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# Canonical angles and limits of sequences of EP and co-EP matrices

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## Abstract

Let  $A$  be a square complex matrix. Let  $\mathcal{P}$  be one of the following properties: a)  $A$  is an EP matrix, b) the column space of  $A$  is complementary to the column space of  $A^*$ , and c) the orthogonal complement of the column space of  $A$  is the column space of  $A^*$ . We study the canonical angles between the column space of  $A$  and the column space of  $A^*$  when  $A$  satisfies property  $\mathcal{P}$ . Also, we research the following problem: Let  $\{A_m\}_{m=1}^{\infty}$  be a sequence of matrices satisfying property  $\mathcal{P}$  that converges to some matrix  $A$ . When does  $A$  satisfy property  $\mathcal{P}$ ?

Keywords: Canonical angles of subspaces, EP matrices.

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## 1 Introduction, notation, and preliminary results

The canonical angles (also called principal angles) between two subspaces provide the best available characterization of the relative position of two given subspaces. This concept allows us to characterize or measure, in a natural way, how two subspaces differ, which is the main connection with perturbation theory. In [1, 2, 3] we can find how these angles were discovered and rediscovered again several times. Computation of canonical angles between subspaces is important in many applications including statistics [4, 5], information retrieval [6], and analysis of algorithms [7].

There are many equivalent definitions of the canonical angles (see [8]). For our purposes, the most convenient is the following:

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**Definition 1.1.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two nontrivial subspaces of  $\mathbb{C}^n$  and  $r = \min\{\dim \mathcal{X}, \dim \mathcal{Y}\}$ . We define the canonical angles  $\theta_1, \dots, \theta_r \in [0, \pi/2]$  between  $\mathcal{X}$  and  $\mathcal{Y}$  by

$$\cos \theta_i = \sigma_i(P_{\mathcal{X}}P_{\mathcal{Y}}), \quad i = 1, \dots, r, \quad (1)$$

where the real numbers  $\sigma_1(P_{\mathcal{X}}P_{\mathcal{Y}}), \dots, \sigma_r(P_{\mathcal{X}}P_{\mathcal{Y}}) \geq 0$  are the singular values of the matrix  $P_{\mathcal{X}}P_{\mathcal{Y}}$ , and  $P_{\mathcal{S}}$  stands for the orthogonal projector onto the subspace  $\mathcal{S} \subset \mathbb{C}^n$ .

See for example [8, 9, 10, 11] for a deeper insight into this definition and [12] for the simultaneous study of two orthogonal projectors.

Let us remark that the greatest singular value of a matrix  $X$  is  $\|X\|$ , where  $\|\cdot\|$  denotes the matrix 2-norm or the matrix norm induced by the Euclidean vector norm. Accordingly to Definition 1.1, the minimal angle between nontrivial subspaces  $\mathcal{X}$  and  $\mathcal{Y}$  of  $\mathbb{C}^n$  is defined to be the number  $\theta_{\mathcal{X}, \mathcal{Y}} \in [0, \pi/2]$  for which  $\cos \theta_{\mathcal{X}, \mathcal{Y}} = \|P_{\mathcal{X}}P_{\mathcal{Y}}\|$ . It can be proved the following result (see [13, Section 5.15]):

**Theorem 1.2.** Let  $\mathcal{X}, \mathcal{Y}$  be two nontrivial subspaces of  $\mathbb{C}^n$ . Then  $\mathcal{X}$  and  $\mathcal{Y}$  are complementary if and only if  $P_{\mathcal{X}} - P_{\mathcal{Y}}$  is nonsingular, and in this case

$$\sin \theta_{\mathcal{X}, \mathcal{Y}} = \frac{1}{\|(P_{\mathcal{X}} - P_{\mathcal{Y}})^{-1}\|} = \frac{1}{\|P_{\mathcal{X}\mathcal{Y}}\|},$$

where  $P_{\mathcal{X}\mathcal{Y}}$  is the oblique projector onto  $\mathcal{X}$  along  $\mathcal{Y}$ .

Before establishing the aim of this paper, let us fix some standard notations. Let  $\mathbb{C}_{m,n}$  denote the set composed of  $m \times n$  complex matrices. Throughout this paper we consider the vectors of  $\mathbb{C}^n$  as columns, thus we shall identify  $\mathbb{C}^n$  with  $\mathbb{C}_{n,1}$ . The symbols  $K^*$ ,  $\text{rk}(K)$ ,  $\mathcal{R}(K)$  will denote the conjugate transpose, the rank, and the column space, respectively, of  $K \in \mathbb{C}_{m,n}$ . Furthermore,  $K^\dagger$  will stand for the Moore–Penrose inverse of  $K \in \mathbb{C}_{m,n}$ , i.e., the unique matrix satisfying the four equations

$$KK^\dagger K = K, \quad K^\dagger KK^\dagger = K^\dagger, \quad (KK^\dagger)^* = KK^\dagger, \quad (K^\dagger K)^* = K^\dagger K. \quad (2)$$

It is known that any matrix  $K \in \mathbb{C}_{n,m}$  has a Moore–Penrose inverse (and trivially,  $K^\dagger \in \mathbb{C}_{m,n}$ ). Moreover, it can be easily proved that  $KK^\dagger$  is the orthogonal projector onto  $\mathcal{R}(K)$  and  $K^\dagger K$  is the orthogonal projector onto  $\mathcal{R}(K^*)$ .

A square matrix  $A \in \mathbb{C}_{n,n}$  is said to be EP when when  $AA^\dagger = A^\dagger A$ . A useful characterization of the EP matrices (the name comes from Equal Projection) is the following (see [14, Chapter 4.4] or [15, Theorem 4.3.1]):

**Theorem 1.3.** *Let  $A \in \mathbb{C}_{n,n}$  have rank  $r$ . The following statements are equivalent:*

(i)  $AA^\dagger = A^\dagger A$ .

(ii) *There exist a unitary matrix  $U \in \mathbb{C}_{n,n}$  and a nonsingular matrix  $K \in \mathbb{C}_{r,r}$ , such that*

$$A = U(K \oplus 0)U^*. \quad (3)$$

(iii)  $\mathcal{R}(A) = \mathcal{R}(A^*)$ .

In some sense, the subset of  $\mathbb{C}_{n,n}$  composed of matrices  $A$  such that  $AA^\dagger - A^\dagger A$  is nonsingular is complementary to the subset of EP matrices. Following [16] we say that a matrix  $A \in \mathbb{C}_{n,n}$  is co-EP when  $AA^\dagger - A^\dagger A$  is nonsingular. Among several characterizations of the co-EP matrices, in [16] the authors proved the following result:

**Theorem 1.4.** *Let  $A \in \mathbb{C}_{n,n}$  and  $r$  be the rank of  $A$ . The following statements are equivalent:*

(i)  $AA^\dagger - A^\dagger A$  is nonsingular.

(ii)  $\mathcal{R}(A) \oplus \mathcal{R}(A^*) = \mathbb{C}_{n,1}$ .

(iii) *There exist a unitary matrix  $U \in \mathbb{C}_{n,n}$ , a nonsingular matrix  $M \in \mathbb{C}_{r,r}$ , and  $\theta_1, \dots, \theta_r \in ]0, \pi/2[$  such that*

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*, \quad (4)$$

where  $C = \text{diag}(\cos \theta_1, \dots, \cos \theta_r)$ , and  $S = \text{diag}(\sin \theta_1, \dots, \sin \theta_r)$ .

Obviously, we have  $C^2 + S^2 = I_r$  and  $CS = SC$ . These two equalities will be used many times in the sequel. Moreover, let us observe that if  $A \in \mathbb{C}_{n,n}$  is co-EP matrix, then  $n$  must be even and  $\text{rk}(A) = n/2$ .

When  $\mathcal{R}(A) = \mathcal{R}(A^*)^\perp$  (see item (ii) of the previous theorem) holds, then we have the following result [16]:

**Theorem 1.5.** *Let  $A \in \mathbb{C}_{n,n}$  and  $r$  be the rank of  $A$ . The following statements are equivalent:*

(i)  $\mathcal{R}(A) = \mathcal{R}(A^*)^\perp$ .

(ii) *There exist a unitary matrix  $U \in \mathbb{C}_{n,n}$ , a nonsingular matrix  $M \in \mathbb{C}_{r,r}$  such that*

$$A = U \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} U^*.$$

(iii)  $AA^\dagger + A^\dagger A = I_n$ .

(iv) *There exists a unitary matrix  $U \in \mathbb{C}_{n,n}$  such that  $AA^\dagger = U(I_r \oplus 0)U^*$  and  $A^\dagger A = U(0 \oplus I_r)U^*$ .*

We shall say that a matrix  $A \in \mathbb{C}_{n,n}$  is co-EP $_\perp$  if it satisfies any condition of Theorem 1.5.

The purpose of this paper is twofold: Firstly we study the canonical angles of the EP, co-EP, and co-EP $_\perp$  matrices. Lastly we get some results concerning limits of sequences of matrices which are EP, co-EP, co-EP $_\perp$ . These last results have a strong relation with the canonical angles between the column space of some matrices and the column space of their conjugate transposes.

## 2 EP, co-EP matrices, and their canonical angles

If a co-EP matrix  $A$  is represented as in (4), then we can know explicitly the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ .

**Theorem 2.1.** *Let  $r \in \mathbb{N}$ . If  $A \in \mathbb{C}_{2r,2r}$  is represented as*

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*,$$

where  $U \in \mathbb{C}_{2r,2r}$  is unitary,  $M \in \mathbb{C}_{r,r}$  is nonsingular,

$$C = \text{diag}(\cos \theta_1, \dots, \cos \theta_r), \quad S = \text{diag}(\sin \theta_1, \dots, \sin \theta_r),$$

and  $\theta_1, \dots, \theta_r \in [0, \pi/2]$ , then the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are  $\theta_1, \dots, \theta_r$ .

*Proof.* First of all, it is easily checked (by using the four conditions in (2)) that

$$A^\dagger = U \begin{bmatrix} CM^{-1} & 0 \\ SM^{-1} & 0 \end{bmatrix} U^*.$$

Obvious computations show

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad A^\dagger A = U \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} U^*.$$

Since  $AA^\dagger = U(I_r \oplus 0)U^*$  we get  $\dim \mathcal{R}(A) = \text{rk}(AA^\dagger) = r$  because  $AA^\dagger$  is the orthogonal projector onto  $\mathcal{R}(A)$ . Also we have  $\dim \mathcal{R}(A^*) = \text{rk}(A^*) = \text{rk}(A) = r$ . We shall use (1) in order to find the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ .

$$P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)} = (AA^\dagger)(A^\dagger A) = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} U^* = U \begin{bmatrix} C^2 & CS \\ 0 & 0 \end{bmatrix} U^*.$$

It is simple to prove that the matrix  $V \in \mathbb{C}_{2r,2r}$  defined by

$$V = U \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

is unitary. Since

$$U \begin{bmatrix} C^2 & CS \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C & S \\ -S & C \end{bmatrix} U^* = U \text{diag}(\cos \theta_1, \dots, \cos \theta_r, 0, \dots, 0)V^*,$$

the singular value decomposition of  $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}$  is

$$P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)} = U \text{diag}(\cos \theta_1, \dots, \cos \theta_r, 0, \dots, 0)V^*.$$

Hence the singular values of  $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}$  are  $\cos \theta_1, \dots, \cos \theta_r$ . The theorem is proved.  $\square$

In order to prove some results of this paper, let us permit to write the following factorization valid for all square complex matrix ([17, Cor. 6]).

**Lemma 2.2.** *Any matrix  $A \in \mathbb{C}_{n,n}$  of rank  $r$  can be represented as*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{5}$$

where  $U \in \mathbb{C}_{n,n}$  is unitary,  $\Sigma = \sigma_1 I_{r_1} \oplus \dots \oplus \sigma_t I_{r_t}$  is the diagonal matrix of singular values of  $A$ ,  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_t > 0$ ,  $r_1 + r_2 + \dots + r_t = r = \text{rk}(A)$ , and  $K \in \mathbb{C}_{r,r}$ ,  $L \in \mathbb{C}_{r,n-r}$  satisfy

$$KK^* + LL^* = I_r. \tag{6}$$

It is easily seen that if a matrix  $A$  is represented as in Lemma 2.2, then

$$A^\dagger = U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^*.$$

The following result permits to find the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  when  $A$  is a matrix represented as in Lemma 2.2.

**Theorem 2.3.** *Let  $A \in \mathbb{C}_{n,n}$  be represented as in Lemma 2.2. Then  $\theta \in [0, \pi/2]$  is a canonical angle between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  if and only if  $\cos \theta$  is a singular value of  $K$ .*

*Proof.* Let us denote  $X = P_{\mathcal{R}(A)} P_{\mathcal{R}(A^*)}$ . Since  $P_{\mathcal{R}(A)} = AA^\dagger$  and  $P_{\mathcal{R}(A^*)} = A^\dagger A$ , using (6) we easily have

$$\begin{aligned} X &= (AA^\dagger)(A^\dagger A) \\ &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K^* K & K^* L \\ L^* K & L^* L \end{bmatrix} U^* \\ &= U \begin{bmatrix} K^* K & K^* L \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

It is well known that for every complex square matrix  $M$  we have that if  $\sigma \geq 0$ , then

$$\sigma \text{ is a singular value of } M \Leftrightarrow \sigma^2 \text{ is an eigenvalue of } MM^* \Leftrightarrow \sigma^2 \text{ is an eigenvalue of } M^*M.$$

Thus, we shall calculate  $XX^*$ . Using (6) leads to

$$XX^* = U \begin{bmatrix} K^* K & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Let  $\theta \in [0, \pi/2[$ . The following chain can be easily followed recalling that  $X = P_{\mathcal{R}(A)} P_{\mathcal{R}(A^*)}$ , Definition 1.1, and  $r = \text{rk}(A)$  and therefore, there are exactly  $r$  canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ :

$$\begin{aligned} \theta \text{ is a canonical angle between } \mathcal{R}(A) \text{ and } \mathcal{R}(A^*) &\Leftrightarrow \cos \theta \text{ is a singular value of } X \\ &\Leftrightarrow \cos^2 \theta \text{ is an eigenvalue of } XX^* \\ &\Leftrightarrow \cos^2 \theta \text{ is an eigenvalue of } K^* K \\ &\Leftrightarrow \cos \theta \text{ is a singular value of } K. \end{aligned}$$

The proof is finished. □

The following result links the EP matrices, co-EP matrices, and co-EP $_{\perp}$  matrices with the canonical angles between the column space of these matrices and of their conjugate transposes.

**Theorem 2.4.** *Let  $A \in \mathbb{C}_{n,n}$ .*

- (i) *A is EP if and only if all the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are zero.*
- (ii) *If A is co-EP, then none of the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are zero.*
- (iii) *If A is co-EP $_{\perp}$ , then all the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are  $\pi/2$ .*

*Proof.* It is well known that  $P_{\mathcal{R}(A)} = AA^{\dagger}$  and  $P_{\mathcal{R}(A^*)} = A^{\dagger}A$ . If  $r$  is the rank of  $A$ , then  $r = \dim \mathcal{R}(A) = \dim \mathcal{R}(A^*)$ .

(i) Suppose that  $A$  is EP. Using the representation (3) of  $A$  given in Theorem 1.3 we easily get  $AA^{\dagger} = A^{\dagger}A = U(I_r \oplus 0)U^*$ . Hence  $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)} = U(I_r \oplus 0)U^*$ . Using (1) proves that all the  $r$  canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are zero.

Suppose that all the  $r$  canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are zero. By Theorem 2.3, the scalar 1 is an eigenvalue of  $K^*K$  with multiplicity  $r$ ; but since  $K^*K \in \mathbb{C}_{r,r}$  and  $K^*K$  is diagonalizable, we have  $K^*K = I_r$ . This implies that  $K^* = K^{-1}$  because  $K$  is a square matrix. Using (6) we get  $L = 0$ . Representation (5) reduces to  $A = U(\Sigma K \oplus 0)U^*$ . Recall that  $\Sigma$  and  $K$  are nonsingular, hence item (ii) of Theorem 1.3 yields that  $A$  is EP.

(ii) The result follows from Theorem 1.4 and Theorem 2.1.

(iii) If  $A$  is co-EP $_{\perp}$ , by item (iv) of Theorem 1.5, we get

$$P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)} = (AA^{\dagger})(A^{\dagger}A) = 0,$$

hence all the singular values of  $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}$  are zero and Definition 1.1 entails that all the  $r$  canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are  $\pi/2$ . □

Items (ii) and (iii) of the Theorem 2.4 are not equivalences as the following examples show:

**Example 1.** Let  $B \in \mathbb{C}_{n,n}$  have rank  $r$  and let us define  $A = B \oplus 0 \in \mathbb{C}_{n+k,n+k}$  with  $k > 0$ . Obviously, we have  $\text{rk}(A) = \text{rk}(B) = \text{rk}(B^*) = \text{rk}(A^*)$  and the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are equal to the canonical angles between  $\mathcal{R}(B)$  and  $\mathcal{R}(B^*)$ .

If  $B$  is co-EP, then none of the  $r$  canonical angles between  $\mathcal{R}(B)$  and  $\mathcal{R}(B^*)$  are zero. Hence, none of the  $r$  canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are zero. However,  $A$  is not co-EP because  $AA^\dagger - A^\dagger A = (BB^\dagger - B^\dagger B) \oplus 0$  is a singular matrix.

If  $B$  is co-EP $_\perp$ , all the  $r$  canonical angles between  $\mathcal{R}(B)$  and  $\mathcal{R}(B^*)$  equal to  $\pi/2$ . Hence, all the  $r$  canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are  $\pi/2$ . However,  $A$  is not co-EP $_\perp$  because  $AA^\dagger + A^\dagger A = (BB^\dagger + B^\dagger B) \oplus 0 = I_n \oplus 0 \neq I_{n+k}$  (see item (iii) of Theorem 1.5).

### 3 Limits of EP, co-EP matrices and canonical angles

In this section we shall research the following challenging problem: Let  $\mathcal{P}$  be the property of being EP, co-EP, or co-EP $_\perp$  and let  $\{A_m\}_{m=1}^\infty \subset \mathbb{C}_{n,n}$  be a sequence of matrices satisfying property  $\mathcal{P}$  that converges to some matrix  $A$ . When does  $A$  satisfy property  $\mathcal{P}$ ?

The following simple result is concerned with the above question when the involved matrices are EP. Observe that we must use some kind of continuity of the Moore–Penrose inverse.

**Theorem 3.1.** *Let  $\{A_m\}_{m=1}^\infty$  be a sequence of EP matrices that converges to  $A$ . If  $A_m^\dagger \rightarrow A^\dagger$ , then  $A$  is EP.*

*Proof.* Since  $A_m$  is EP for each  $m \in \mathbb{N}$  we have  $A_m A_m^\dagger = A_m^\dagger A_m$ , hence

$$AA^\dagger = (\lim_m A_m)(\lim_m A_m^\dagger) = \lim_m (A_m A_m^\dagger) = \lim_m (A_m^\dagger A_m) = (\lim_m A_m^\dagger)(\lim_m A_m) = A^\dagger A.$$

The proof is finished. □

The converse of this result is false as the following example shows:

**Example 2.** For each natural number, let  $A_m = 1/m \in \mathbb{C}_{1,1}$ . Every  $A_m$  is evidently EP, the sequence  $\{A_m\}_{m=1}^\infty$  is convergent, and  $\lim_m A_m = 0$  is obviously EP. However,  $\{A_m^\dagger\}_{m=1}^\infty$  does not converge to  $0^\dagger$ .

Contrarily to the usual inverse of a square matrix, it is well known that the Moore–Penrose inverse is not necessarily a continuous function of the elements of the matrix. The following classical theorem gives an equivalent condition for the continuity of the Moore–Penrose inverse of a matrix.

**Theorem 3.2** ([18, 19]). *If  $\{A_m\}_{m=1}^\infty \subset \mathbb{C}_{n,k}$  is a convergent sequence to  $A \in \mathbb{C}_{n,k}$ , then*

$$A_m^\dagger \rightarrow A^\dagger \quad \iff \quad \exists m_0 \in \mathbb{N} : \text{rk}(A_m) = \text{rk}(A) \quad \forall m \geq m_0.$$

The following examples show that the set of  $n \times n$  co-EP matrices and the set of  $n \times n$  co-EP $_\perp$  matrices are not closed in  $\mathbb{C}_{n,n}$ .

**Example 3.** For  $m \in \mathbb{N}$ , let us define

$$A_m = \begin{bmatrix} 0 & 1/m \\ 0 & 0 \end{bmatrix}.$$

All matrices  $A_m$  are co-EP $_\perp$  and obviously  $\lim_m A_m = 0$  is not co-EP $_\perp$ .

**Example 4.** For  $m \in \mathbb{N}$ , let us define

$$B_m = \begin{bmatrix} \cos(1/m) & \sin(1/m) \\ 0 & 0 \end{bmatrix}.$$

By Theorem 1.4 we have that  $B_m$  is co-EP for all  $m \in \mathbb{N}$ . Evidently one has  $B_m \rightarrow \text{diag}(1, 0)$ . But  $\lim_m B_m$  is not co-EP (by item (ii) of Theorem 1.3),

Example 3 invites to think about the stability of the rank. This assumption is related with the continuity of the Moore–Penrose inverse.

Let us think about Example 4. Evidently, we have that  $\dim \mathcal{R}(B_m) = \dim \mathcal{R}(B_m^*) = 1$  and  $\mathcal{R}(B_m) \oplus \mathcal{R}(B_m^*) = \mathbb{C}_{2,1}$  for every  $m \in \mathbb{N}$  (this situation is depicted in Figure 1) and  $1/m$  is the angle between the lines  $\mathcal{R}(B_m)$  and  $\mathcal{R}(B_m^*)$ . When  $m \rightarrow \infty$  it is intuitive that these aforementioned lines converge to the same line, say  $r$ . We have (in a very extremely informal parlance), by denoting  $B = \lim_m B_m = \text{diag}(1, 0)$ ,

$$\mathcal{R}(B) = \mathcal{R}(\lim_{m \rightarrow \infty} B_m) = \lim_{m \rightarrow \infty} \mathcal{R}(B_m) = r = \lim_{m \rightarrow \infty} \mathcal{R}(B_m^*) = \mathcal{R}(\lim_{m \rightarrow \infty} B_m^*) = \mathcal{R}(B^*).$$

Thus it is impossible that  $B$  is a co-EP matrix. In fact,  $B$  is an EP matrix.

Let us recall the following fact that we will use in Theorem 3.3 below: If  $\{X_m\}_{m=1}^\infty$  is a sequence of nonsingular matrices in  $\mathbb{C}_{n,n}$  converging to  $X$  such that  $\{X_m^{-1}\}_{m=1}^\infty$  is bounded, then  $X$  is nonsingular and  $\{X_m^{-1}\}_{m=1}^\infty$  converges to  $X^{-1}$ . In fact, the equality  $\|X_k^{-1} - X_m^{-1}\| = \|X_k^{-1}(X_m - X_k)X_m^{-1}\|$  implies that the sequence  $\{X_m^{-1}\}_{m=1}^\infty$  is a Cauchy sequence, hence convergent, say to  $Y$ . Now,  $I_n = X_m X_m^{-1}$  implies  $I_n = XY$ .

The following result establishes precise conditions (and a rigorous argument) in order to the limit of co-EP matrices is co-EP.

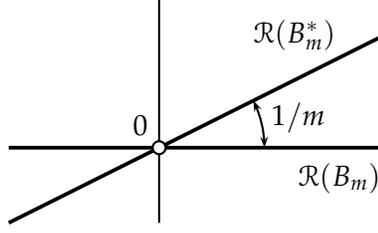


Figure 1: The column space of  $B$  is  $\text{span}\{[1, 0]^*\}$  and the column space of  $B_m^*$  is  $\text{span}\{[\cos(1/m), \sin(1/m)]^*\}$ .

**Theorem 3.3.** *Let  $\{A_m\}_{m=1}^\infty \subset \mathbb{C}_{n,n}$  be a sequence of co-EP matrices that converges to  $A$ . Then the following statements are equivalent:*

- (i)  $A_m^\dagger \rightarrow A^\dagger$  and there exists  $\theta_0 > 0$  and  $m_0 \in \mathbb{N}$  such that the minimal angle between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$  is greater than  $\theta_0$  for all  $m \geq m_0$ .
- (ii)  $A$  is a co-EP matrix.

*Proof.* (i)  $\Rightarrow$  (ii): Since  $A_m \rightarrow A$  and  $A_m^\dagger \rightarrow A^\dagger$  we get

$$A_m A_m^\dagger - A_m^\dagger A_m \rightarrow A A^\dagger - A^\dagger A. \quad (7)$$

On the other hand, matrices  $A_m$  are co-EP for all  $m \in \mathbb{N}$ , and thus  $A_m A_m^\dagger - A_m^\dagger A_m$  are nonsingular matrices. By using Theorem 1.2

$$\|(A_m A_m^\dagger - A_m^\dagger A_m)^{-1}\|_2 = \frac{1}{\sin \theta_{\mathcal{R}(A_m), \mathcal{R}(A_m^*)}} \leq \frac{1}{\sin \theta_0}$$

holds for  $m \geq m_0$ . Therefore, the sequence  $\{(A_m A_m^\dagger - A_m^\dagger A_m)^{-1}\}_{m=1}^\infty$  is bounded. The use of (7) ensures now that  $A A^\dagger - A^\dagger A$  is nonsingular, or in another words,  $A A^\dagger - A^\dagger A$  is co-EP.

(ii)  $\Rightarrow$  (i): Since  $A$  and  $A_m$  are co-EP matrices, we get  $\text{rk}(A_m) = \text{rk}(A) = n/2$  for all  $m \in \mathbb{N}$ . By Theorem 3.2 we get  $A_m^\dagger \rightarrow A^\dagger$ , hence  $A_m A_m^\dagger - A_m^\dagger A_m \rightarrow A A^\dagger - A^\dagger A$ . Since  $A_m A_m^\dagger - A_m^\dagger A_m$  and  $A A^\dagger - A^\dagger A$  are nonsingular matrices, the continuity of the inverse permits to affirm that

$$\lim_{m \rightarrow \infty} (A_m A_m^\dagger - A_m^\dagger A_m)^{-1} = (A A^\dagger - A^\dagger A)^{-1},$$

in particular, the sequence  $\{\|(A_m A_m^\dagger - A_m^\dagger A_m)^{-1}\|\}_{m=1}^\infty$  is bounded, say by  $K$ . Using Theorem 1.2 we get

$$\sin \theta_{\mathcal{R}(A_m), \mathcal{R}(A_m^*)} = \frac{1}{\|(A_m A_m^\dagger - A_m^\dagger A_m)^{-1}\|} \geq \frac{1}{K}.$$

The proof is completed.  $\square$

A related result holds when the involved matrices are  $\text{co-EP}_\perp$ :

**Theorem 3.4.** *Let  $\{A_m\}_{m=1}^\infty \subset \mathbb{C}_{n,n}$  be a sequence of  $\text{co-EP}_\perp$  matrices that converges to  $A$ . Then the following statements are equivalent:*

(i)  $A_m^\dagger \rightarrow A^\dagger$ .

(ii)  $A$  is  $\text{co-EP}_\perp$ .

*Proof.* (i)  $\Rightarrow$  (ii): By item (iv) of Theorem 1.5 we get  $A_m A_m^\dagger + A_m^\dagger A_m = I_n$  for each  $m \in \mathbb{N}$ . Since  $A_m \rightarrow A$  and  $A_m^\dagger \rightarrow A^\dagger$ , the results follows.

(ii)  $\Rightarrow$  (i): Since  $A$  and  $A_m$  are  $\text{co-EP}_\perp$ , we have  $\text{rk}(A) = n/2$  and  $\text{rk}(A_m) = n/2$  for all  $m \in \mathbb{N}$ . Theorem 3.2 ends the proof.  $\square$

The following simple lemmas will be used in the sequel.

**Lemma 3.5.** *Let  $K \in \mathbb{C}_{r,r}$  and  $L \in \mathbb{C}_{r,n-r}$  be such  $KK^* + LL^* = I_r$ , then  $\|K\| \leq 1$ ,  $\|L\| \leq 1$ .*

*Proof.* Let us define

$$A = \begin{bmatrix} K & L \\ 0 & 0 \end{bmatrix}.$$

It is easily checked that  $AA^* = I_r \oplus 0$ . Hence  $1 = \|I_r \oplus 0\| = \|AA^*\| = \|A\|^2$ , and thus  $\max\{\|K\|, \|L\|\} \leq \|A\| = 1$  (the norm of a submatrix cannot be greater than the norm of a whole matrix, see e.g. [20, Lemma 2]).  $\square$

**Lemma 3.6.** *The set  $\{(A, B) \in \mathbb{C}_{r,r} \times \mathbb{C}_{r,n-r} : AA^* + BB^* = I_r\}$  is a compact subset of  $\mathbb{C}_{r,r} \times \mathbb{C}_{r,n-r}$ .*

*Proof.* Let us prove that  $\mathcal{H} = \{(A, B) \in \mathbb{C}_{r,r} \times \mathbb{C}_{r,n-r} : AA^* + BB^* = I_r\}$  is a compact subset of  $\mathbb{C}_{r,r} \times \mathbb{C}_{r,n-r}$ . By Lemma 3.5 we get that  $\mathcal{H}$  is bounded. Now, let us consider the continuous map  $\Phi : \mathbb{C}_{r,r} \times \mathbb{C}_{r,n-r} \rightarrow \mathbb{C}_{r,r}$  given by  $\Phi(X, Y) = XX^* + YY^*$ . Since  $\mathcal{H} = \Phi^{-1}(I_r)$  we get that  $\mathcal{H}$  is closed in  $\mathbb{C}_{r,r} \times \mathbb{C}_{r,n-r}$ . This finishes the proof.  $\square$

Let us recall that the set composed of unitary  $n \times n$  matrices is compact in  $\mathbb{C}_{n,n}$ . Also, let us recall that any sequence in a compact set has a convergent subsequence.

The following result concerns EP matrices.

**Theorem 3.7.** Let  $\{A_m\}_{m=1}^\infty \subset \mathbb{C}_{n,n}$  be a convergent sequence of matrices such that  $A_m \rightarrow A$  and  $A_m^\dagger \rightarrow A^\dagger$ . Let  $\phi_m$  be the greatest canonical angle between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$ . The following statements are equivalent:

(i)  $\phi_m \rightarrow 0$ .

(ii)  $A$  is EP.

*Proof.* By Theorem 3.2 there exists  $m_0 \in \mathbb{N}$  such that  $\text{rk}(A_m) = \text{rk}(A)$  for  $m \geq m_0$ . Let us denote  $r = \text{rk}(A)$ . By Lemma 2.2, we can write for  $m \geq m_0$

$$A_m = U_m \begin{bmatrix} \Sigma_m K_m & \Sigma_m L_m \\ 0 & 0 \end{bmatrix} U_m^* \quad (8)$$

where  $\Sigma_m$  are diagonal and invertible,  $U_m$  unitary, and

$$K_m K_m^* + L_m L_m^* = I_r. \quad (9)$$

Also, we have

$$A_m^\dagger = U_m \begin{bmatrix} K_m^* \Sigma_m^{-1} & 0 \\ L_m^* \Sigma_m^{-1} & 0 \end{bmatrix} U_m^*,$$

By passing to a subsequence, we can suppose that  $\{U_m\}_{m=1}^\infty$  is convergent to the unitary matrix  $U$ . Since the sequences  $\{A_m\}_{m=1}^\infty$ ,  $\{A_m^\dagger\}_{m=1}^\infty$ , and  $\{U_m\}_{m=1}^\infty$  are convergent, from representations (8) and (9) we get that the following four sequences  $\{\Sigma_m K_m\}_{m=1}^\infty$ ,  $\{\Sigma_m L_m\}_{m=1}^\infty$ ,  $\{K_m^* \Sigma_m^{-1}\}_{m=1}^\infty$ , and  $\{L_m^* \Sigma_m^{-1}\}_{m=1}^\infty$  are convergent, say to  $X, Y, Z$ , and  $T$ , respectively. Since we have  $A_m \rightarrow A$  and  $A_m^\dagger \rightarrow A^\dagger$  we get

$$A = U \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} U^*, \quad A^\dagger = U \begin{bmatrix} Z & 0 \\ T & 0 \end{bmatrix} U^*. \quad (10)$$

We will prove that the sequences  $\Sigma_m$  and  $\Sigma_m^{-1}$  are bounded: In fact by Lemma 3.5 and recalling that  $\{\Sigma_m L_m\}_{m=1}^\infty$ ,  $\{K_m^* \Sigma_m^{-1}\}_{m=1}^\infty$ , and  $\{L_m^* \Sigma_m^{-1}\}_{m=1}^\infty$  are convergent,

$$\|\Sigma_m\| = \|\Sigma_m K_m K_m^* + \Sigma_m L_m L_m^*\| \leq \|\Sigma_m K_m\| + \|\Sigma_m L_m\|$$

and

$$\|\Sigma_m^{-1}\| = \|K_m K_m^* \Sigma_m^{-1} + L_m L_m^* \Sigma_m^{-1}\| \leq \|K_m^* \Sigma_m^{-1}\| + \|L_m^* \Sigma_m^{-1}\|.$$

Thus, denoting by  $\theta_1, \dots, \theta_r$  the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ , from Theorem 2.3 and (9), we have

$$\begin{aligned} \phi_m \rightarrow 0 &\Leftrightarrow \cos^2 \theta_i \rightarrow 1 \text{ for all } i = 1, \dots, r \\ &\Leftrightarrow K_m K_m^* \rightarrow I_r \Leftrightarrow L_m L_m^* \rightarrow 0 \Leftrightarrow L_m \rightarrow 0 \Leftrightarrow \Sigma_m L_m \rightarrow 0 \Leftrightarrow Y = 0. \end{aligned} \quad (11)$$

Therefore, to demonstrate this theorem, it is enough to prove that  $Y = 0$  if and only if  $A$  is EP.

(i)  $\Rightarrow$  (ii): If  $Y = 0$ , then from

$$I_r = \Sigma_m K_m K_m^* \Sigma_m^{-1} + \Sigma_m L_m L_m^* \Sigma_m^{-1} \rightarrow XZ + 0 \cdot T = XZ,$$

we deduce  $XZ = I_r$ , hence  $X$  is nonsingular and the first representation in (10) and Theorem 1.3 yields that  $A$  is EP.

(ii)  $\Rightarrow$  (i): If  $A$  is EP, then  $AA^\dagger = A^\dagger A$ . Now, equality (10) implies

$$XZ + YT = ZX, \quad 0 = ZY, \quad 0 = TX, \quad 0 = TY. \quad (12)$$

Moreover, since  $I_r = \Sigma_m K_m K_m^* \Sigma_m^{-1} + \Sigma_m L_m L_m^* \Sigma_m^{-1}$ , making  $k \rightarrow \infty$  we get  $I_r = XZ + YT$ . Thus, the first equality of (12) leads to  $ZX = I_r$ . Let us remark that both matrices  $Z$  and  $X$  are square, thus  $X$  and  $Z$  are nonsingular and from the second equality of (12) we get  $Y = 0$ . The proof is finished  $\square$

Let  $\{A_m\}_{m=1}^\infty \subset \mathbb{C}_{n,n}$  be a convergent sequence to  $A$ . Let  $\phi_m$  the greatest canonical between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$ . If  $\phi_m \rightarrow 0$ , it is reasonable to think that the canonical angles between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  are 0, or according to Theorem 2.4,  $A$  is EP. But, the following example shows that this intuition is false:

**Example 5.** Let  $\{a_m\}_{m=1}^\infty$  be a sequence of numbers in  $]0, 1[$  converging to 1, and define  $b_m = \sqrt{1 - a_m^2}$ ,

$$K_m = \begin{bmatrix} 0 & a_m \\ a_m & 0 \end{bmatrix}, \quad L_m = \begin{bmatrix} 0 & b_m \\ b_m & 0 \end{bmatrix}, \quad \Sigma_m = \begin{bmatrix} 1 & 0 \\ 0 & 1/m \end{bmatrix}, \quad A_m = \begin{bmatrix} \Sigma_m K_m & \Sigma_m L_m \\ 0 & 0 \end{bmatrix}.$$

It is straightforward to prove  $K_m K_m^* = a_m^2 I_2$ , and thus  $K_m$  has only one singular value, namely  $a_m$ . Moreover,  $L_m L_m^* = b_m^2 I_2$ , and therefore  $K_m K_m^* + L_m L_m^* = I_2$  holds. By Theorem 2.3, the angle  $\arccos(a_m)$  is the unique canonical angle between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$ . Using the notation

of Theorem 3.7, one has  $\phi_m = \arccos(a_m)$ . Since  $a_m \rightarrow 1$  we get  $\phi_m \rightarrow 0$ . However, the sequence  $\{A_m\}_{m=1}^{\infty}$  converges to

$$A = \begin{bmatrix} \Sigma K & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where } K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

However, matrix  $A$  is not EP because if  $A$  were EP, then matrix  $\Sigma K$  would be also EP; but since

$$\Sigma K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$\Sigma K$  is not EP. In fact  $\Sigma K$  is co-EP $_{\perp}$ . Observe that this last statement does not contradict Theorem 2.4.

The following result does not require the continuity of the Moore–Penrose inverse. Let us remark that any Hermitian matrix is EP.

**Theorem 3.8.** *Let  $\{A_m\}_{m=1}^{\infty} \subset \mathbb{C}_{n,n}$  be a convergent sequence to  $A$ . Let  $\phi_m$  the greatest canonical between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$ . If  $\phi_m \rightarrow 0$  and  $P_{\mathcal{R}(A_m)}(A_m - A_m^*)P_{\mathcal{R}(A_m)} \rightarrow 0$ , then  $A$  is Hermitian.*

*Proof.* Let  $r_m = \text{rk}(A_m)$ . Since  $r_m \in \{1, \dots, n\}$  for all  $m \in \mathbb{N}$ , by passing to a subsequence, we can suppose that  $\{r_m\}_{m=1}^{\infty}$  is constant, say to  $r$ . By Lemma 2.2, for every  $m \in \mathbb{N}$ , there exist a unitary matrix  $U_m \in \mathbb{C}_{n,n}$ , a diagonal and positive matrix  $\Sigma_m \in \mathbb{C}_{r,r}$ , and  $(K_m, L_m) \in \mathbb{C}_{r,r} \times \mathbb{C}_{r,n-r}$  such that

$$A_m = U_m \begin{bmatrix} \Sigma_m K_m & \Sigma_m L_m \\ 0 & 0 \end{bmatrix} U_m^* \quad (13)$$

and

$$K_m K_m^* + L_m L_m^* = I_r.$$

Since the subset of unitary matrices is compact and by Lemma 3.6, by passing again to another subsequence, we can suppose that  $\{U_m\}_{m=1}^{\infty}$ ,  $\{K_m\}_{m=1}^{\infty}$ , and  $\{L_m\}_{m=1}^{\infty}$  are convergent sequences to, respectively,  $U, K, L$ . These matrices satisfy

$$UU^* = U^*U = I_n, \quad KK^* + LL^* = I_r. \quad (14)$$

By the proof of Theorem 3.7 (see (11)), since  $\phi_m \rightarrow 0$  we get  $L_m \rightarrow 0$  i.e.,  $L = 0$ . From the second equality of (14) we get  $KK^* = I_r$ . Observe that

$$\begin{bmatrix} \Sigma_m K_m & \Sigma_m L_m \\ 0 & 0 \end{bmatrix} = U_m^* A_m U_m \rightarrow U^* A U,$$

hence  $\{\Sigma_m K_m\}_{m=1}^{\infty}$  and  $\{\Sigma_m L_m\}_{m=1}^{\infty}$  are sequences that converge to  $X$  and  $Y$ , respectively. Moreover we get

$$\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} = U^* A U.$$

Since  $KK^* = I_r$ , in particular  $K$  is nonsingular (recall that  $K$  is a square matrix) and for big enough  $m$ , one has that  $K_m$  is nonsingular. Hence, the continuity of the inverse in the subset composed of nonsingular matrices permits assure that  $K_m^{-1} \rightarrow K^{-1}$ . Now, for big enough  $m$ , we have

$$\Sigma_m = \Sigma_m K_m K_m^{-1} \rightarrow X K^{-1} \quad (15)$$

and  $\Sigma_m L_m \rightarrow X K^{-1} 0 = 0$ . Since  $Y = \lim_m \Sigma_m L_m$ , by the uniqueness of the limit, we get  $Y = 0$ .

Thus, we get

$$A = U \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad A^\dagger = U \begin{bmatrix} X^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (16)$$

Denote  $\Sigma = X K^{-1}$ . Evidently, we have  $X = \Sigma K$ . From (15), matrix  $\Sigma$  is a limit of diagonal and real matrices, hence  $\Sigma$  is also diagonal and real. In particular we get  $\Sigma^* = \Sigma$ .

Using representation (13) we have

$$P_{\mathcal{R}(A_m)}(A_m - A_m^*)P_{\mathcal{R}(A_m)} = U_m \begin{bmatrix} \Sigma_m K_m - K_m^* \Sigma_m^* & 0 \\ 0 & 0 \end{bmatrix} U_m^* \rightarrow U \begin{bmatrix} \Sigma K - K^* \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*$$

From  $P_{\mathcal{R}(A_m)}(A_m - A_m^*)P_{\mathcal{R}(A_m)} \rightarrow 0$  we get  $\Sigma K = K^* \Sigma$ . Now let us prove that  $X$  is Hermitian:

$$X^* = (\Sigma K)^* = K^* \Sigma^* = K^* \Sigma = \Sigma K = X.$$

The representation of  $A$  given in (16) proves that  $A$  is Hermitian.  $\square$

**Example 6.** The following example shows that we cannot remove the condition

$$\lim_{m \rightarrow \infty} P_{\mathcal{R}(A_m)}(A_m - A_m^*)P_{\mathcal{R}(A_m)} = 0$$

in Theorem 3.8. Let

$$A_m = U \begin{bmatrix} 1 & 0 \\ 0 & 1/m \end{bmatrix}, \quad A = U \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Note that  $U$  is unitary. Since  $A_m$  is nonsingular, the maximal angle between  $\mathcal{R}(A_m)$  and  $\mathcal{R}(A_m^*)$  is zero for all  $m \in \mathbb{N}$ . Clearly  $A_m \rightarrow A$ . Now, let us compute the maximal angle between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ : Since

$$P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)} = A A^\dagger A^\dagger A = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 0 \end{bmatrix},$$

Definition 1.1 shows that if  $\phi$  the maximal canonical angle between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$  (in fact is the only canonical angle between  $\mathcal{R}(A)$  and  $\mathcal{R}(A^*)$ ), then  $\cos^2 \phi = 1/2$ , i.e.,  $\phi = \pi/4$ .

Since  $A_m$  is nonsingular for each  $m$ , we have that  $P_{\mathcal{R}(A_m)} = P_{\mathcal{R}(A_m^*)} = I_2$ . Thus

$$P_{\mathcal{R}(A_m)}(A_m - A_m^*)P_{\mathcal{R}(A_m)} = U \begin{bmatrix} 1 & 0 \\ 0 & 1/m \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1/m \end{bmatrix} U^* \rightarrow U \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^* \neq 0.$$

**Example 7.** Let  $\{A_m\}_{m=1}^{\infty}$  be a sequence of matrices converging to  $A$ . The following conditions

(a)  $\lim_{m \rightarrow \infty} A_m^\dagger = A^\dagger$ .

(b)  $\lim_{m \rightarrow \infty} P_{\mathcal{R}(A_m)}(A_m - A_m^*)P_{\mathcal{R}(A_m)} = 0$ .

are independent, as the following examples show.

Let  $A_m = 1/m \in \mathbb{C}_{1,1}$  and  $A = 0$ . Obviously, the sequence  $\{A_m\}_{m=1}^{\infty}$  satisfy condition (b) and does not satisfy condition (a).

Now let  $B \in \mathbb{C}_{n,n}$  be any nonsingular and non Hermitian matrix. For any  $m \in \mathbb{N}$  let us define  $B_m = B$ . The sequence  $\{B_m\}_{m=1}^{\infty}$  satisfy condition (a). However, this sequence does not satisfy condition (b): Observe that  $P_{\mathcal{R}(B_m)} = I_n$  since  $B_m$  is nonsingular, therefore,  $\lim_{m \rightarrow \infty} P_{\mathcal{R}(B_m)}(B_m - B_m^*)P_{\mathcal{R}(B_m)} = B - B^* \neq 0$ .

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