Radii of Starlikeness Associated with the Lemniscate of Bernoulli and the Left-Half Plane

Rosihan M. Ali, Naveen Jain, and V. Ravichandran

Abstract. A normalized analytic function f defined on the open unit disk in the complex plane is in the class $\mathcal{SL}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2-1|<1$. In the present investigation, the $\mathcal{SL}$-radii for certain well-known classes of functions are obtained. Radius problems associated with the left-half plane are also investigated for these classes.

1. Introduction

Let $\mathcal{A}_n$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ of the form $f(z) = z + \sum_{k=n+1} a_k z^k$, and let $\mathcal{A} := \mathcal{A}_1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{SL}$ be the class of functions defined by

$$\mathcal{SL} := \left\{ f \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\} \quad (z \in \mathbb{D}).$$

Thus a function $f \in \mathcal{SL}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2-1|<1$. For two functions f and g analytic in $\mathbb{D}$, the function f is said to be *subordinate* to g, written $f(z) \prec g(z) \quad (z \in \mathbb{D})$, if there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. In terms of subordination, the class $\mathcal{SL}$ consists of normalized analytic functions f satisfying $zf'(z)/f(z) \prec \sqrt{1+z}$. This class $\mathcal{SL}$ was introduced by Sokół and Stankiewicz [**20**]. Paprocki and Sokół[**10**] discussed a more general class $\mathcal{S}^*(a,b)$ consisting of normalized analytic functions f satisfying $|[zf'(z)/f(z)]^a - b| < b$, $b \geq \frac{1}{2}$, $a \geq 1$.

Recall that a function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is starlike with respect to 0. Similarly, a function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex. Analytically, a function $f \in \mathcal{A}$ is starlike or convex if the following respective subordinations hold:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}.$$

2010 *Mathematics Subject Classification.* Primary 30C80; Secondary 30C45.

Key words and phrases. Starlike functions, radius of starlikeness, lemniscate of Bernoulli.

The work presented here was supported in part by grants from Universiti Sains Malaysia and University of Delhi.

Ma and Minda [6] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $(1+z)/(1-z)$ by a more general function φ. They considered analytic functions φ with positive real part that map the unit disk $\mathbb{D}$ onto regions starlike with respect to 1, symmetric with respect to the real axis and normalized by $\varphi(0) = 1$. They introduced the following classes that include several well-known classes as special cases:

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} \,\middle|\, \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} \,\middle|\, 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

For $0 \leq \alpha < 1$,

$$\mathcal{ST}(\alpha) := \mathcal{ST}((1 + (1-2\alpha)z)/(1-z)), \quad \mathcal{CV}(\alpha) := \mathcal{CV}((1 + (1-2\alpha)z)/(1-z))$$

are the subclasses of $\mathcal{S}$ consisting of starlike and convex functions of order α in $\mathbb{D}$ respectively. Then $\mathcal{ST} := \mathcal{ST}(0)$, $\mathcal{CV} := \mathcal{CV}(0)$ are the well-known classes of starlike and convex functions respectively. Also let

$$\mathcal{ST}_n(\alpha) := \mathcal{A}_n \cap \mathcal{ST}(\alpha), \quad \mathcal{CV}_n(\alpha) := \mathcal{A}_n \cap \mathcal{CV}(\alpha), \quad \mathcal{SL}_n := \mathcal{A}_n \cap \mathcal{SL}.$$

Since $\mathcal{SL} = \mathcal{ST}(\sqrt{1+z})$, distortion, growth, and rotation results for the class $\mathcal{SL}$ can conveniently be obtained by applying the corresponding results in [6].

The radius of a property P in a set of functions $\mathcal{M}$, denoted by $R_P(\mathcal{M})$, is the largest number R such that every function in the set $\mathcal{M}$ has the property P in each disk $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$ for every $r < R$. For example, the radius of convexity in the class $\mathcal{S}$ is $2 - \sqrt{3}$. Sokół and Stankiewicz [20] determined the radius of convexity for functions in the class $\mathcal{SL}$. They have also obtained structural formula, growth and distortion theorems for these functions. Estimates for the first few coefficients of functions in this class can be found in [21]. Recently, Sokół [22] determined various radii for functions belonging to the class $\mathcal{SL}$; these include the radii of convexity, starlikeness and strong starlikeness of order α. In contrast, in our present investigation, we compute the $\mathcal{SL}$-radius for functions belonging to several interesting classes. Unlike the radii problems associated with starlikeness and convexity, where a central feature is the estimates for the real part of the expressions $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ respectively, the $\mathcal{SL}$-radius problems for classes of functions are tackled by first finding the disk that contains the values of $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$. This technical result will be presented in the next section.

Another interesting class is $\mathcal{M}(\beta)$, $\beta < 1$, defined by

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad z \in \Delta \right\}.$$

The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi *et al.* [23], while its subclass was investigated by Owa and Srivastava [9]. We let $\mathcal{M}_n(\beta) := \mathcal{A}_n \cap \mathcal{M}(\beta)$. In the present paper, radius problems related to $\mathcal{M}(\beta)$ will also be investigated. Related radius problem for this class can be found in [1] and [11]. The following definitions and results will be required.

An analytic function $p(z) = 1 + c_n z^n + \cdots$ is a function with positive real part if $\operatorname{Re} p(z) > 0$. The class of all such functions is denoted by $\mathcal{P}_n$. We also denote the subclass of $\mathcal{P}_n$ satisfying $\operatorname{Re} p(z) > \alpha$, $0 \leq \alpha < 1$, by $\mathcal{P}_n(\alpha)$. More generally, for $-1 \leq B < A \leq 1$, the class $\mathcal{P}_n[A, B]$ consists of functions p of the form $p(z) = 1 + c_n z^n + \cdots$ satisfying

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Lemma 1.1. [**7**] *If $p \in \mathcal{P}_n$, then*

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}} \quad (|z| = r < 1).$$

Lemma 1.2. [**12**] *If $p \in P_n[A, B]$, then*

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leq \frac{(A - B)r^n}{1 - B^2 r^{2n}} \quad (|z| = r < 1).$$

In particular, if $p \in P_n(\alpha)$, then

$$\left| p(z) - \frac{1 + (1 - 2\alpha)r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(1 - \alpha)r^n}{1 - r^{2n}} \quad (|z| = r < 1).$$

2. The $\mathcal{SL}_n$-Radius Problems

In this section, three special classes of functions will be considered. First is the class

$$\mathcal{S}_n := \left\{ f \in \mathcal{A}_n : \frac{f(z)}{z} \in \mathcal{P}_n \right\}.$$

For this class, we shall find its $\mathcal{SL}_n$-radius, denoted by $R_{\mathcal{SL}_n}(\mathcal{S}_n)$.

Theorem 2.1. *The $\mathcal{SL}_n$-radius for the class $\mathcal{S}_n$ is*

$$R_{\mathcal{SL}_n}(\mathcal{S}_n) = \left\{ \frac{\sqrt{2} - 1}{n + \sqrt{n^2 + (\sqrt{2} - 1)^2}} \right\}^{1/n}.$$

This radius is sharp.

Proof. Define the function h by

$$h(z) = \frac{f(z)}{z}.$$

Then the function $h \in \mathcal{P}_n$ and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zh'(z)}{h(z)}.$$

Applying Lemma 1.1 to the function h yields

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{2nr^n}{1 - r^{2n}}.$$

Notice that if $|w-1| < \sqrt{2}-1$, then $|w+1| \leq \sqrt{2}+1$ and hence $|w^2-1| \leq 1$. Thus the above disk lies inside the lemniscate $|w^2-1| < 1$ if

$$\frac{2nr^n}{1-r^{2n}} \leq \sqrt{2}-1.$$

Solving this inequality for r yields the desired $\mathcal{SL}_n$-radius for the class $\mathcal{S}_n$.

Now consider the function f defined by

$$f(z) = \frac{z+z^{n+1}}{1-z^n}.$$

Clearly the function f satisfies the hypothesis of the theorem and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2nz^n}{1-z^{2n}}.$$

At $z = R$ where R is the $\mathcal{SL}_n$-radius for the class $\mathcal{S}_n$ given in the theorem, routine computations show that

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| = 1.$$

This proves that the result is sharp. ∎

The following technical lemma will be useful in our subsequent investigations.

Lemma 2.2. *For $0 < a < \sqrt{2}$, let r_a be given by*

$$r_a = \begin{cases} \left(\sqrt{1-a^2} - (1-a^2)\right)^{1/2} & (0 < a \leq 2\sqrt{2}/3) \\ \sqrt{2}-a & (2\sqrt{2}/3 \leq a < \sqrt{2}), \end{cases}$$

and for $a > 0$, let R_a be given by

$$R_a = \begin{cases} \sqrt{2}-a & (0 < a \leq 1/\sqrt{2}) \\ a & (1/\sqrt{2} \leq a). \end{cases}$$

Then

$$\{w : |w-a| < r_a\} \subseteq \{w : |w^2-1| < 1\} \subseteq \{w : |w-a| < R_a\}.$$

Proof. The equation of the lemniscate of Bernoulli is

$$(x^2+y^2)^2 - 2(x^2-y^2) = 0$$

and the parametric equations of its right-half is given by

$$x(t) = \frac{\sqrt{2}\cos t}{1+\sin^2 t}, \qquad y(t) = \frac{\sqrt{2}\sin t \cos t}{1+\sin^2 t}, \quad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right).$$

The square of the distance from the point $(a, 0)$ to the points on the lemniscate is given by

$$\begin{aligned} z(t) &= (a-x(t))^2 + (y(t))^2 \\ &= a^2 + \frac{2(\cos^2 t - \sqrt{2}a\cos t)}{1+\sin^2 t}, \end{aligned}$$

and its derivative is

$$z'(t) = 2\frac{(-4\cos t + \sqrt{2}a(2+\cos^2 t))\sin t}{(1+\sin^2 t)^2}.$$

Clearly $z'(t) = 0$ if and only if

$$t = 0 \quad \text{or} \quad \cos t = \frac{\sqrt{2}(1 \pm \sqrt{1-a^2})}{a}.$$

Note that for $a > 1$, the numbers $\sqrt{2}(1 \pm \sqrt{1-a^2})/a$ are complex and for $0 < a \le 1$, the number $\sqrt{2}(1+\sqrt{1-a^2})/a > 1$. For $0 < a < 1$, the number $\sqrt{2}(1-\sqrt{1-a^2})/a$ lies between -1 and 1 if and only if $0 < a \le 2\sqrt{2}/3$.

Let us first assume that $0 < a \le 2\sqrt{2}/3$ and $t = t_0$ be given by

$$\cos t_0 = \frac{\sqrt{2}(1-\sqrt{1-a^2})}{a}.$$

Since

$$\min\{z(\pi/2), z(-\pi/2), z(0), z(t_0)\} = z(t_0),$$

it follows that $\min\sqrt{z(t)} = \sqrt{z(t_0)}$. A calculation shows that

$$z(t_0) = \sqrt{1-a^2} - (1-a^2).$$

Hence

$$r_a = \min\sqrt{z(t)} = \sqrt{\sqrt{1-a^2} - (1-a^2)}.$$

Let us next assume that $2\sqrt{2}/3 \le a < \sqrt{2}$. In this case,

$$\min\{z(\pi/2), z(-\pi/2), z(0)\} = z(0),$$

and thus $z(t)$ attains its minimum value at $t = 0$ and

$$r_a = \min\sqrt{z(t)} = \sqrt{2} - a.$$

Now consider $0 < a \le 1/\sqrt{2}$ and $t = t_0$ be given by

$$\cos t_0 = \frac{\sqrt{2}(1-\sqrt{1-a^2})}{a}.$$

It is easy to see that

$$\max\{z(\pi/2), z(-\pi/2), z(0), z(t_0)\} = z(0),$$

and thus

$$R_a = \max\sqrt{z(t)} = \sqrt{2} - a.$$

Similarly, for $a \ge 1/\sqrt{2}$,

$$\max\{z(\pi/2), z(-\pi/2), z(0)\} = z(\pi/2),$$

and hence

$$R_a = \max\sqrt{z(t)} = a. \qquad \blacksquare$$

Now consider the subclass $\mathcal{CS}_n(\alpha)$ consisting of close-to-starlike functions of type α defined by

$$\mathcal{CS}_n(\alpha) := \left\{ f \in \mathcal{A}_n : \frac{f}{g} \in \mathcal{P}_n, \quad g \in \mathcal{ST}_n(\alpha) \right\}.$$

The $\mathcal{SL}_n$-radius for this class is given in the following theorem.

Theorem 2.3. *The $\mathcal{SL}_n$-radius for the class $\mathcal{CS}_n(\alpha)$ is given by*

$$R_{\mathcal{SL}_n}(\mathcal{CS}_n(\alpha)) = \left(\frac{\sqrt{2}-1}{(1+n-\alpha) + \sqrt{(1+n-\alpha)^2 + (1-2\alpha+\sqrt{2})(\sqrt{2}-1)}} \right)^{1/n}$$

This radius is sharp.

Proof. Let g be a starlike function of order α with $h(z) = f(z)/g(z) \in \mathcal{P}_n$. Then $zg'(z)/g(z)$ is in $\mathcal{P}_n(\alpha)$ and from Lemma 1.2,

$$\left| \frac{zg'(z)}{g(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}} \right| \leq \frac{2(1-\alpha)r^n}{1-r^{2n}}. \tag{2.1}$$

Applying Lemma 1.1 yields

$$\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2nr^n}{1-r^{2n}}. \tag{2.2}$$

Now

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}, \tag{2.3}$$

and using (2.1)–(2.3), it follows that

$$\left| \frac{zf'(z)}{f(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}} \right| \leq \frac{2(1+n-\alpha)r^n}{1-r^{2n}}. \tag{2.4}$$

Since the center of the disk in (2.4) is greater than 1, from Lemma 2.2, it is seen that the points w are inside the lemniscate $|w^2-1| < 1$ if

$$\frac{2(1+n-\alpha)r^n}{1-r^{2n}} \leq \sqrt{2} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}.$$

The last inequality reduces to $(1-2\alpha+\sqrt{2})r^{2n} + 2(1+n-\alpha)r^n - (\sqrt{2}-1) \leq 0$. Solving this latter inequality results in the value of $R = R_{\mathcal{SL}_n}(\mathcal{CS}_n(\alpha))$.

The function f given by

$$f(z) = \frac{z(1+z^n)}{(1-z^n)^{(n+2-2\alpha)/n}}$$

satisfies the hypothesis of Theorem 2.3 with $g(z) = z/(1-z^n)^{(2-2\alpha)/n}$. It is easy to see that, for $z = R = R_{\mathcal{SL}_n}(\mathcal{CS}_n(\alpha))$,

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| = \left| \frac{[1+(1-2\alpha)R^{2n} + 2(1+n-\alpha)R^n]^2}{(1-R^{2n})^2} - 1 \right| = 1.$$

This shows that the result is sharp. ∎

For $-1 \leq B < A \leq 1$, define the class

$$\mathcal{ST}_n[A,B] := \left\{ f \in \mathcal{A}_n : \frac{zf'(z)}{f(z)} \in \mathcal{P}_n[A,B] \right\}.$$

This is the well-known class of Janowski starlike functions. For this class, we have the following results.

Theorem 2.4. *Let $-1 < B < A \leq 1$ and either (i) $1 + A \leq \sqrt{2}(1 + B)$ and $2\sqrt{2}(1 - B^2) \leq 3(1 - AB) < 3\sqrt{2}(1 - B^2)$, or (ii) $(A - B)(1 - B^2) + (1 - B^2)^2 \leq (1 - B^2)\sqrt{(1 - B^2) - (1 - AB)^2} + (1-AB)^2$ and $2\sqrt{2}(1-B^2) \geq 3(1-AB)$. Then $\mathcal{ST}_n[A,B] \subset \mathcal{SL}_n$.*

Proof. Since $\frac{zf'(z)}{f(z)} \in P_n[A,B]$, Lemma 1.2 gives

$$\left| \frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2} \right| \leq \frac{A-B}{1-B^2} \quad (|z| < 1). \tag{2.5}$$

Let $a = (1 - AB)/(1 - B^2)$, and suppose the two conditions in (i) hold. By multiplying the inequality $1 + A \leq \sqrt{2}(1 + B)$ by the positive constant $1 - B$ and rewriting, it is seen that the given inequality is equivalent to $A - B \leq \sqrt{2}(1 - B^2) - (1 - AB)$. A division by $1 - B^2$ shows that the condition $1 + A \leq \sqrt{2}(1 + B)$ is equivalent to the condition $(A - B)/(1 - B^2) \leq \sqrt{2} - a$. Similarly, the condition $2\sqrt{2}(1 - B^2) \leq 3(1 - AB) < 3\sqrt{2}(1 - B^2)$ is equivalent to $2\sqrt{2}/3 \leq a < \sqrt{2}$. In view of these equivalences, it follows from (2.5) that the quantity $w = zf'(z)/f(z)$ lies in the disk $|w-a| < r_a$ where $r_a = \sqrt{2}-a$. Since $2\sqrt{2}/3 \leq a < \sqrt{2}$ and $|w - a| < r_a$, Lemma 2.2 shows that $|w^2 - 1| < 1$ or

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1.$$

This proves that $f \in \mathcal{SL}_n$. The proof is similar if the conditions in (ii) hold, and is therefore omitted. ∎

Theorem 2.5. *Let $-1 \leq B < A \leq 1$, with $B \leq 0$. Then the $\mathcal{SL}_n$-radius for the class $\mathcal{ST}_n[A,B]$ is*

$$R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B]) = \min\left(1, \left(\frac{2(\sqrt{2}-1)}{(A-B) + \sqrt{(A-B)^2 + 4(\sqrt{2}B - A)B(\sqrt{2}-1)}} \right)^{\frac{1}{n}} \right).$$

In particular, if $1 + A < \sqrt{2}(1 + B)$, then $\mathcal{ST}_n[A,B] \subseteq \mathcal{SL}_n$. Also the $\mathcal{SL}$-radius for the class consisting of starlike functions is $3 - 2\sqrt{2}$.

Proof. Since $\frac{zf'(z)}{f(z)} \in P_n[A,B]$, Lemma 1.2 yields

$$\left| \frac{zf'(z)}{f(z)} - \frac{1-ABr^{2n}}{1-B^2r^{2n}} \right| \leq \frac{(A-B)r^n}{1-B^2r^{2n}}.$$

Since $B \leq 0$, it follows that

$$a := \frac{1-ABr^{2n}}{1-B^2r^{2n}} \geq 1.$$

Using Lemma 2.2, the function f satisfies

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1$$

provided

$$\frac{(A-B)r^n}{1-B^2r^{2n}} < \sqrt{2} - \frac{1-ABr^{2n}}{1-B^2r^{2n}},$$

that is,

$$(\sqrt{2}B - A)Br^{2n} + (A-B)r^n - (\sqrt{2}-1) < 0.$$

Solving the inequality, we get $r \leq R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B])$. The result is sharp for the function given by $f(z) = z(1+Bz^n)^{\frac{A-B}{nB}}$ for $B \neq 0$ and $f(z) = z\exp(Az^n/n)$ for $B = 0$. Such function f satisfies the equation $zf'(z)/f(z) = (1+Az^n)/(1+Bz^n)$, and therefore the function $f \in \mathcal{ST}_n[A,B]$. ∎

Theorem 2.6. *Assume that $f \in \mathcal{ST}_n[A,B]$ and $0 < B < A \leq 1$. Let R_1 be given by*

$$R_1 = \left(\frac{2\sqrt{2}-3}{(2\sqrt{2}B-3A)B}\right)^{1/(2n)},$$

and let R_2 be the number $R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B])$ as given in Theorem 2.5. Let R_3 be the largest number in $(0,1]$ such that

$$(A-B)r^n(1-B^2r^{2n}) + (1-B^2r^{2n})^2 - (1-ABr^{2n})^2 - \sqrt{(1-B^2r^{2n})^2 - (1-ABr^{2n})^2} \leq 0$$

for all $0 \leq r \leq R_3$. Then the $\mathcal{SL}_n$-radius for the class $\mathcal{ST}_n[A,B]$ is given by

$$R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B]) = \begin{cases} R_2 & (R_2 \leq R_1) \\ R_3 & (R_2 > R_1). \end{cases}$$

Proof. From the proof of the previous theorem, it easy to see that the quantity $w = zf'(z)/f(z)$ lies in the disk $|w-a| \leq R$ where

$$a := \frac{1-ABr^{2n}}{1-B^2r^{2n}}, \quad R = \frac{(A-B)r^n}{1-B^2r^{2n}}.$$

Let us first assume that $R_2 \leq R_1$ where R_1, R_2 are as defined in the statement of the theorem. In this case, $r \leq R_1$ if and only if $a \geq 2\sqrt{2}/3$ and in particular, for $0 \leq r \leq R_2$, we have $a \geq 2\sqrt{2}/3$. Lemma 2.2 shows that $f \in \mathcal{SL}_n$ in $|z| \leq r$ if $R \leq \sqrt{2} - a$ or equivalently if $r \leq R_2$.

Let us now assume that $R_2 > R_1$. In this case, $r \geq R_1$ if and only if $a \leq 2\sqrt{2}/3$ and in particular for $r \geq R_2$, we have $a \leq 2\sqrt{2}/3$. Lemma 2.2 shows that $f \in \mathcal{SL}_n$ in $|z| \leq r$ if $R \leq \left(\sqrt{1-a^2} - (1-a^2)\right)^{1/2}$ or equivalently if $r \leq R_3$. The sharpness follows because $w = zf'(z)/f(z)$ with $z \in \mathbb{D}$ fills the entire disk $|w-a| < R$ where a and R are as given above. ∎

3. The $\mathcal{M}_n(\beta)$-Radius Problems

In this section, we compute the $\mathcal{M}_n(\beta)$-radii for the classes $\mathcal{S}_n$ and $\mathcal{CS}_n(\alpha)$.

Theorem 3.1. *The $\mathcal{M}_n(\beta)$-radius of functions in $\mathcal{S}_n$ is given by*

$$R_{\mathcal{M}_n(\beta)}(\mathcal{S}_n) = \left[\frac{\beta-1}{n+\sqrt{n^2+(\beta-1)^2}}\right]^{1/n}.$$

PROOF. Since $h(z) = f(z)/z \in \mathcal{P}_n$, Lemma 1.1 yields

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2nr^n}{1-r^{2n}}.$$

Therefore

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \leq \frac{1+2nr^n-r^{2n}}{1-r^{2n}} \leq \beta$$

for $r \leq R_{\mathcal{M}_n(\beta)}(\mathcal{S}_n)$.

The result is sharp for the function

$$f(z) = \frac{z(1+z^n)}{1-z^n}$$

which satisfies the hypothesis of Theorem 3.1. ∎

For the class $\mathcal{CS}_n(\alpha)$, the following radius is obtained.

Theorem 3.2. *The $\mathcal{M}_n(\beta)$-radius of functions in $\mathcal{CS}_n(\alpha)$ is given by*

$$R_{\mathcal{M}_n(\beta)}(\mathcal{CS}_n(\alpha)) = \frac{\beta-1}{(1+n-\alpha)+\sqrt{(1+n-\alpha)^2+(\beta-1)(1+\beta-2\alpha)}}.$$

PROOF. Define the function h by

$$h(z) := \frac{f(z)}{g(z)}.$$

Then $h \in \mathcal{P}_n$ and by Lemma 1.1,

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2nr^n}{1-r^{2n}}. \tag{3.1}$$

Since $g \in \mathcal{ST}_n(\alpha)$, it follows that $zg'(z)/g(z)$ is in $\mathcal{P}_n(\alpha)$ and therefore, by Lemma 1.2,

$$\left|\frac{zg'(z)}{g(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right| \leq \frac{2(1-\alpha)r^n}{1-r^{2n}}. \tag{3.2}$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)},$$

in view of (3.1) and (3.2), it is seen that

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right| \leq \frac{2(1+n-\alpha)r^n}{1-r^{2n}}.$$

This represents a circular disk intersecting the real axis at

$$x_0 = \frac{1-2(1+n-\alpha)r^n+(1-2\alpha)r^{2n}}{1-r^{2n}} \text{ and } x_1 = \frac{1+2(1+n-\alpha)r^n+(1-2\alpha)r^{2n}}{1-r^{2n}},$$

and therefore

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \leq \frac{1+2(1+n-\alpha)r^n+(1-2\alpha)r^{2n}}{1-r^{2n}} \leq \beta$$

for $r \leq R$.

The function

$$f(z) = \frac{z(1+z^n)}{(1-z^n)^{(n+2-2\alpha)/n}}$$

satisfies the hypothesis of Theorem 3.2 with

$$g(z) = \frac{z}{(1-z^n)^{(2-2\alpha)/n}}.$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{1+2(1+n-\alpha)z^n+(1-2\alpha)z^{2n}}{1-z^{2n}} = \beta$$

for $z = R = R_{\mathcal{M}_n(\beta)}(\mathcal{CS}_n(\alpha))$, the result is sharp. ∎

References

[1] R. M. Ali, M. H. Mohd, S. K. Lee and V. Ravichandran, Radii of starlikeness, parabolic starlikeness and strong starlikeness for Janowski starlike functions with complex parameters, *Tamsui Oxford J. Math. Sci.*, to appear.

[2] A. W. Goodman, *Univalent Functions*, Vol. 1 & 2, Polygonal Publ. House, Washington, New Jersey, 1983.

[3] A. W. Goodman, On uniformly starlike functions, *J. Math. Anal. Appl.*, **155** (1991), 364-370.

[4] A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.* **57** (1991), 87-92.

[5] W. Ma, and D. Minda, Uniformly convex functions, *Ann. Polon. Math.*, **57** (1992), 165-175.

[6] W. Ma and D. Minda, A Unified treatment of some special classes of univalent functions, in: *Proceedings of the Conference on Complex Analysis,* Z. Li, F. Ren, L. Yang, and S. Zhang(Eds.), Int. Press (1994), 157–169.

[7] T.H. MacGregor, The radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.*, 14(1963), 514-520.

[8] T.H. MacGregor,, The radius of univalence of certain analytic functions,II., *Proc. Amer. Math. Soc.*, 14(1963), 521-524.

[9] S. Owa and H. M. Srivastava, Some generalized convolution properties associated with certain subclasses of analytic functions, *J. Inequal. Pure Appl. Math.* **3** (2002), no. 3, Article 42, 13 pp. (electronic).

[10] E. Paprocki and J. Sokół, The extremal problems in some subclass of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* No. 20 (1996), 89–94.

[11] V. Ravichandran, H. Silverman, M. Hussain Khan, K. G. Subramanian, Radius problems for a class of analytic functions, *Demonstrat. Math.*, **39**(1) (2006), 67–74.

[12] V. Ravichandran, F. Rønning and T. N. Shanmugam, Radius of convexity and radius of starlikeness for some classes of analytic functions, *Complex Variables*, **33**(1997), 265–280.

[13] Rønning, F., On uniform starlikeness and related properties of univalent functions, *Complex Variables*, **24**, (1994), 233-239.

[14] F. Rønning, Some radius results for univalent functions, *J. Math. Anal. Appl.*, **194**(1995), 319–327.

[15] Rønning, F., On starlike functions associated with parabolic regions, *Ann. Univ. Mariae Curie-Sklodowska*, Sect A, XLV, **14** (1991), 117-122.

[16] Rønning, F., Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118**, (1993), 189-196.

[17] F. Rønning, A survey on uniformly convex and uniformly starlike functions, *Ann. Univ. Mariae Curie-Sklodowska,* Sect A, **47**(1993), 123–134.

[18] T. N. Shanmugam and V. Ravichandran, Certain properties of uniformly convex functions, In: *Computational Methods and Function Theory 1994*, (R. M. Ali, St. Ruscheweyh & E. B. Saff (Eds.)), World Scientific Publ. Co., Singapore, 1995, pp. 319–324.

[19] T. N. Shanmugam and V. Ravichandran, Radius problems for analytic functions, *Chinese J. Math.*, **23**(1995), 343–351.

[20] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* No. 19 (1996), 101–105.

[21] J. Sokół, Coefficient estimates in a class of strongly starlike functions, *Kyungpook Math. J.* **49** (2009), no. 2, 349–353.

[22] J. Sokół, Radius problems in the class $\mathcal{SL}$, *Appl. Math. Comput.* **214** (2009), no. 2, 569–573.

[23] B. A. Uralegaddi, M. D. Ganigi and S. M. Sarangi, Univalent functions with positive coefficients, *Tamkang J. Math.* **25** (1994), no. 3, 225–230.

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
E-mail address: rosihan@cs.usm.my

Department of Mathematics, University of Delhi, Delhi-110007, India
E-mail address: naveenjain05@gmail.com

Department of Mathematics, University of Delhi, Delhi-110007, India
E-mail address: vravi@maths.du.ac.in