Status of the differential transformation method

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Abstract

Further to a recent controversy on whether the differential transformation method (DTM) for solving a differential equation is purely and solely the traditional Taylor series method, it is emphasized that the DTM is currently used, often only, as a technique for (analytically) calculating the power series of the solution (in terms of the initial value parameters). Sometimes, a piecewise analytic continuation process is implemented either in a numerical routine (e.g., within a shooting method) or in a semi-analytical procedure (e.g., to solve a boundary value problem). Emphasized also is the fact that, at the time of its invention, the currently-used basic ingredients of the DTM (that transform a differential equation into a difference equation of same order that is iteratively solvable) were already known for a long time by the "traditional"-Taylor-method users (notably in the elaboration of software packages –numerical routines– for automatically solving ordinary differential equations). At now, the defenders of the DTM still ignore the, though much better developed, studies of the "traditional"-Taylor-method users who, in turn, seem to ignore similarly the existence of the DTM. The DTM has been given an apparent strong formalization (set on the same footing as the Fourier, Laplace or Mellin transformations). Though often used trivially, it is easily attainable and easily adaptable to different kinds of differentiation procedures. That has made it very attractive. Hence applications to various problems of the Taylor method, and more generally of the power series method (including noninteger powers) has been sketched. It seems that its potential has not been exploited as it could be. After a discussion on the reasons of the "misunderstandings" which have caused the controversy, the preceding topics are concretely illustrated. It is concluded that, for the sake of clarity, when the DTM is applied to ODEs, it should be mentioned that the DTM exactly coincides with the traditional Taylor method, contrary to what is currently written.

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1 Introduction

The differential transformation method (DTM) of Pukhov [1]–[5] and Zhou [6] is frequently presented as a (relatively) new method for solving differential equations[1]. Though based on Taylor series, it would be different from the traditional Taylor (or power) series method presented in usual textbooks as e.g., [7]. This distinction has been the object of a recent dispute [8,9]. Independently of whether this distinction was present in the original ideas[2], it is at least seemingly clearly expressed and often repeated since the second half of the 90's when the DTM has more systematically been used "to solve differential equations" [10]–[16]. For example, in [14] the *"differential transformation technique"*, is presented *"as an extended Taylor series method"* and in [15], where the differential equation referred to is

$$\frac{dx}{dt} = f\left(x,t\right) \ , \tag{1}$$

one can read (in addition):

"*The differential transformation technique is one of the numerical methods for ordinary differential equations. It uses the form of polynomials as the approximation to exact solutions which are sufficiently differentiable. This is in contrast to the traditional high-order Taylor series method, which requires the computation of the necessary derivatives of $f(x,t)$ and is computationally intensive as the order becomes large. Instead, the differential transformation technique provides an iterative procedure to get the high-order Taylor series. Therefore, it can be applied to the case with high order.*" [15, p. 25] (see also [16])

That statement is important since then the idea that the "traditional" Taylor series method would require the explicit calculation of high-order derivatives of $f\left(x,t\right)$ is repeated in many articles on the DTM as if it would be a convincing reason to make a definitive distinction between the two methods.

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[1] An extensive presentation of the DTM is given in section 2.2.

[2] The most frequently cited original works on DTM are inaccessible to me because they are written either in Russian [4,5] or in Chinese [6]. Except those two references, I have systematically solely cited articles written in English and, hopefully, accessible to all.

Before going any further, it is worth recalling what is the Taylor series method [3] :

Definition 1 ((Formal or raw) Taylor series method) *The Taylor series method consists in expressing the solution of (1) as a power series expansion about the initial time* t_0*:*

$$\left.\begin{array}{l} x\left(t\right)=\sum_{k=0}^{\infty}\left(t-t_0\right)^k \frac{1}{k!}\left.\frac{d^k x}{dt^k}\right|_{t=t_0} , \\ x\left(t_0\right)=\alpha_0 \,. \end{array}\right\} \tag{2}$$

in which the derivatives $\left.\frac{d^k x}{dt^k}\right|_{t=t_0}$ *are such that (1) is satisfied order by order in powers of* $(t-t_0)$*. As consequence, the Taylor coefficients of the expansion are completely determined once the initial parameter* α_0 *is fixed.*

Notice that this definition specifies neither the way the derivatives $\left.\frac{d^k x}{dt^k}\right|_{t=t_0}$ are calculated nor the convergent property of the series. Thus it is surely clear to all that the *formal* Taylor series method (described in textbooks such as [7]) requires only that the series of $x\left(t\right)$ be obtained by any means whatsoever. Consequently, it is obvious that if the DTM is a "*procedure to get the high-order Taylor series*" [15,16], then the method used to solve the differential equation is not the DTM but the Taylor series method (at least formally). Consequently to really appreciate the above quotation of [15], it is necessary to understand what actually is the "traditional" Taylor series method. No doubt that for the DTM users, the naive explicit computation of high order derivatives of $f(x,t)$ is an integral part of the "traditional" method. One of the object of the present article is to show that this allegation is false.

In fact, as one may anticipate from the presence of the terms "*technique*" and "*numerical method*", the Taylor series method mentioned in the above quotation of [15], though not explicitly defined, is not the *formal* one [4] . It would be possible that it is a problem of semantics which has generated misunderstanding.

Those words (technique, numerical) have wittingly been used to qualify the DTM. Indeed, some authors would like to see the DTM not as a formal method, but exclusively as a new *numerical* approach:

"... the construction of power-series solutions has been generally thought of as an analytic tool and not as the basis for numerical algorithms. This is changing, and algorithmically constructing power-series solutions to ordinary

[3] For the sake of a convenient writing, extensive descriptions of the "traditional" Taylor series method and of the DTM are postponed to section 2.
[4] Which is the best known formulation of the Taylor method.

differential equations is gaining in popularity. This is now often called the Differential Transform(ation) Method (DTM)." [17]

However, if this view corresponds well to Pukhov's initial aim of "*consider*[ing] *the problem of feasibility of constructing computer-specialized procedures oriented toward automatic solution of Taylor equations*"[1], the actual use of power-series solution as the basis for numerical algorithms is not at all new, see, for example, [18]–[22]. Actually:

Remark 1 ((Numerical) Taylor series method) *Taylor series may be used as a tool to numerically solve the initial value problem associated with (1). To this end the convergence of the Taylor series (2) must be controlled. The most frequently used procedure is the stepwise (or piecewise) procedure described in section 2.1.2.1.*

In this respect of the numerical treatment of ODEs, and considering the high level of numerical development of the traditional Taylor method (see section 2.1.2.3), one is forced to acknowledge that the current use of the DTM cannot be seen as a real technological break-through (despite the huge number of publications on the subject). One may easily verify that, the DTM (as it is currently used) brings no new capability into the numerical treatment of (at least ordinary) differential equations (see section 2). Consequently, to possibly distinguish it from the traditional Taylor series method only remains the way the expansion coefficients are calculated, in apparent accordance with the above quotation of [15].

Unfortunately, when looking at the "old papers" using the Taylor method, in that respect of calculating the terms of the Taylor expansion, one quickly realizes that the basical tricks of the DTM (that transform the differential equation into an algebraic iteration equation, see section 2) were already known and used (for a long time, indeed, see section 2.1.1) without having any need to change the name of the Taylor series method (e.g., see [18,19,21]). Even some efficient procedures of the traditional approach have remained ignored by the DTM users (see section 2.2.2). Several articles that present numerical softwares using the traditional Taylor series method with a view to automatically solve systems of ODEs are still currently published [23]–[33]. Based on an old and well established method (see, e.g. [32]), the authors of these articles have no need to mention the existence of the DTM, though they use exactly the same process! In fact, the reverse should be the rule: the DTM users should cite traditional studies, what they never do, probably because they do not know their existence.

The status of the DTM is made very confused because that aspect of the DTM to be potentially a *computer-specialized procedure* is often ignored by several DTM users. Most authors using the DTM do not implement its numerical ability, they provide the solutions to the problems studied under merely raw Taylor series. When such studies are directed towards systems of pure ODEs, the original contribution of the DTM is practically vacuous since it obviously coincides with extremely simple, to not say trivial, traditional studies (see section 2).

This explains and justifies the recent biting comment [8] in which one may read: "*the DTM is the familiar Taylor series method with a fancy name*" and also:

"*There is no doubt that the DTM is merely the textbook Taylor series approach that some authors have disguised as a fancy discrete transformation.*" [8]

The sharpness of the misunderstanding is well emphasized by the reply [9] to that comment:

"*Several revered authors and reputed journals* [would] *have nurtured this 'misconception' over the years. It* [would be] *a challenge to such authors and journals who have published several works on DTM.*" [9]

It is time to clarify the status of the DTM. This is the tentative object of the present article.

As one may have already foreseen, there are two aspects of the DTM, it may be seen either

(A) as a "*computer-specialized procedure*" (as proposed in [1] for example),

or

(B) as a technique for determining the terms of the raw Taylor[5] series of solutions of various problems (see [13] as an example among several others),

or both, of course.

Concerning interpretation A, it is worth recalling that, in general, the process of solving a differential equation using a Taylor (power) series requires two steps:

(1) Obtening the solution as a formal power series (Taylor series, whatever the technique used to get it).
(2) The eventual implementation of an analytic continuation process when the range of convergence of the direct sum of the series is too limited to

[5] More generally of power series including non-integer power series.

reproduce the solution sought in its expected domain of definition. This procedure may be used:

(a) either, in a numerical routine to solve an initial value problem (or within a shooting method procedure) with fixed initial parameters
(b) or in a semi-analytic approach of, e.g., a boundary value problem (the series is expressed in terms of the unknown –yet undetermined– initial parameters, see e.g., [34]).

Now, it is true that, from its origin, the DTM implicitly involves the possibility of automatically implementing an analytic continuation via a piecewise process (although it is said in terms closer to the numerical version of item 2a, as a multistage numerical method, see for example [1]). This possibility has been exploited by several authors (see [12,14,16,17] and [35]–[40]). In that case, the method is truly different from the raw Taylor series method which, stricto sensu, does not involve any consideration of analytic continuation (see for example [34]). One may then rightly consider the method as being "*an extended Taylor series method*"[14]. But, as explicitly shown in section 2, in the circumstances, the DTM merely uses the "*analytic continuation*" process of the Taylor series, as already Euler did it (see [41]) and which is described in textbooks under the name of "*continuous analytic continuation*" [42, chap. 9] or of "*Taylor algorithm*" [43, p. 267]. This is a technique included for a long time and still currently used in numerical softwares for solving ODEs by the "traditional" Taylor series method (see, [18]–[33] and [44,45]).

There is a more serious problem with the way that the DTM is viewed by some authors. This problem is linked with the interpretation B and the analytic continuation presently in question.

Many studies of ODEs using the DTM have been applied according to interpretation B to so simple problems that the solutions sought were analytical (at least in the range of supposed interest of the independent variable, it is, in particular, the case of the study criticized in [8]). In such circumstances, where no analytic continuation was needed or performed, one has employed the expression DTM (the name of the technique used to get the raw Taylor series) as if it was a new (formal) method in place of the Taylor series method itself. This unfortunate view of the DTM is, by now, so common that when an analytic continuation process of the series has been realized by means of Padé approximants, one has changed the name of the DTM into the *modified* differential transformation method (MDTM or also the "DTM-Padé method") [46]–[51]. Such names are not justified because the DTM enables, in principle[6], the automatic implementation of Padé approximants (see section 2.2.3).

[6] It is possible (see footnote 2) that this eventuality has been forseen in the original presentation of the method, but –to my knowledge– this has never been used by the defenders of the DTM in the current literature, hence the new name of MDTM.

Anyway, in the circumstances, the (formal) method used is merely the Taylor series analytically continued by Padé approximants (see, for example, [34]) a method which has been used in numerical routines for a long time [52].

Even more serious is the recent and repeated proposal of introducing, as a novelty, a piecewise procedure within the DTM and of naming this "new method" again MDTM (with "M" for modified or multistage or multistep) [53]–[58]. The authors of such proposals (and the referees who have allowed their publications) were manifestly unaware of the fact that the procedure was already clearly foreseen in the original DTM (see, for example, [1]) and already implemented by several DTM users [12,14,16,17], [35]–[40], not to mention the previous utilizations (some are cited above) by the traditional-Taylor-method users.

Extensions of interpretation B to problems which are potentially more tricky than a simple system of ODEs are seemingly more interesting (despite the standing formal character of this interpretation). As a matter of fact, the DTM has been used as a technique for determining the terms of the raw power [7] series of solutions of various problems. Its basic ingredients, though not new in the current use, have been presented in such a manner that it has appeared very attractive (see section 2.2). Rid of the original constraining goal of performing finite precision calculations, its implementation, sufficiently easy and clearly presented, has been formally extended to several different kinds of equations. Thus, equations of the following types have been considered using the DTM: integro-differential [59,60], partial-differential [39], [61]–[70], differential-difference [51,71,72], differential-algebraic [73,74], fractional order differential [58], [75]–[90], fuzzy differential [91], q-partial differential [92], and so on. Of course, some of those subjects had already been similarly (i.e. analytically) addressed by Taylor-method users, but the lack of a systematic formalization of the traditional approach could have limited their impact (see [93]–[103], for purely analytical approaches). However, if one considers studies with finite precision goal then, again, the traditional method has reached a higher level of development such as in the treatment of the differential algebraic case –a particular system of ODEs already well treated [23,32], [104]–[107]– or of integral and integro-differential equations (see, e.g., [96,108]). I will not discuss further the use of the DTM in such problems. Nevertheless, it seems that the DTM could deserve its name (as a technique) when it extends the Taylor method to new kinds of expansions (different from a Taylor expansion, see e.g., [109]) and to unusual derivation processes such as in the q-calculus. Nevertheless, in such studies, considering their formal character limited to the obtaining the algebraic iteration equations for the coefficients of the series expansion, it should be clearly stated that the *method* used (to solve the primary equation) remains merely the Taylor (or power) series method.

[7] Including non-integer power series.

Moreover, with interpretation B, the series is calculated only once (and most often it is limited to low orders). Consequently the recourse to an iterative procedure to determine the raw power series is even not necessary. Thus, in this meaning, the recourse to DTM may appear entirely artificial (a "fancy" name). As already said, that version of the DTM is so commonly used that it is by now often presented as "the" DTM. That interpretation, surely, has induced the polemic and the confusions mentioned above. This does however not give more credit to interpretation A, as shown in what follows.

The following section is limited to the problem of solving ODEs (numerically or not). It is explicitly shown that the traditional Taylor method should take precedence over the DTM. It is also indicated why it is not acceptable to modify the name "DTM" when the resulting Taylor series is analytically continued.

2 The Taylor series method vs. the differential transform method

In this section, a comparison is made between the most elaborated current formulations of the Taylor series method and of the DTM when they are applied to solve an initial (or boundary) value problem of ODEs (for recent reviews on the use of the Taylor method see, e.g., [24]–[33] and [110] and for the most elaborated presentations of the DTM see, e.g., [11,12,14,40,111]). Only the one dimensional case of the first order ODE (1) is considered here, the generalization to several variables and to systems of ODEs of higher orders is straightforward.

2.1 The Taylor series method

The recourse to "infinite series" to solve differential equations is an old method which, as mentioned in [41, p. 4], may be attributed to Newton in [112]. Since then, the method has been rediscovered several times (as well illustrated, notably, in [113]). By now, it is named the Taylor series method. Several current papers may be found that present this method in details (for example: [32,33,110], see also [41, pp. 47–49]). The following considerations are thus far from being original.

For the sake of considering practical solutions of (1), and with a view to describe the "traditional" Taylor method, definition 1 given p. 3 must be slightly modified.

With the Taylor series method, the generic solution of (1) is expressed locally as a *truncated* Taylor expansion about a time t_i:

$$\left.\begin{array}{l} x(t) = \sum_{k=0}^{N} (t-t_i)^k X_k + O\left[\left((t-t_i)^{N+1}\right)\right] , \\ x(t_i) = \alpha_i , \\ X_k = \frac{1}{k!} \left.\frac{d^k x}{dt^k}\right|_{t=t_i} , \quad k = 1, 2, \cdots, N . \end{array}\right\} \qquad (3)$$

In the following it is assumed that $t > t_i$, and α_i is supposed to be known either because t_i is the initial time (say t_0) at which the initial condition is provided or because it is an intermediate time value ($t_i = t_0 + i\,h$) reached by a stepwise analytic continuation procedure (see section 2.1.2.1). Although not indicated here and in the following (except where noted), the coefficients X_k depend on t_i. The dependence is implicit via the initial value α_i, but it may also be explicit if the ODE considered is non-autonomous.

As said in the introduction, the method involves two steps: first the obtaining the series for some (high?[8]) value of N, and second a process of analytic continuation of the series. Stricto sensu, the second step is not a part of the formal Taylor series method. But, because it is by now difficult to distinguish between studies based on Taylor series which are formal (getting only the raw series) or numerical (using, in addition, analytic continuations), I temporarily include the two steps in the *traditional* Taylor series method (awaiting to show below why that is justified), reserving the name *raw* or *formal* Taylor series method exclusively to the first step (in accordance with definition 1 p. 3).

2.1.1 Getting the raw Taylor series

The coefficients X_k must be determined such that the ODE (1) be formally satisfied order by order up to $O\left[\left((t-t_i)^N\right)\right]$, one condition[9] ($x(t_i) = \alpha_i$) is required so that the resulting system of N equations for the X_k ($k = 0, 1, \cdots, N$) has a definite solution.

Establishing and solving that system of equations is not the most efficient way of determining the coefficients X_k (in particular this would imply the explicit calculation of "*the necessary derivatives of* $f(x,t)$"). In fact, it is known for a long time by the (thus traditionnal) Taylor-series-method users that the X_k may be determined more efficiently by iteration provided $f(x,t)$ be simple enough (see, e.g. [114,115,116], see also [41, pp. 47–49]). Instead

[8] Actually, it is rarely necessary to consider high values of N, as is shown in the following.
[9] In general the order is $N - p$, with p the order of the ODE, and p conditions are required to unambiguously determine the X_k.

of N equations, only one equation (the iteration equation) has, then, to be effectively considered.

Assuming, as a simple illustration, that $f(x,t) = \lambda\, x(t)$ with λ a constant, it is easy to show that the generic Taylor coefficients X_k satisfy the following iteration equation:

$$\left.\begin{array}{l} X_0 = \alpha_i\,, \\ (k+1)\; X_{k+1} = \lambda\, X_k\,, \quad k = 0, 1, \cdots, \infty\,. \end{array}\right\} \tag{4}$$

This is a difference equation of same order (first) as the ODE under study. With the initial condition, the coefficients X_k are obtained easily by iteration of (4) up to high orders, with e.g.:

$$X_0 = \alpha_i\,, \quad X_1 = \alpha_i \lambda\,, \quad X_2 = \alpha_i \frac{\lambda^2}{2}\,, \quad \cdots, \quad X_N = \alpha_i \frac{\lambda^N}{N!}\,, \quad \cdots.$$

The terms of the Taylor series of an exponential is recognized giving the exact solution (which satisfies the right initial condition):

$$x(t) = \alpha_i\, \mathrm{e}^{\lambda(t-t_i)}\,.$$

Although that example is trivial, it indicates the way towards a more elaborated procedure when $f(x,t)$ is more complicated. The above example may be presented as follows. One constructs a priori a table of correspondence:

$$\begin{array}{rl} x(t) & \rightarrow X_k\,, \\ \frac{dx(t)}{dt} & \rightarrow (k+1)\; X_{k+1}\,, \\ \lambda\, x(t) & \rightarrow \lambda\, X_k\,, \end{array}$$

so that the ODE of interest is automatically transformed into a difference equation:

$$\tfrac{dx(t)}{dt} = \lambda\, x(t) \rightarrow (k+1)\; X_{k+1} = \lambda\, X_k\,.$$

Generalizing to the k^{th} Taylor coefficient of $f(x,t)$ (denoted below by F_k) then it comes:

$$(k+1)\; X_{k+1} = F_k\,. \tag{5}$$

To construct the F_k's, the Taylor-series-method users have considered supplementary obvious (or already known) correspondences such as (see [117, p. 525] but also [118]):

$$\begin{array}{ll} \frac{d^n x(t)}{dt^n} & \to (k+n)\cdots(k+1)\,X_{k+n}\,, \\ x(t)+y(t) & \to X_k+Y_k\,, \\ x(t)\;y(t) & \to \sum_{i=0}^{k} X_i\,Y_{k-i}\,, \\ z(t)=\frac{x(t)}{y(t)} & \to \frac{1}{Y_0}\left(X_k-\sum_{i=0}^{k-1} Z_i Y_{k-i}\right)\,, \\ z(t)=[x(t)]^{\beta} & \to \sum_{i=1}^{k}\left(\frac{\beta+1}{k}i-1\right)\frac{X_i}{X_0}Z_{k-i}, \quad k\geq 1, \quad Z_0=X_0^{\beta}\,. \end{array} \tag{6}$$

One has then considered more complicated forms of $f(x,t)$ such as exponential (similar to $[x(t)]^{\beta}$), logarithm, trigonometric functions etc...[18,19] (see also [41, pp. 48, 49]) so as to find similar simple formulas for their images. The aim was, notably, to include them into numerical softwares [119] with a view to solve automatically various kinds of systems of ODEs [18,19,120].

As noted in [22], such recurrence schemes for the terms of a Taylor series in solving a differential equation was used as early as 1946 by J.C.P. Miller [114]. Some authors (see, e.g. [121]) attribute the first use of such recurrences to the computation of derivatives by J.R. Airy in 1932. Actually, the iterative procedure was known already at the time of Newton (see, [112, Prob. II, Sol. Case II, Ex.I, p. 33] and also, e.g., [122, p. 116]), and the last line of (6), often attributed to J.C.P. Miller [117, p. 507 of second ed.] and [123], had been established by L. Euler in 1748 [124], cf [117, p. 526]. Such a formula is the result of the application of the Leibnitz rule for the derivative of a product of functions. This rule, and others, is at the basis of an efficient procedure for numerically calculating high order derivatives named automatic differentiation (for extensive bibliographies, see [125,126]), which is used in turn to get efficient iterative formulas for complicated forms of $f(x,t)$. This technique is used for a while in numerical routines [23], [25]–[31], [33,121], [127]–[130] and in regular studies (e.g., [45]). More recently, similar considerations have led to a proposal [131] for efficiently implementing the Picard iteration method.

It is worth indicating that such recurrences (and others) have been established also for functions of two variables (see e.g., [20] for analytical expressions) and that the automatic differentiation process in many variables is also well developed for numerical applications (see, e.g., [132,133,134]).

In the following, the name "Taylor transformation" will designate the above described automatic procedure for obtaining the iteration equation (the image) of a given ODE (by the way, it was the primary name of the DTM [1] a name rightly re-used in the first papers of the 90's [10,11,12]).

It is important to keep in mind that, up to now, the above rules of calculation of the terms of the Taylor series is purely formal. In particular it does not anticipate the nature of computation (theoretical exact or finite precision) which is to be done. In the following subsections, the considerations are less formal and are progressively more and more oriented towards numerical applications.

2.1.2 Analytic continuations

Sometimes the generic Taylor series is convergent in the domain of definition of interest (such as in the preceding trivial case of the exponential). The (formal) method then provides an exact solution (when $N = \infty$ and if the generic term is identified) or, at least, a convergent form towards the exact solution (for N finite). Those simple cases, often linked to linear ODEs, are not of great interest.

More interesting are the nonlinear ODEs, the generic Taylor series of the solutions of which have finite radius of convergence due to the presence of singularities in the complex plane of the independent variable t. The efficiency of the Taylor expansion thus is limited to a finite domain (which may be small) about the expansion point t_i. To enlarge this domain, the recourse to analytic continuation procedures is often required. (Notice however that in practice, one may need to use analytic continuation even for series with an infinite radius of convergence, if only to reduce the length of the series.)

There are several kinds of analytic continuations. Let us consider only the two most often used in conjunction with the Taylor series method.

2.1.2.1 Stepwise procedure The stepwise procedure, also named the "*continuous analytic continuation*" [42, chap. 9] or the "*Taylor algorithm*" [43, p. 267], is particularly interesting because, beyond its character of analytic continuation procedure, it is structurally well adapted to numerical routines.

Let the domain of definition of the solution sought, say $t \in [t_0, t_0 + H]$, be cut in $n \in \mathbb{N}$ pieces: $H = n\,h$. Then the times of reference read $t_i = t_0 + i\,h$, $i = 0, 1, \ldots, n$. If h is chosen sufficiently small (at each step, smaller than the distance to the closest complex singularity of the solution), then the Taylor series converges in the domain $[t_i, t_{i+1}]$, and its sum provides the solution in this domain with a sensible accuracy even if N is not very large. If this is true for each value t_i then one gets, step by step, an approximate solution in the whole domain $[0, H]$, each initial value α_i being (approximately) provided by the sum of the Taylor series at the second boundary of the preceding subdomain, namely

$$\alpha_i = \sum_{k=0}^{N} h^k \left(X_k\right)_{(i-1)} + O\left(h^{N+1}\right) , \tag{7}$$

in which I have indicated the dependence of X_k on the step $i-1$.

This stepwise procedure has been (and is currently) successfully used in numerical routines by Taylor-series-method users to solve initial value problems (for reviews see, e.g. [24,32,33]). Its first use in conjunction with power series goes back to Euler [10] [41]. For that reason, it is justified to consider it as being an integral part of the traditional Taylor series method as the name "*Taylor algorithm*" [43, p. 267] suggests it.

The procedure has also been used to solve boundary value problems [16,23,120,135]. In that case, the unknown α_0 of the initial boundary is transmitted, step by step via the parameters α_i, up to the second boundary $t_0 + H$ where the desired condition is imposed. The value of the unknown is chosen among the acceptable solutions of the resulting equation. The method is limited by two aspects:

(1) The dependence of α_i on the unknown becomes rapidly very complicated. (However, see [136,137] in which supplementary Taylor expansions in powers of the initial unknown are used as generally done when using the Taylor method in a sensitivity analysis, see e.g., [133,134].)
(2) When $H \to \infty$ the method no longer works because, when $t \to \infty$ the truncated Taylor series goes to $\pm\infty$ according to the sign of the last term

In the last case, where the domain of definition of the problem is infinite, it is better to consider other kinds of analytic continuation such as, for example, rational fractions (Padé approximants).

2.1.2.2 Padé approximants The use of Padé approximants is well known [138,139,140], and will not be described here. It suffices to recall that it is a procedure that transforms a truncated series (a polynomial) into a ratio of polynomials. It is a so drastic modification of the Taylor series of the generic solution that it cannot be easily seen as the "inverse" of an "image" resulting from some "modified" Taylor transformation. However, it is shown in section 2.2.3 that, similarly to the step-wise procedure, the recourse to Padé-approximant seems to have been also foreseen in the DTM but, seemingly, has never been implemented in the current literature.

2.1.2.3 High level numerical development There is no doubt that, as a numerical tool, the traditional Taylor series method has reached a high level of developement. I have already mentioned several numerical routines that

[10] Euler's motivation was not to perform an analytic continuation but, precisely in opposition to the statement of [15] quoted in the introduction, to avoid the calculation of high-order derivatives of $f(x,t)$.

have been developed with a view to automatically solve the initial (and boundary) value problems of systems of ODEs [23]–[33]. I must mention also the existence of adjustments of the Taylor method to treat stiff systems [23],[25]–[32], [141]–[145] and the possibility that this method offers to get high-precision solution for ODE [146]. In the same vein, it is worth mentioning the use of Taylor series methods in interval (or validated) solutions, that provide a garantee of existence of (and bounds on) the solution in a given interval [129,130], [147]–[151] (for a bibliography on the subject of validated solutions see, e.g., the paper by Nedialkov et al in [129,130]). As an extension to those latter studies, it is fair to mention also the developement of new methods for obtaining garanteed bounds on the expansion [136], [152]–[154], for a review see Makino and Berz in [155].

In this respect of the quality of the numerical treatment of ODEs the difference is glaring with studies using the DTM in which almost never are raised questions like, e.g., selecting the stepsize (or the order of development) or discussing the computational complexity of the system considered. In particular, despite the original Pukhov's aim, I have not seen any paper that proposes a numerical routine with a view to automatically solve systems of ODEs nor a possible adaptation of the DTM to treat stiff systems.

2.2 The differential transform method

In this section the most elaborated version [11] of the DTM and its usual version are presented and compared with the Taylor series method presented in the preceding sections.

2.2.1 General procedure

The DTM [1]–[6] is a formalized modified version of the Taylor transformation described in section 2.1. The basic difference is that instead of X_k defined in (3), one considers the image $\breve{X}_k$ of $x(t)$ defined as [11,12,14,40,111]:

$$\breve{X}_k = M_k \left. \frac{d^k \left(q\left(t \right) x\left(t \right) \right)}{dt^k} \right|_{t=t_i} , \tag{8}$$

in which the auxiliary function $q(t)$ and the infinite set M_k are chosen a priori. M_k is called the weighting factor and $q(t)$ is named the kernel corresponding to $x(t)$ (see, e.g. [111]).

[11] Accessible to me, see footnote 2.

One recovers $x(t)$ through the following series (the inverse transform):

$$x(t) = \frac{1}{q(t)} \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\breve{X}_k}{M_k} . \tag{9}$$

2.2.2 Usual procedure

To my knowledge (see footnote 2), the freedom in the choice of $q(t)$ has never been exploited in the current literature, where it is usually set equal to one (for possible uses of $q(t)$, see section 2.2.3.1). Moreover, the choice of M_k is currently limited exclusively to the two following cases (except in [40] where an adaptive grid size mechanism is employed):

- $M_k = \frac{1}{k!}$, then $\breve{X}_k \equiv X_k$ (since $q(t) \equiv 1$), the DTM coincides with the Taylor transformation and the method (to solve a differential equation) to the formal Taylor method (without any analytic continuation –i.e., the interpretation B mentioned in the introduction).
- $M_k = \frac{h^k}{k!}$ as proposed in [1,3], where h has the same meaning as in section 2.1.2.1 and is effectively used in the same way (see, e.g. [3]). One has $\breve{X}_k = h^k X_k$, and eq. (7) is slightly simplified:

$$\alpha_i = \sum_{k=0}^{N} \left(\breve{X}_k\right)_{(i-1)} + O\left(h^{N+1}\right) . \tag{10}$$

  The drawback is that one must account for the presence of h in the definitions of the images such as, e.g.

$$\begin{aligned} &x(t) \to \breve{X}_k , \\ &\tfrac{dx(t)}{dt} \to \tfrac{(k+1)}{h} \breve{X}_{k+1} . \end{aligned}$$

  It is easy to verify that one obtains the same result by merely rescaling the time variable $t \to h\,t$ within the traditional Taylor approach. Although, in practice, there may be advantages one way or the other [12], this is too little to justify a drastic change of designation of the method used.

For the remaining aspects of the DTM, that is to say the way that $f(x,t)$ is treated, there is no difference from the Taylor transformation described in section 2.1.1. Only simple forms of $f(x,t)$ are considered, and each case is treated separately. Finally a table of correspondence similar to (6) is constructed.

[12] Indeed, this form has been included in some routines of Taylor-series-method users to reduce underflow and overflow in practical computation by choosing h of the order of the radius of convergence of the series [22].

It is important to mention that, concerning the construction of the table of correspondence, DTM users have recently been reminded of the existence of the exponentiation recurrence [last line of (6)] with a view to improve the efficiency of their method [156]. The same kind of remark had already been made some time before in [157,158] where the recourse to the Leibnitz rule of derivation is used to get the image of the exponential or logarithm functions through "*new algorithms*" [13] which are more efficient than those of the standard DTM. These proposals for improving the DTM clearly contradict the claim quoted in the introduction that the traditional Taylor series method would be *"computationally intensive as the order becomes large"* compared to the DTM since, for a long time, those recurrences were known (see, e.g. [118] and section 2.1.1) and already included in routines to solve ODEs by Taylor series (see, e.g. [20,22]). These late propositions for improving the DTM, and the succes of the DTM itself, show enough that, as the traditional Taylor method, the automatic differentiation procedure [125]–[130] is largely unrecognized in the current litterature, this is a pity.

2.2.3 The modified differential transform methods

2.2.3.1 Modification by Padé approximants When the domain of definition of the solution sought is infinite ($H = \infty$), some authors have proposed substituting for the stepwise procedure foreseen in the DTM by using Padé approximants [46]–[51]. To this end one first chooses $M_k = \frac{1}{k!}$, whereas $q\,(t)$ is still set equal to one (i.e. the raw Taylor series is obtained first). One then transforms the Taylor series (the inverse image) into a rational fraction (according to the rule of Padé calculus [138]–[140]). Now, because the recourse to Padé approximants (eventually after a Laplace transform of the series [159,160]) has been implemented at the level of the inverse of the image of $x\,(t)$, then this is no longer a differential transformation (modified or not).

In fact, eq. (9) shows that the DTM enables the automatic implementation of the Padé-approximant technique. Indeed, it suffices to decide that $q\,(t)$ is a polynomial in $(t - t_i)$ of order N_2 (with coefficients $Q_k,\ k = 1, \cdots, N_2$ to be determined and $Q_0 = 1$) and to limit the $\breve{X}_k$-series to be of order N_1 with $N_1 + N_2 = N$, to get a balanced system of equations for the Q_k and $\breve{X}_k$ expressed directly in terms of the "image" of $f\,(x, t)$ and of the initial value α_i. To be concrete, the system of equations for the Q_k's and $\breve{X}_k$'s is:

$$\begin{aligned} &\textstyle\sum_{i=0}^{N_2} Q_i\, X_{k-i} = 0, &&k = N_1 + 1, \cdots, N_1 + N_2\,, \\ &\breve{X}_k = \textstyle\sum_{i=0}^{N_2} Q_i\, X_{k-i}, &&k = 0, \cdots, N_1\,, \end{aligned} \tag{11}$$

in which $Q_0 = 1$ and the X_k's are implicitly known from (5) in terms of the

[13] Already known by the users of the traditional-Taylor-series method.

initial value α_i.

Notice that, because the first line of (11) is a system of N_2 linear equations which cannot be solved iteratively, it is necessary to determine the X_k's explicitly. Consequently, the process is not more efficient than applying the Padé-technique on the Taylor series and, in any way, the method used remains merely the Taylor series analytically continued by Padé approximants.

An other possible use[14] of $q(t)$, in the case of an infinite value of the range H, could be as follows. Sometimes one knows that, when $t \to \infty$, the solution sought behaves as t^{ν} with $\nu \in \mathbb{Q}$, and this behavior is hardly reproduced by a Taylor series even continued by a Padé approximant. To circumvent this difficulty, one may choose (assuming $t_i = 0$):

$$q(t) = \left(1 + \frac{t}{r}\right)^{-\nu} \tag{12}$$

in which r is a parameter to be adjusted in order that the radius of convergence $|r|$ of the series expansion of (12) be larger than that (a priori unknown) of the Taylor series of $x(t)$. With such a choice for $q(t)$, the behavior at infinity of the transformed function is a simple constant reproducible by a diagonal Padé approximant.

2.2.3.2 Modification by the stepwise procedure I have already expressed, in the introduction, what must be thought of the proposal of improving the DTM by introducing a stepwise analytic continuation of the Taylor series, when this procedure was already clearly foreseen in the DTM and used several times as recalled in section 2.2.2.

2.2.4 Formalization

With a bit of emphasis, the interpretation B of the DTM has been presented on the same footing as the Laplace, Fourier, and Mellin transforms where the integration procedure used to construct the image of a given function is replaced by a derivation procedure [3] (see also [161]). The inverse transformation gives a power series (when $q(t) = 1$ and $M_k = \frac{1}{k!}$, it is merely the Taylor series of the function). By similitude with the Laplace etc. transformations, a list of properties of the DTM with respect to the addition, multiplication, etc. has been given, as well as the images of current functions. The resulting transformation of, say, an ODE into an algebraic equation easily solvable, has been presented as being formally equivalent to that of the Laplace (or

[14] Notice also that M_k could be used to implement a Borel transform of the Taylor series of $x(t)$.

Fourier) transform with the additional advantage that the DTM applies also to nonlinear ODEs.

Of course, the depth of the considerations is lesser than in the traditional Taylor series approach. But this latter has been almost exclusively developed as a numerical tool. This ability of the Taylor series method being not very well publicized, the formal presentation of the DTM has appeared (wrongly) as a new and elegant way of constructing a table of correspondence like (6).

As emphasized enough in the present article, given the high level of developement of the traditional Taylor series method (as a numerical tool), the DTM contributes nothing new in the way systems of ODEs are solved. Nevertheless, the DTM has allowed an easy generalization of the Taylor method to various derivation procedures. For example, fractional differential equations have been considered using the DTM extended to the fractional derivative procedure [58], [75]–[90] via a modified version of the Taylor series [109]. In similar situations –such are also fuzzy differential [91], q-differential [92], etc.– the DTM might deserve its name (although traditional Taylor series treatments also exist that deal with fractional derivative [103] or fuzzy equations [101,102]).

Less evident is the original contribution of the DTM to the study of partial differential, integro-differential, integral, etc. equations where the usual derivative procedure is in action (e.g., integral and integro-differential equations have been studied by Taylor series methods in [93]–[98], [108]).

In general, it would be fair to not forget the limit of the formal version of the DTM which is not a method to solve the initial problem but only a convenient way of obtaining the iteration equation for the power series coefficients (provided the rules of the automatic derivation be well accounted for). For the sake of clarity, in such situations, a clear reference to the traditional Taylor (or power) method should be mandatory.

3 Conclusion

A misunderstanding on what the "traditional" Taylor series method is has grown out since the 90's of a desire to promote an attractive method named the differential transformation method. Actually, when the DTM was being born, the Taylor series method had developed for quite a while the treatment of ODEs exactly on the same basis and, concerning this treatment, absolutely nothing new has originated from the new method. Even some efficient recurrence formulas (known for a while) have been unknown to the DTM users.

The misunderstanding has increased over time because the DTM (and also the Taylor series method, of course) has two possible uses: as an analytic tool or as a numerical tool. Although it was created with this latter use in mind, the DTM has most often been used as an analytic tool, so that some authors have even forgotten the initial aim and proposed to reinvent the method.

Beyond the treatment of ODEs with the DTM –in which case one should at least clearly refer to the numerous studies done with the Taylor series method– it seems that the major contribution of the DTM is in the easy generalization of the Taylor method (either analytical or numerical) to problems involving unusual derivative procedures such as fractional, fuzzy or q-derivative.

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