ and dashed line) with the exact solution (○ and continuous line) for A = 0.9999.

FIGURE 1

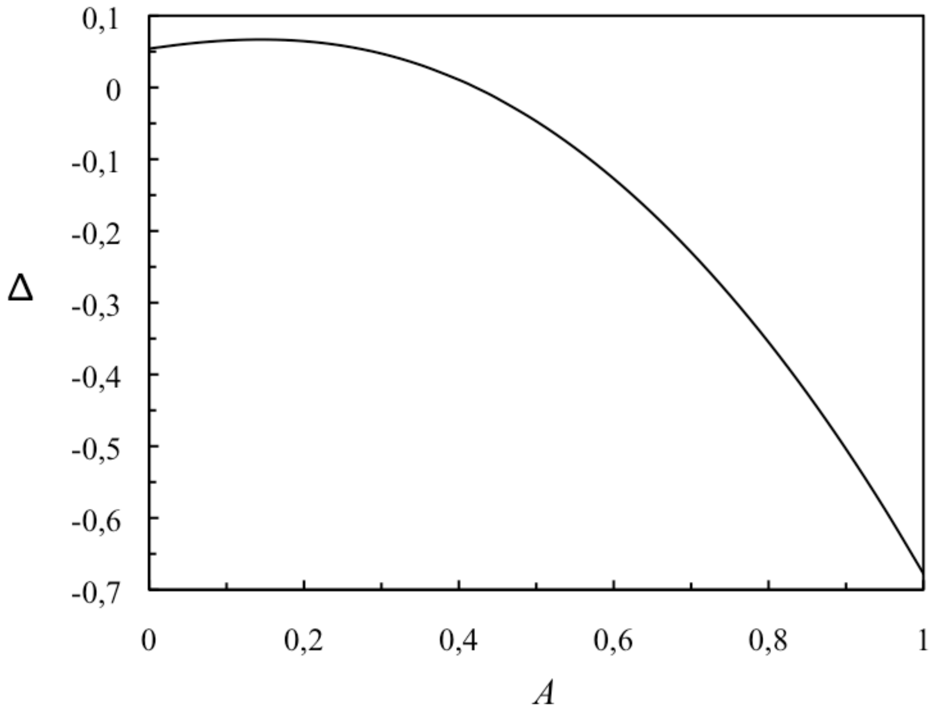


FIGURE 2

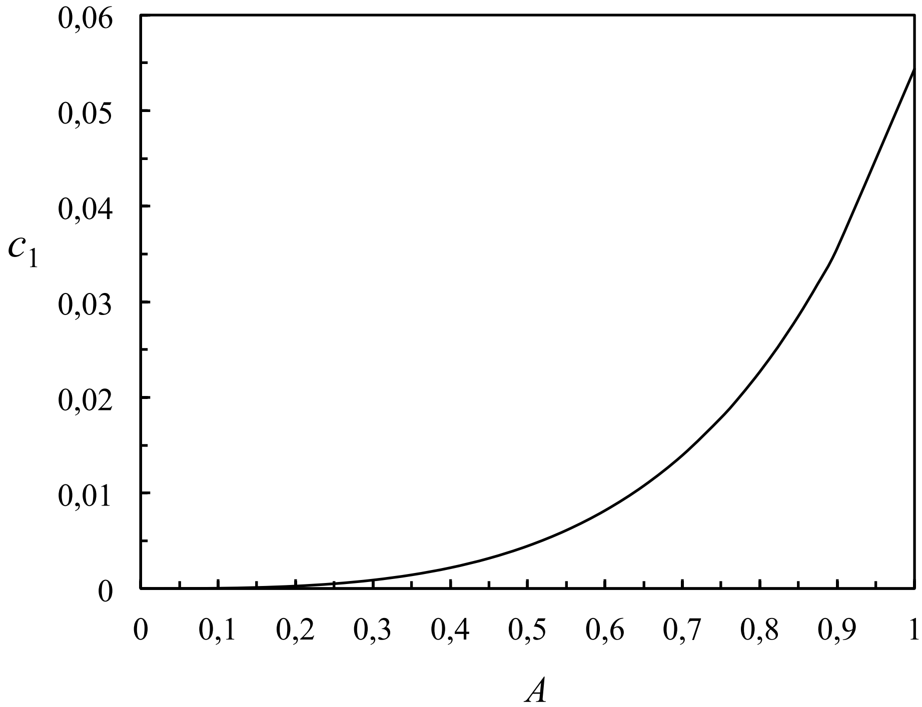
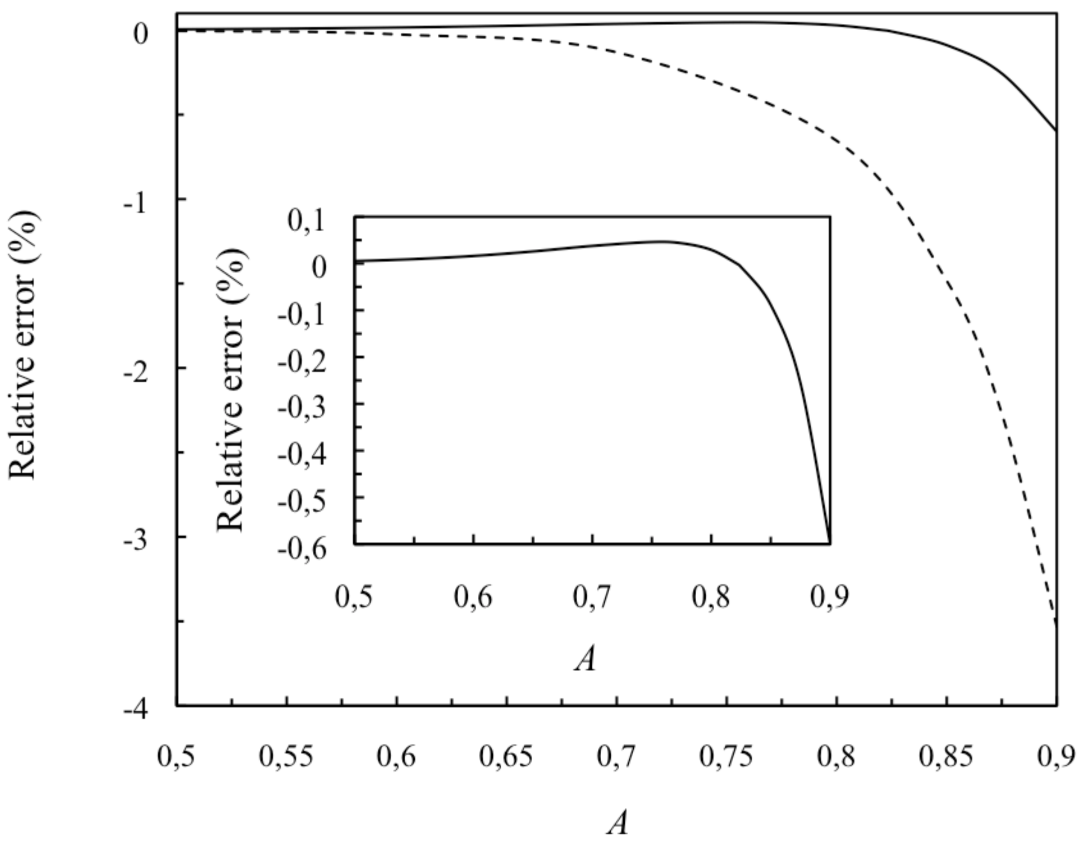


FIGURE 3



0
-1
-2
-3
-4
Relative error (%)
0,5
0,55
0,6
0,65
0,7
0,75
0,8
0,85
0,9
A
0,1
0
-0,1
-0,2
-0,3
-0,4
-0,5
-0,6
Relative error (%)
0,5
0,6
0,7
0,8
0,9
A

FIGURE 4

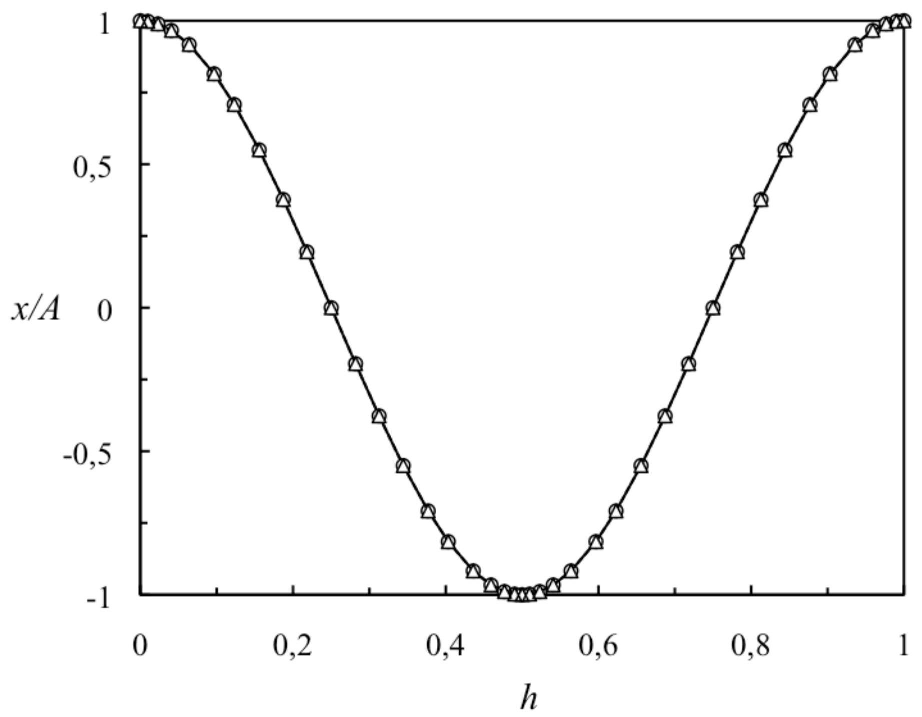


FIGURE 5

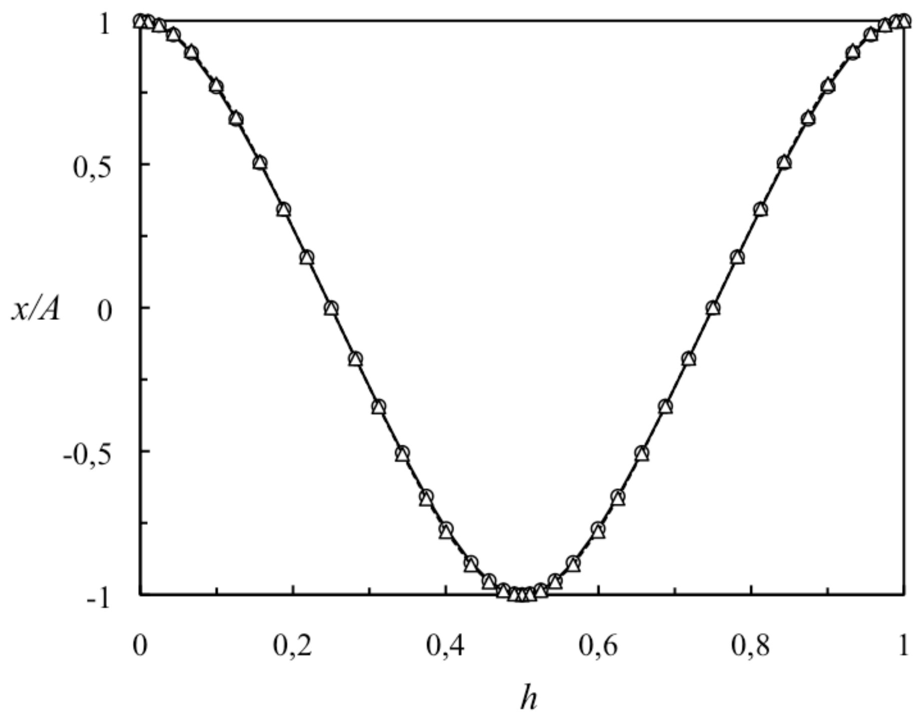


FIGURE 6

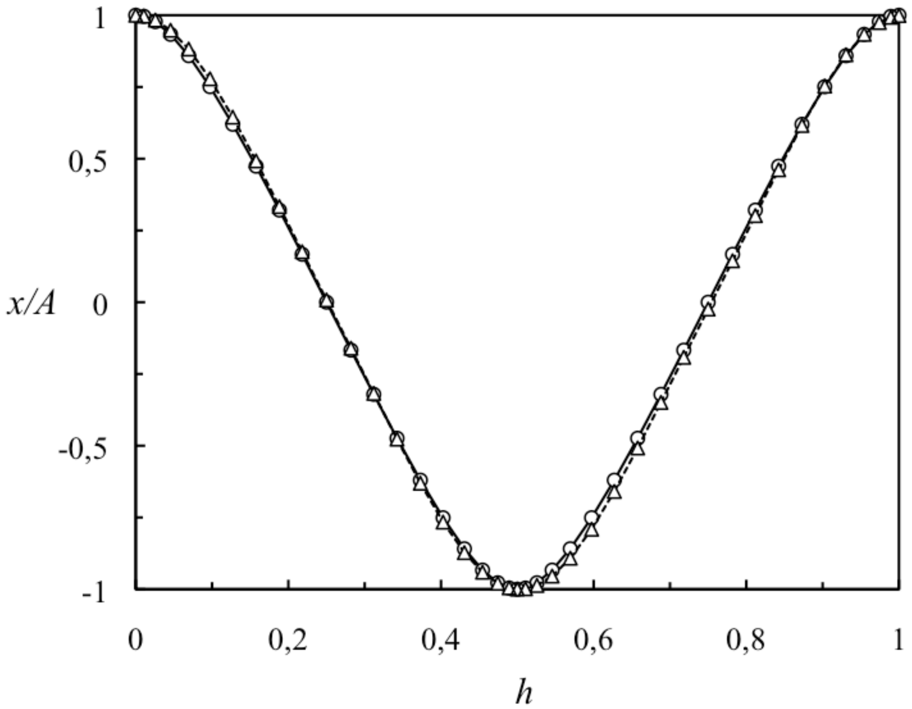


FIGURE 7

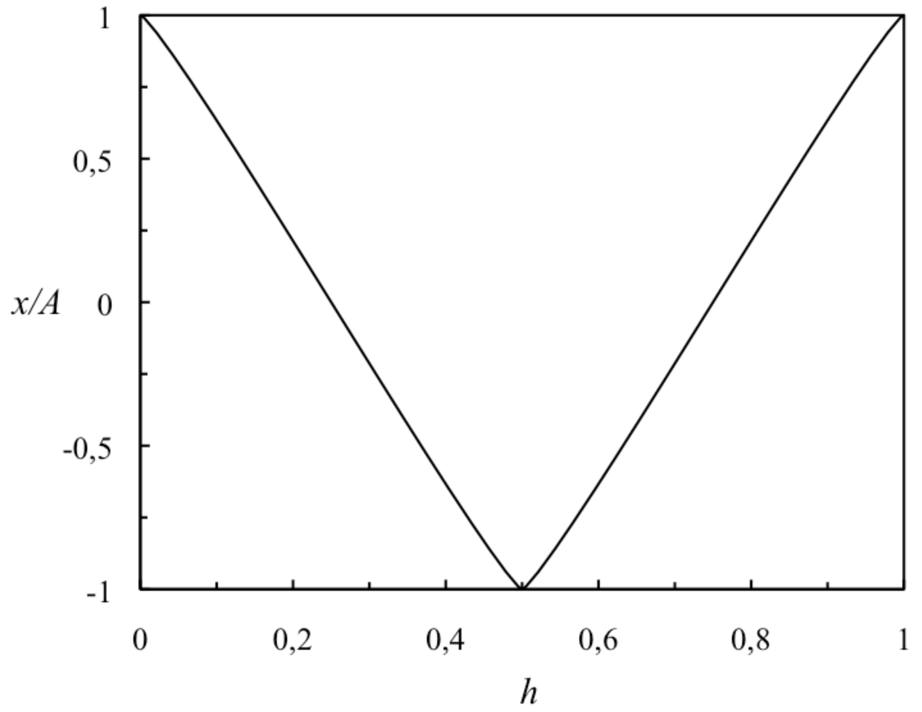


FIGURE 8

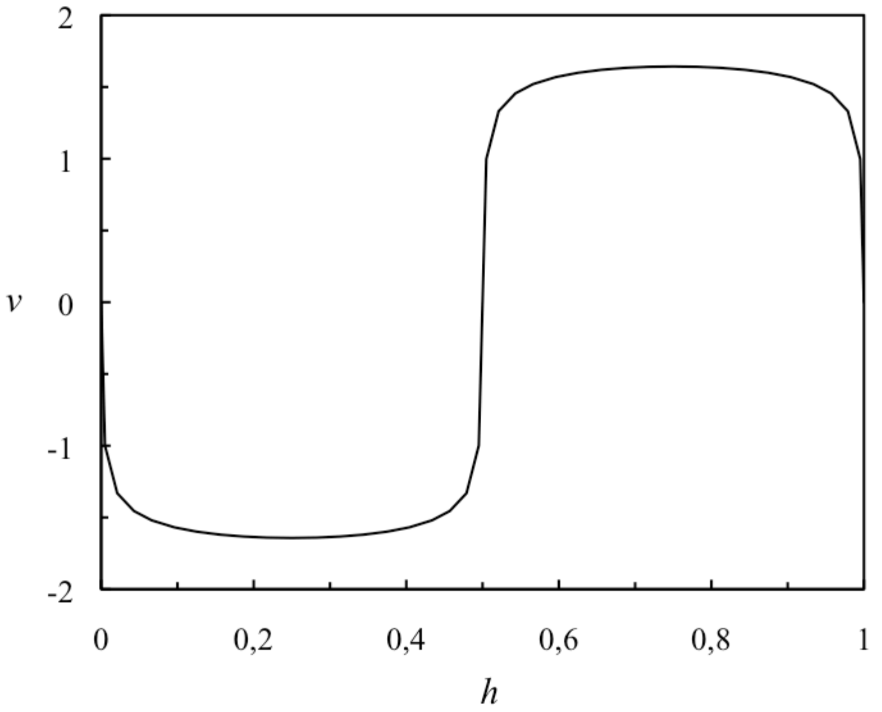


FIGURE 9

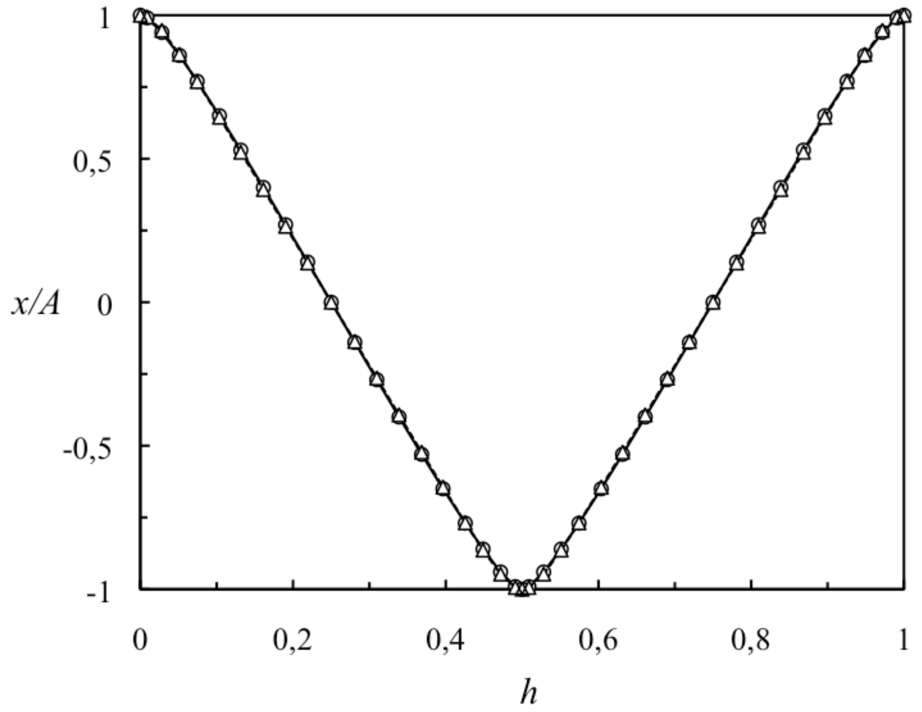


FIGURE 10

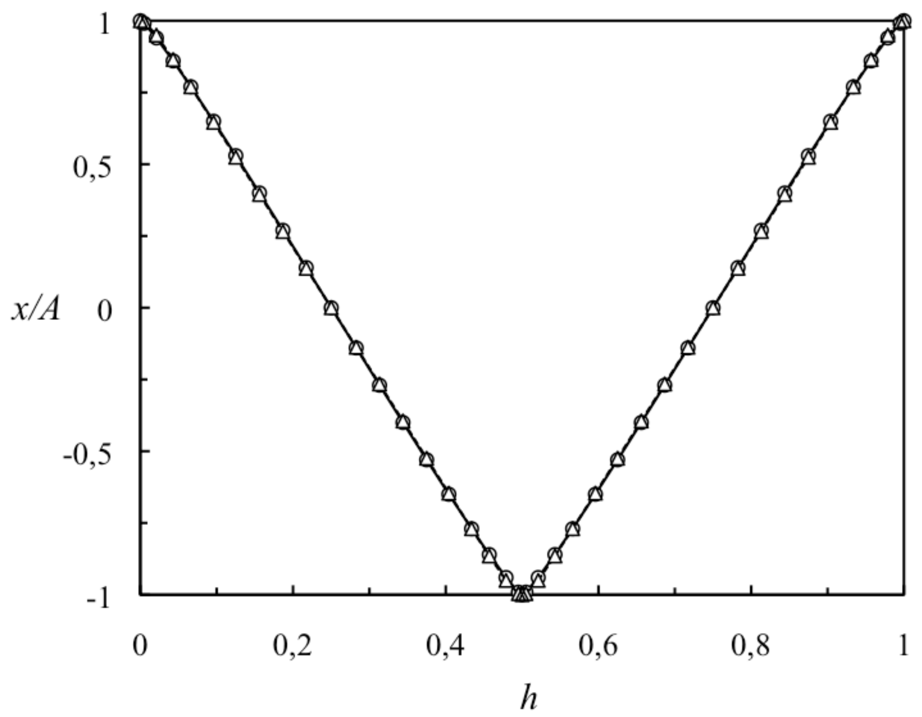


FIGURE 11