Rank-one Solutions for Homogeneous Linear Matrix Equations over the Positive Semidefinite Cone

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Abstract. The problem of finding a rank-one solution to a system of linear matrix equations arises from many practical applications. Given a system of linear matrix equations, however, such a low-rank solution does not always exist. In this paper, we aim at developing some sufficient conditions for the existence of a rank-one solution to the system of homogeneous linear matrix equations (HLME) over the positive semidefinite cone. First, we prove that an existence condition of a rank-one solution can be established by a homotopy invariance theorem. The derived condition is closely related to the so-called $P_\emptyset$ property of the function defined by quadratic transformations. Second, we prove that the existence condition for a rank-one solution can be also established through the maximum rank of the (positive semidefinite) linear combination of given matrices. It is shown that an upper bound for the rank of the solution to a system of HLME over the positive semidefinite cone can be obtained efficiently by solving a semidefinite programming (SDP) problem. Moreover, a sufficient condition for the nonexistence of a rank-one solution to the system of HLME is also established in this paper.

Key words. Linear matrix equation, semidefinite programming, rank-one solution, rank maximization, homotopy invariance theorem, $P_\emptyset$-function.

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1 Introduction

Let R^n denote the n-dimensional Euclidean space, with the standard inner product, and let S^n denote the set of real symmetric matrices. For a given $A \in S^n$, $A \succeq 0$ ($\succ 0$) means that A is positive semidefinite (positive definite). For two $n \times n$ matrices X and Y, $\langle X, Y \rangle = \mathrm{tr}(X^T Y)$ denotes the inner product of X and Y, where $\mathrm{tr}(\cdot)$ stands for the trace of a square matrix. We use $\|X\|$ and $\|X\|_*$ to denote the spectral norm and the nuclear norm (i.e., the sum of singular values), respectively, of matrix X.

Stimulated by the recent work on compressed/compressive sensing (e.g. [22, 21, 14, 15]), the study of finding a low-rank solution to optimization problems with linear matrix (in)equality constraints has recently become intensive [43, 50]. Many practical problems across disciplines (such as sparse signal recovery [21, 15], system control [23, 9, 29, 25, 26], matrix completion [12, 13], machine learning [1, 35], quadratic equation [50], Euclidean distance geometry [47, 44, 17] and combinatorial optimization [2]) can be formulated as the following problem:

$$\min \{\mathrm{rank}(X) : \ \mathcal{A}X = b, \ X \in C\},$$

where X is an $n \times p$ matrix, b is a vector in R^m, $\mathcal{A} : R^{n\times p} \to R^m$ is a linear operator, and $C \subseteq R^{n\times p}$ is a convex set. Various heuristic methods for such problems have been proposed and investigated (e.g. [25, 43, 34, 46, 3]). Among all low-rank solutions, a rank-one solution is particularly useful in many situations, especially in system control and quadratic optimization [23, 9, 29, 2, 49]. Locating a low-rank solution (especially a rank-one solution) is not only motivated by these practical applications, but also motivated naturally by the structure of the solution set of semidefinite programming (SDP) problems. In fact, an optimal solution of an SDP problem, if attained, usually lies on the boundary of its feasible set. Note that the nontrivial extreme rays of the positive semidefinite cone are generated by rank-one matrices.

However, for a given system of linear matrix equalities, a rank-one solution with a desired structure does not always exist. In this paper, we focus on the existence issue of a rank-one solution to the homogeneous system:

$$\langle A_i, \ X \rangle = 0, \ i = 1, \ldots, m, \ X \succeq 0, \tag{1}$$

where $A_i \in S^n, i = 1, \ldots, m$, are given $n \times n$ matrices. This system is referred to as the *homogeneous linear matrix equations (HLME) over the positive semidefinite cone.* We aim at addressing the following fundamental question: When does system (1) possess a rank-one solution? The main motivation to the study of the system (1) is that it is closely related to the system of quadratic equations (see e.g., [50]):

$$x^T A_i x = 0, \ i = 1, \ldots, m, \ x \in R^n. \tag{2}$$

Hence it is closely linked to various linear algebra and optimization topics, such as the simultaneous diagonalizability of a given set of matrices $(A_1, \ldots, A_m)$, the convexity property of the field of values associated with a finite number of matrices [10, 32, 31, 30], and the widely used 'S-Lemma' or 'S-Procedure' in system control and optimization [8, 9, 6, 41, 42].

In this paper, we investigate the existence of a rank-one solution to the system (1) from the viewpoint of nonlinear analysis and rank optimization. Clearly, the system (1) has a rank-one solution if and only if (2) has a nonzero solution. Thus, the existence of a rank-one solution of the system (1) is equivalent to that of a nonzero solution of the system (2). Except for some special cases, however, a general and complete characterization of the existence condition for a rank-one solution to the system (1) remains open (see, for instance, the open "Problem 12" and

"Problem 13" in [30]). The study of (2) can date back to 1930s. Thanks to the early work of Dines [18, 19, 20], Brickman [10], Calabi [11], and the work of Finsler [27], a complete characterization of the system (2) with $m = 2$ and $n \geq 3$ is clear: *$x = 0$ is the only solution to the system $x^T A_1 x = 0$ together with $x^T A_2 x = 0$ if and only if $t_1 A_1 + t_2 A_2 \succ 0$ for some $t_1, t_2 \in R$.* A good survey on the historical development of this result can be found in [48, 31, 42]. This result is closely related to S-Lemma/S-procedure [41], and related to the approximate S-lemma and low rank issues discussed in [5]. It can be related to the trust region subproblem in nonlinear optimization as well (see e.g. [45, 37]).

Let us restate the above classical result as follows, in terms of rank-one solutions to the system of HLME over the positive semidefinite cone. We call it "Dines-Brickman's Theorem".

Theorem 1.1 (Dines-Brickman) *When $m = 2$ and $n \geq 3$, the system*

$$\langle A_1, X \rangle = 0, \ \langle A_2, X \rangle = 0, \ X \succeq 0 \tag{3}$$

has a rank-one solution if and only if $t_1 A_1 + t_2 A_2 \not\succ 0$ for any $t_1, t_2 \in R$, in other words,

$$\max_{t_1, t_2} \ \{ \operatorname{rank}(t_1 A_1 + t_2 A_2) : \ t_1 A_1 + t_2 A_2 \succeq 0 \} \ \leq \ n - 1.$$

Unfortunately, such a complete characterization does not hold in general for $m \geq 3$. The first purpose of this paper is to establish a general sufficient condition for the existence of a rank-one solution to the system (1) by using a homotopy invariance theorem. To this end, we introduce a class of functions called $P_\emptyset$, and show that the system (1) has a rank-one solution if the quadratic image function has a $P_\emptyset$ property. This analysis is the first to link $P_\emptyset$-functions and the existence of a rank-one solution to the system of HLME over the positive semidefinite cone. The second purpose of this paper is to develop an existence condition for the rank-one solution of (1) via a rank optimization approach. To this end, we introduce the following rank maximization problem

$$r^* = \max_{t_1, \dots, t_m \in R} \left\{ \operatorname{rank} \left(\sum_{i=1}^m t_i A_i \right) \ : \ \sum_{i=1}^m t_i A_i \succeq 0 \right\},$$

which turns out to be an important factor for the existence of a rank-one solution of the system (1) (this has already been observed in the aforementioned Theorem 1.1). In section 3, we also point out that the value of r^*, combined with Barvinok-Pataki's bound [4, 40], can be used to determine an upper bound for the rank of the solution of (1).

It is also important to understand when the system (1) does not have a rank-one solution. In other words, we study the question: When is $x = 0$ the only solution to the quadratic system (2)? This was posted as the open "Problem 13" in [30]. For $m \geq 3$, the well-known condition $\sum_{i=1}^m t_i A_i \succ 0$ does imply that $x = 0$ is the only solution of the quadratic system (2) (and hence the system (1) has no rank-one solution). However, this condition is too strong (see Lemma 3.2 for details). This motivates us to investigate the above-mentioned open question, and to develop another sufficient condition for the nonexistence of a rank-one solution to the system (1) from a nonlinear analysis point of view.

This paper is organized as follows. In section 2, we develop a sufficient condition for the existence of a rank-one solution to (1) by using a homotopy invariance theorem. In section 3, we establish other sufficient conditions by means of rank optimization. A nonexistence condition for the rank-one solution of (1) is provided in section 4, and conclusions are given in the last section.

2 Existence of a rank-one solution: homotopy invariance

Note that any rank-one matrix in a positive semidefinite cone must be of the form $X = xx^T$ with $x \neq 0$. Thus the system (1) has a rank-one solution if and only if there is an $x \neq 0$ such that $\langle A_i, xx^T \rangle = 0$, $i = 1, \ldots, m$, which is nothing but the system (2). By homogeneity, this is equivalent to saying that the system (1) has a rank-one solution if and only if there is a solution to the system

$$x^T A_i x = 0, \ i = 1, \ldots, m, \ x^T x = 1, \tag{4}$$

which has $m+1$ equations and n variables. It is worth mentioning that the system (4) is related to the structure of the set $\{x : \ x^T A_i x \leq 1, \ i = 1, \ldots, m\}$ which has been considered by Nemirovski, Roos and Terlaky [36]. For instance, if (4) has no solution, then the set $\{x : \ x^T A_i x \leq 1, \ i = 1, \ldots, m\}$ contains only a single point $x = 0$.

In this section, we assume that $m + 1 \leq n$. Let $\Phi : R^n \to R^{m+1}$ be the mapping defined by

$$\Phi(x) = (x^T A_1 x, \ \cdots, \ x^T A_m x, \ x^T x - 1)^T. \tag{5}$$

When $m + 1 < n$, we may introduce the extra matrices $A_{m+1} = \cdots = A_{n-1} = 0$ into (1) without any change of the system. Thus, without loss of generality, we assume that $m + 1 = n$ in the remainder of this section so that $\Phi(x) = 0$ consists of n equations with n variables. An immediate observation is given as follows.

Lemma 2.1. *The system (1) has a rank-one solution if and only if $\Phi(x) = 0$ has a solution.*

This observation combined with the next result (homotopy invariance theorem of topological degree) is a key to developing our first existence result. For a bounded subset D of R^n, $\overline{D}$ and ∂D denote the closure and the boundary of D, respectively. Let f be a continuous function from $\overline{D}$ into R^n. For $y \in R^n$ such that $y \notin f(\partial D) = \{f(x) : x \in \partial D\}$, $\deg(f, D, y)$ denotes the topological degree associated with f, D and y, which has been widely used in the existence analysis of nonlinear equations (see, [39, 33, 38]). The following result is called the homotopy invariance theorem. Part (i) is due to Poincaré and Bohl, while part (ii) is attributed to Kronecker [39, Theorems 6.2.4 and 6.3.1].

Lemma 2.2. (Poincaré-Bohl-Kronecker) *Let $D \subset R^n$ be a nonempty open bounded set and F, G be two continuous functions from $\overline{D}$ into R^n. Let $H(x,t) = tG(x) + (1-t)F(x)$, $0 \leq t \leq 1$, and let y be an arbitrary point in R^n.*

(i) *If $y \notin \{H(x,t) : x \in \partial D$ and $t \in [0,1]\}$, then* $\deg(G, D, y) = \deg(F, D, y)$.

(ii) *If $y \notin F(\partial D)$ and $\deg(F, D, y) \neq 0$, then the equation $F(x) = y$ has a solution in D.*

First, we prove the following technical result.

Lemma 2.3. *If $\Phi(x) = 0$ has no solution, where Φ is defined by (5), then there exists a sequence $\{x^k\} \subset R^n$ such that $\|x^k\| \to \infty$ as $k \to \infty$, and for each k there exists $\gamma^k \in (0, \infty)$ such that $\Phi(x^k) = -\gamma^k x^k$, i.e., x^k satisfies the following relations:*

$$(x^k)^T A_i x^k = -\gamma^k x_i^k, \quad i = 1, \ldots, m, \tag{6}$$

$$(x^k)^T x^k = 1 - \gamma^k x_n^k, \tag{7}$$

where x_j^k denotes the jth component of x^k.

Proof. Suppose that $\Phi(x) = 0$ has no solution. Consider the homotopy between the identity mapping and $\Phi(x)$, i.e.,

$$\mathcal{H}(x,t) = tx + (1-t)\Phi(x) = \begin{pmatrix} tx_1 + (1-t)x^T A_1 x \\ \vdots \\ tx_m + (1-t)x^T A_m x \\ tx_n + (1-t)(x^T x - 1) \end{pmatrix} \in R^n.$$

Let

$$\mathcal{Q} = \{x \in R^n : \ \mathcal{H}(x,t) = 0 \text{ for some } t \in [0,1]\}.$$

First, we prove that $\mathcal{Q}$ is unbounded. In fact, if $\mathcal{Q}$ is bounded, then there exists an open bounded ball $D \subset R^n$ that contains the set $\mathcal{Q}$, and D can be chosen large enough so that the boundary ∂D does not touch the set $\mathcal{Q}$, i.e., $\partial D \bigcap \mathcal{Q} = \emptyset$. By the definition of $\mathcal{Q}$, we deduce that

$$0 \notin \{\mathcal{H}(x,t) : \quad x \in \partial D, \quad 0 \le t \le 1\}.$$

(Indeed, if 0 is an element of the above set, then $\mathcal{H}(x,t) = 0$ for some $x \in \partial D$ and $t \in [0,1]$, which implies $\partial D \bigcap \mathcal{Q} \neq \emptyset$, a contradiction.) Thus, $\deg(I, D, 0)$ and $\deg(\Phi, D, 0)$ are well defined, and by Lemma 2.2 (i), we have

$$\deg(\Phi, D, 0) = \deg(I, D, 0).$$

For the identity mapping, we have $|\deg(I, D, 0)| = 1$. Thus, we have $\deg(\Phi, D, 0) \neq 0$, which along with Lemma 2.2 implies that there exists a solution to $\Phi(x) = 0$, contradicting the assumption. So $\mathcal{Q}$ must be unbounded, and there exists an unbounded sequence $\{x^k\}$ in $\mathcal{Q}$. Without loss of generality, let $x^k \neq 0$ for all k, and $\|x^k\| \to \infty$ as $k \to \infty$. Since $\{x^k\} \subseteq \mathcal{Q}$, there is a sequence $\{t^k\} \subseteq [0,1]$ such that

$$\mathcal{H}(x^k, t^k) = \begin{pmatrix} t^k x_1^k + (1-t^k)(x^k)^T A_1 x^k \\ \vdots \\ t^k x_m^k + (1-t^k)(x^k)^T A_m x^k \\ t^k x_n^k + (1-t^k)((x^k)^T x^k - 1) \end{pmatrix} = 0. \tag{8}$$

Since $x^k \neq 0$, it follows from (8) that $t^k \neq 1$. By assumption, there is no solution to the equation $\Phi(x) = 0$, so it follows from (8) that $t^k \neq 0$. As a result, (8) can be written as

$$\Phi(x^k) = -\left(\frac{t^k}{1-t^k}\right) x^k, \quad t^k \in (0,1), \quad k \ge 1.$$

By the definition of Φ, we have

$$\begin{aligned} (x^k)^T A_i x^k &= -\frac{t^k}{1-t^k} x_i^k, \quad i = 1, \ldots, m, \\ (x^k)^T x^k &= 1 - \left(\frac{t^k}{1-t^k}\right) x_n^k, \end{aligned}$$

where $t_k \in (0,1)$ for all $k \ge 1$. The desired result then follows by setting $\gamma^k = \frac{t^k}{1-t^k}$. $\square$

A similar analysis to the above (by degree theory) has been used in the existence analysis for the solution of finite-dimensional variational inequalities and complementarity problems (see e.g. [28, 51, 52, 53]). It is also worth mentioning that the identity mapping in $\mathcal{H}(x,t)$ can be replaced by a general invertible mapping $\varphi(x)$ with $|\deg(\varphi, D, 0)| \neq 0$, and thus the result developed in this paper can be easily adapted to this case. However, we choose to use the identity mapping throughout this paper in order to keep results as simple as possible.

From Lemma 2.3, we can prove the next result.

Lemma 2.4. *If $\Phi(x) = 0$ has no solution, where Φ is defined by (5), then there exists a vector $\widehat{x}$ with $\|\widehat{x}\| = 1$ such that*

$$\widehat{x}^T A_i \widehat{x} = -\frac{1}{\delta}\widehat{x}_i, \quad i = 1, \ldots, m, \quad -\widehat{x}_n = \delta \in (0,1), \tag{9}$$

and hence

$$\widehat{x}^T \left(\sum_{i=1}^{m} -\widehat{x}_i A_i \right) \widehat{x} = \frac{1}{\delta}(1-\delta^2) > 0. \tag{10}$$

Proof. Suppose that $\Phi(x) = 0$ has no solution. Then, by Lemma 2.3, there exists a sequence $\{x^k\}$ satisfying (6) and (7), and $\|x^k\| \to \infty$ as $k \to \infty$. We see from (7) that $x_n^k < 0$ for all sufficiently large k. It is not difficult to show that any accumulation point of the sequence $\left\{x_n^k/\|x^k\|\right\}$ is in $(-1,0)$. In fact, since x_n^k is negative for all sufficiently large k, any accumulation point of the sequence $\left\{x_n^k/\|x^k\|\right\}$ must be in $[-1,0]$. So it is sufficient to prove that it is not equal to -1 or 0. Let $\widehat{x}$ be an arbitrary accumulation point of the sequence $\left\{x^k/\|x^k\|\right\}$. By passing to a subsequence if necessary, we may assume that $\lim_{k\to\infty} x^k/\|x^k\| = \widehat{x}$. By (7), we have

$$1 = \frac{1}{\|x^k\|^2} - \frac{\gamma^k}{\|x^k\|}\left(\frac{x_n^k}{\|x^k\|}\right) \tag{11}$$

and, by (6), for every $i = 1, \ldots, m$, we have

$$\left(-\frac{\gamma^k}{\|x^k\|}\right)\left(\frac{x_i^k}{\|x^k\|}\right) = \frac{(x^k)^T A_i x^k}{\|x^k\|^2} \to \widehat{x}^T A_i \widehat{x}. \tag{12}$$

Case 1: Assume that $\widehat{x}_n = -1$. Then, from (11), we see that $\gamma^k/\|x^k\| \to 1$. Since $\widehat{x}_n = -1$ and $\|\widehat{x}\| = 1$, we must have $x_i^k/\|x^k\| \to \widehat{x}_i = 0$ for all $i = 1, \ldots, m$. Thus, it follows from (6) that

$$\widehat{x}^T A_i \widehat{x} = -\widehat{x}_i = 0, \quad i = 1, \ldots, m.$$

This implies that the unit vector $\widehat{x}$ is a solution of the system (2), i.e., $\Phi(\widehat{x}) = 0$, contradicting the assumption of this lemma.

Case 2: Assume that $\widehat{x}_n = 0$. Then it follows from (11) that $\gamma^k/\|x^k\| \to \infty$ as $k \to \infty$. Thus, from (12) we have $x_i^k/\|x^k\| \to \widehat{x}_i = 0$, $i = 1, \ldots, m$. This contradicts the fact that $\|\widehat{x}\| = 1$.

Both cases above yield a contradiction. Thus we conclude that for any accumulation point $\widehat{x}$ of $\{x^k/\|x^k\|\}$, its last component $\widehat{x}_n$ satisfies $\widehat{x}_n \in (-1,0)$. We now show that such an accumulation point satisfies (9) and (10). Indeed, without loss of generality, we assume that $\lim_{k\to\infty} x^k/\|x^k\| =$

$\widehat{x}$ and $-\widehat{x}_n = \delta \in (0,1)$. From (11), we see that $\gamma^k/\|x^k\| \to 1/\delta$, and thus (9) follows directly from (12). Multiplying (6) by x_i^k for each i and adding them up yield

$$(x^k)^T \left(\sum_{i=1}^m x_i^k A_i\right) x^k = -\gamma^k \sum_{i=1}^m (x_i^k)^2 = -\gamma^k \left(\|x^k\|^2 - (x_n^k)^2\right).$$

Dividing this equality by $\|x^k\|^3$ and taking the limit yield (10). $\square$

An immediate consequence of Lemmas 2.1 and 2.4 is the following sufficient condition.

Corollary 2.5. *If the system*

$$x^T A_i x = \frac{x_i}{x_n}, \quad i = 1, \ldots, m, \ x_n \in (-1, 0), \ \|x\| = 1$$

is inconsistent, then the system (1) has a rank-one solution.

Based on this fact, we have the following result.

Theorem 2.6. *If $A_i, i = 1, \ldots, m$, satisfy the condition*

$$\max_{1 \le i \le m, \ x^T A_i x \neq 0} x_i (x^T A_i x) \ge 0 \ \text{ for all } x \text{ with } \|x\| = 1 \text{ and } x_n < 0, \tag{13}$$

then the system (1) has a rank-one solution.

Proof. We prove this result by contradiction. Assume that the system (1) has no rank-one solution. Then, by Lemma 2.1, the equation $\Phi(x) = 0$ has no solution. It follows from Lemma 2.4 that there exists a unit vector $\widehat{x}$ satisfying (9). Multiplying both sides of (9) by $\widehat{x}_i$ yields

$$\widehat{x}_i (\widehat{x}^T A_i \widehat{x}) = -\frac{1}{\delta} \widehat{x}_i^2, \quad i = 1, \ldots, m, \quad -\widehat{x}_n = \delta \in (0, 1).$$

Note that $\|\widehat{x}\| = 1$ and $|\widehat{x}_n| = \delta < 1$. Thus $(\widehat{x}_1, \ldots, \widehat{x}_m) \neq 0$, which implies that

$$\max_{1 \le i \le m, \ \widehat{x}^T A_i \widehat{x} \neq 0} \widehat{x}_i (\widehat{x}^T A_i \widehat{x}) = \max_{1 \le i \le m, \ \widehat{x}_i \neq 0} (-\widehat{x}_i^2/\delta) < 0.$$

This contradicts (13). $\square$

Corollary 2.7. *If $A_i, i = 1, \ldots, m$, satisfy the condition*

$$\max_{1 \le i \le m, \ x^T A_i x \neq 0} x_i (x^T A_i x) \ge 0 \ \text{ for all } x \neq 0, \tag{14}$$

then the system (1) has a rank-one solution.

Clearly, the condition (14) is stronger than (13). So it implies the existence of a rank-one solution to the system (1). Motivated by conditions (13) and (14), we introduce the class of $P_\emptyset$ functions defined as follows.

Definition 2.8. *Let $D \subseteq R^n$ and $\widehat{x} \in D$. A mapping $F : R^n \to R^n$ is said to be a $P_\emptyset$-function at $\widehat{x}$ over D if*

$$\max_{1 \le i \le n, \ F_i(x) \neq F_i(\hat{x})} (x_i - \widehat{x}_i)(F_i(x) - F_i(\widehat{x})) \ge 0 \ \text{ for all } x \in D, x \neq \widehat{x}.$$

Recall that for a given set $D \subseteq R^n$ and $\widehat{x} \in D$, the mapping $G : R^n \to R^n$ is said to be a P_0-function at $\widehat{x}$ over D if $\max_{1\leq i\leq n, x_i\neq \widehat{x}_i}(x_i - \widehat{x}_i)(G_i(x) - G_i(\widehat{x})) \geq 0$ for all $x \neq \widehat{x}$ and $x \in D$. The class of P_0-functions has been widely used in nonlinear analysis and optimization (see e.g. [16, 24, 52, 53]). There are some relationships between $P_\emptyset$-functions and P_0-functions. In fact, when the inverse of a $P_\emptyset$-function exists, it is easy to see that the inverse is a P_0-function. By Definition 2.8, we can state the following result.

Theorem 2.9. *Let $\mathcal{F} : R^n \to R^n$ be defined by*

$$\mathcal{F}(x) = \left(x^T A_1 x,\ \ \ldots,\ x^T A_m x,\ x^T x\right)^T$$

and let $U = \{x : \|x\| \leq 1\}$. If $\mathcal{F}$ is a $P_\emptyset$-function at $x = 0$ over U, then the system (1) has a rank-one solution.

Proof. Since $\mathcal{F}$ is a $P_\emptyset$-function at $x = 0$ over U, we have for any $x \in U \setminus \{0\}$ that

$$\max_{1\leq i\leq n,\ \mathcal{F}_i(x)\neq\mathcal{F}_i(0)} x_i(\mathcal{F}_i(x) - \mathcal{F}_i(0)) \geq 0. \tag{15}$$

In particular, for any x such that $\|x\| = 1$ and $x_n < 0$, we have $x_n(\mathcal{F}_n(x) - \mathcal{F}_n(0)) = x_n x^T x = x_n < 0$. So (15) is reduced to

$$\max_{1\leq i\leq m,\ x^T A_i x\neq 0} x_i(x^T A_i x) \geq 0.$$

By Theorem 2.6 or its corollary, the system (1) must have a rank-one solution. $\square$

Remark 2.10. In this section, we have seen that the degree-based analysis can provide a sufficient condition for the system (1) to have a rank-one solution. In section 5, we will further show that such a sufficient condition is almost necessary for the system (1) to have a rank-one solution. In section 3, we show that checking whether or not the system (1) has a rank-one solution is equivalent to solving an SDP problem with a rank constraint, which is clearly not an easy problem due to the rank constraint. It is well known that a general rank minimization problem are NP-hard [43, 50] since it includes the so-called cardinality minimization problem as a special case. So roughly speaking, the level of difficulty for checking the conditions developed in this paper, such as the ones in Corollary 2.5, Theorem 2.6 and Theorem 4.5, are almost equivalent to that of the original system (1) which, except for the case of $m = 2$ and $n \geq 3$, is difficult in general. However, these conditions provide a new angle (from degree theory) to understand the system (1). More interestingly, some verifiable sufficient conditions (from a rank optimization point of view) can be also developed for the system (1) to have a rank-one solution, as shown in the next section.

3 Existence of a rank-one solution: rank optimization

We now study the existence of a rank-one solution to the system (1) from a rank optimization point of view. When $m = 2$ and $n \geq 3$, Theorem 1.1 claims that the condition $t_1 A_1 + t_2 A_2 \not\succeq 0$ is a complete characterization of the existence of a rank-one solution to the system (1). However, this result does not hold when $m \geq 3$. It is interesting to note that a complex counterpart of such a result was given in [41]. In this section, we show that some conditions stronger than $\sum_{i=1}^m t_i A_i \not\succeq 0$ are needed in order to ensure the existence of rank-one solutions. Before we proceed, let us first reformulate the problem as a rank constrained optimization problem.

Consider the following rank-constrained optimization problem:

$$\max\{\langle I, X\rangle : \ \ \langle A_i, X\rangle = 0, \ i = 1, \ldots, m, \ \mathrm{rank}(X) \leq 1, \ X \succeq 0\}. \tag{16}$$

If $X = 0$ is the only feasible point to the problem, then the optimal value of the problem is 0. Otherwise, it is ∞. Since the system (1) is homogeneous, normalizing the system does not change its solvability and the rank of its solutions. So adding the constraint $\mathrm{tr}(X) \leq 1$ (i.e., $\langle I, X\rangle \leq 1$) to (16) yields the following problem:

$$\begin{aligned} z^* = \quad & \max \quad \langle I, X\rangle \\ & \text{s.t.} \quad \langle A_i, X\rangle = 0, \ i = 1, \ldots, m, \\ & \qquad\ \ \mathrm{rank}(X) \leq 1, \\ & \qquad\ \ \mathrm{tr}(X) \leq 1, \\ & \qquad\ \ X \succeq 0. \end{aligned} \tag{17}$$

Since the optimal value z^* is either 0 or 1, we have the following observation.

Lemma 3.1. *The system (1) has a rank-one solution if and only if $z^* = 1$ is the optimal value of (17).*

In other words, $z^* = 0$ is the optimal value of (17) if and only if (1) has no rank-one solution. Lemma 3.1 indicates that checking the existence of a rank-one solution to the system (1) is equivalent to solving an SDP problem with the rank constraint $\mathrm{rank}(X) \leq 1$, which is hard to solve in general because of the discontinuity and nonconvexity of $\mathrm{rank}(X)$. Based on the problem obtained by dropping the constraint "$\mathrm{rank}(X) \leq 1$" from (17), we have the following result.

Lemma 3.2. *There exist $t_i \in R, i = 1, \ldots, m$, such that $\sum_{i=1}^m t_i A_i \succ 0$ if and only if $X = 0$ is the only solution to the system (1). In other words, $\sum_{i=1}^m t_i A_i \not\succ 0$ for all $t_i \in R, i = 1, \ldots, m$, if and only if the system (1) has a nontrivial solution, i.e., a solution X with* $\mathrm{rank}(X) \geq 1$.

Proof. The standard SDP duality theory [49] (or the result in [7]) can yield the result of this lemma. In fact, let us consider the SDP problem

$$\max\{\langle I, X\rangle : \ \langle A_i, X\rangle = 0, \ i = 1, \ldots, m, \ \mathrm{tr}(X) \leq 1, \ X \succeq 0\}, \tag{18}$$

and its dual problem

$$\min\left\{\alpha : \ \sum_{i=1}^m t_i A_i + \alpha I \succeq I, \ \alpha \geq 0, \ t_i \in R, i = 1, \ldots, m\right\}. \tag{19}$$

Clearly, (19) satisfies the Slater's condition (for instance, $(\alpha, t_1, \ldots, t_m) = (2, 0, \ldots, 0)$ is a strictly feasible point). The optimal value of (19) is obviously finite. By the duality theory of semidefinite programming, both problems (22) and (19) have finite optimal values and there is no duality gap between them (i.e., their optimal values are equal). If there exist $t_i, i = 1, \ldots, m$ such that $\sum_{i=1}^m t_i A_i \succ 0$, then $\sum_{i=1}^m (\beta t_i) A_i \succeq I$ for some $\beta > 0$, which means that the optimal value of the dual problem (19) is 0. Thus the optimal value of (22) is also 0, implying that $X = 0$ is the only point satisfying the system (1). Conversely, if $X = 0$ is the only solution of (1), then the optimal value of (22) is 0, and thus the dual optimal value is also 0, i.e., $\alpha^* = 0$. This indicates that there exist $t_i \in R, i = 1, \dots, m$, such that $\sum_{i=1}^m t_i A_i \succeq I \succ 0$. $\Box$

As we mentioned in section 1, the result of Theorem 1.1 does not hold for $m \geq 3$. That is, only knowing that the system (1) has no rank-one solution does not give a full picture of the condition $\sum_{i=1}^m t_i A_i \succ 0$. The system (1) may have no rank-one solution, but have a solution with $\text{rank}(X) \geq 2$. For example, let

$$A_1 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

It is easy to see that for this example there is no $x \neq 0$ satisfying $x^T A_i x = 0, i = 1, 2, 3$, and hence the corresponding system (1) has no rank-one solution. However, the system has a higher rank solution, $X = \text{diag}(1, 1, 1, 0)$. Clearly, there exists no (t_1, t_2, t_3) such that $t_1 A_1 + t_2 A_2 + t_3 A_3 \succ 0$ for this example. To ensure the condition $\sum_{i=1}^m t_i A_i \succ 0$, Lemma 3.2 claims that the system (1) must possess not only no rank-one solution but also no solution with rank higher than 1. It should be stressed that the condition $\sum_{i=1}^m t_i A_i \not\succ 0$ for all t_i's implies that there is a nonzero solution X to the system (1), but it cannot ensure that $\text{rank}(X) = 1$. Some stronger conditions than $\sum_{i=1}^m t_i A_i \not\succ 0$ should be imposed in order to guarantee the existence of a rank-one solution.

From Lemma 3.2, we see that the linear combination of A_i's plays an important role in determining the solution structure of the system (1). Given a finite number of matrices $A_i, i = 1, \dots, m$, we use r^* to denote the maximum rank of the linear combination $\sum_{i=1}^m t_i A_i$, where $t_i \in R, i = 1, \dots, m$ are chosen such that the linear combination is positive semidefinite, i.e.,

$$r^* = \max \left\{ \text{rank} \left(\sum_{i=1}^m t_i A_i \right) : \quad \sum_{i=1}^m t_i A_i \succeq 0, \quad t_i \in R, \ i = 1, \dots, m \right\}. \tag{21}$$

Clearly, r^* is finite and attainable. Moreover, $r^* = n$ is equivalent to $\sum_{i=1}^m t_i A_i \succ 0$. The next result shows how r^* affects the existence of a nontrivial solution to the system (1), including low-rank ones. It also indicates when the rank-constrained problem (17) can be reduced to an SDP problem.

Corollary 3.3. (i) *The system (1) has a solution $X \neq 0$ if and only if $r^* \leq n - 1$, where r^* is defined by (21). Moreover, any nonzero solution X of (1) satisfies* $\text{rank}(X) \leq n - r^*$.

(ii) *Particularly, if $r^* = n - 1$, then the system (1) has a rank-one solution, and the rank-one solutions are the only nonzero solutions of (1). In this case, the problem (17) is equivalent to the SDP problem*

$$\max\{\langle I, X \rangle : \ \langle A_i, X \rangle = 0, \ i = 1, \dots, m, \ \text{tr(X)} \leq 1, \ \text{X} \succeq 0\}. \tag{22}$$

Proof. (i) Lemma 3.2 claims that $r^* = n$ if and only if $X = 0$ is the only solution of (1). Thus, the system (1) has a nonzero solution if and only if $r^* \leq n - 1$. It is sufficient to prove that any nonzero solution X of (1) must satisfy $\text{rank}(X) \leq n - r^*$. Indeed, let $(t_1^*, \dots, t_m^*)$ determine the maximum value r^*, i.e.,

$$r^* = \ \text{rank} \left(\sum_{i=1}^m t_i^* A_i \right), \quad \sum_{i=1}^m t_i^* A_i \succeq 0.$$

Let X be an arbitrary solution of (1). Thus, X satisfies

$$\left\langle X,\ \sum_{i=1}^{m} t_i^* A_i \right\rangle = \sum_{i=1}^{m} t_i^* \langle X,\ A_i\rangle = 0, \quad X \succeq 0.$$

Since $X \succeq 0$ and $\sum_{i=1}^m t_i^* A_i \succeq 0$, it implies that

$$\left(\sum_{i=1}^{m} t_i^* A_i\right) X = 0.$$

Thus, $\operatorname{rank}(X) \le n - \operatorname{rank}\left(\sum_{i=1}^m t_i^* A_i\right) = n - r^*$.

We now prove (ii). Suppose $r^* = n-1$. Then the first half follows directly from (i) and Lemma 3.2. Moreover, since any solution of (1) satisfies $\operatorname{rank}(X) \le n - r^*$, it must satisfy $\operatorname{rank}(X) \le 1$. This means that the rank constraint in (17) is redundant. As a result, the problem (17) is reduced to the SDP problem (22). □

From Lemma 3.2 and Corollary 3.3, the cases $r^* = n$ and $r^* = n-1$ are clear. In the remainder of this section, we focus on the case $r^* \le n-2$ for which certain conditions should be imposed in order to ensure the existence of a rank-one solution. In fact, when $r^* \le n-2$, the system (1) may have a solution with $\operatorname{rank}(X) \ge 2$, but no rank-one solution. It is easy to see that $r^* < n-2$ holds in the example (20), since $r^* = 0$. Another simple example is that $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For this example, we have $r^* = n-2$, but the system $\langle A_1, X\rangle = 0$, $\langle A_2, X\rangle = 0$, $X \succeq 0$ has no rank-one solution. We now state an existence condition for the case $r^* \le n-2$.

Theorem 3.4. *Suppose that $r^* \le n-2$. Then the system (1) has a rank-one solution if any of the following conditions holds:*

(i) *For each i, A_i is either positive semidefinite or negative semidefinite.*

(ii) *There exists exactly one indefinite matrix among A_i's.*

(iii) *There are more than one indefinite matrices among A_i's, and there is an indefinite matrix A_k such that*

$$\{x : x^T A_k x = 0\} \subseteq \bigcap_{\text{All indefinite } A_l,\, l\neq k} \{x : x^T A_l x = 0\}. \tag{23}$$

Proof. When $r^* \le n-2$, by Lemma 3.2 the system (1) has a solution with $1 \le \operatorname{rank}(X) \le n - r^*$. So if the system has no solution with rank $(X) > 1$, then it must have a rank-one solution. Thus, without loss of generality, we assume that the system (1) has a solution X^* with $\operatorname{rank}(X^*) = r \ge 2$, and hence X^* can be decomposed as

$$X^* = \lambda_1 u^1 (u^1)^T + \cdots + \lambda_r u^r (u^r)^T,$$

where $\lambda_j > 0$ and $u^j, j = 1, \ldots, r$ are eigenvalues and eigenvectors of X^*, respectively, and u^j's are mutually orthogonal. For every $i = 1, \ldots, m$, we have

$$0 = \langle A_i, X^*\rangle = \sum_{j=1}^{r} \lambda_j (u^j)^T A_i u^j.$$

In particular, if $A_i \succeq 0$ or $A_i \preceq 0$, then the above equality implies

$$(u^j)^T A_i u^j = 0 \ \text{ for all } j = 1, \ldots, r. \tag{24}$$

First we suppose that condition (i) holds. That is, either $A_i \succeq 0$ or $A_i \preceq 0$ holds for each i. Then (24) implies that any of the matrices $u^j(u^j)^T, j = 1, \ldots, r$ is a rank-one solution to the system (1). This shows that condition (i) ensures the existence of a rank-one solution.

Next, we suppose that condition (ii) holds and let A_k be the only indefinite matrix. If $(u^j)^T A_k u^j = 0$ for some j, then $X = u^j(u^j)^T$ readily gives a rank-one solution to the system (1), since (24) holds for all $i \neq k$. On the other hand, if $(u^j)^T A_k u^j \neq 0$ for all $j = 1, \ldots, r$, then from the fact that $\lambda_j > 0$ for all j and

$$0 = \langle A_k, X^* \rangle = \sum_{j=1}^{r} \lambda_j (u^j)^T A_k u^j,$$

it follows that there exist two indices p and q such that

$$\left((u^p)^T A_k u^p\right) \left((u^q)^T A_k u^q\right) < 0.$$

By continuity, there exists a $\gamma \in (0, 1)$ such that

$$w = \gamma u^p + (1 - \gamma) u^q, \quad w^T A_k w = 0. \tag{25}$$

Since u^p and u^q are orthogonal, it is evident that $w \neq 0$. If $A_i \succeq 0$ or $A_i \preceq 0$, then (24) implies that $A_i u^j = 0$ for all $j = 1, \ldots, r$. Thus we have

$$w^T A_i w = \gamma^2 (u^p)^T A_i u^p + (1 - \gamma)^2 (u^q)^T A_i u^q + 2\gamma(1 - \gamma)(u^p)^T A_i u^q = 0 \tag{26}$$

for all $i \neq k$. Therefore, $X = ww^T$ is a rank-one solution of (1). Consequently, condition (ii) ensures the existence of a rank-one solution.

Finally, we suppose that condition (iii) holds. If $(u^j)^T A_k u^j = 0$ for some j, then it follows from (23) that $(u^j)^T A_l u^j = 0$ for all other indefinite matrices A_l. Since (24) holds for all matrices A_i such that $A_i \succeq 0$ or $A_i \preceq 0$, we may deduce that $X = u^j(u^j)^T$ is a rank-one solution of the system (1). If $(u^j)^T A_k u^j \neq 0$ for all $j = 1, \ldots, r$, then by the same reasoning as above, we can find a vector $w \neq 0$ that satisfies (25). We can also repeat the same argument as above to show that w satisfies (26) for all $A_i \succeq 0$ or $A_i \preceq 0$. Moreover, by (23), we have $w^T A_l w = 0$ for all indefinite A_l with $l \neq k$. Thus, the nonzero vector w satisfies $w^T A_i w = 0$ for all $i = 1, \ldots, m$, implying that $X = ww^T$ is a rank-one solution to the system (1). The proof is complete. $\square$

Remark 3.5. Given a set of matrices $A_i, i = 1, ..., m$, conditions (i) and (ii) in Theorem 3.4 can be verified straightaway. A simple (and trivial) example satisfying the condition (iii) of Theorem 3.4 is as follows: Consider the system (1) with $m = 3$ and $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $A_2 = 2A_1$ and $A_3 = 3A_1$. Then the condition (23) holds trivially, and $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is a rank-one solution of the system (1).

Remark 3.6. Theorem 3.4 shows that the number of indefinite matrices among A_i's and their relationships are closely related to the existence of a rank-one solution to the system (1). While this result gives some sufficient conditions for the system (1) to have a rank-one solution, it is worth noting that these conditions remain not tight, as shown by the following example: Consider the system (1) with $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which are both indefinite. It is easy to see that

$$\{x : x^T A_1 x = 0\} \not\subseteq \{x : x^T A_2 x = 0\}, \ \{x : x^T A_2 x = 0\} \not\subseteq \{x : x^T A_1 x = 0\}.$$

So all conditions (i), (ii) and (iii) in Theorem 3.4 do not hold for this example. However, the system (1) with these two matrices has a rank-one solution, for instance $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is a rank-one solution.

From Corollary 3.3(i), the rank of the solution of (1) is at most $n - r^*$, which is a uniform bound for all solutions. From a practical viewpoint, it is important to compute the value r^*. This motivates us to study the rank maximization problem (21), which can be rewritten as

$$r^* = \max\left\{\lambda : \ \operatorname{rank}\left(\sum_{i=1}^m t_i A_i\right) \geq \lambda, \ \sum_{i=1}^m t_i A_i \succeq 0\right\}. \tag{27}$$

Since $\operatorname{rank}(X)$ is a discontinuous function (in fact, a lower semi-continuous function) of X, the set $\{X : \operatorname{rank}(X) \geq \lambda\}$ is not closed in general. This makes the problem (27) (or (21)) difficult to solve directly. In what follows, we propose a method to estimate r^* from below. The following lemma will be used in our analysis.

Lemma 3.7. [25, 26, 43] *The convex envelope of* $\operatorname{rank}(X)$ *on the set* $\{X \in R^{m\times n} : \|X\| \leq 1\}$ *is the nuclear norm* $\|X\|_*$.

Note that for any matrix $Y \neq 0$, we have $\operatorname{rank}(Y) = \operatorname{rank}(\alpha Y)$ for any $\alpha \neq 0$. Thus problem (17) can be rewritten as

$$r^* = \max\left\{\operatorname{rank}\left(\sum_{i=1}^m t_i A_i\right) : \ \sum_{i=1}^m t_i A_i \succeq 0, \ \sum_{i=1}^m t_i A_i \preceq I\right\}. \tag{28}$$

Since $0 \preceq \sum_{i=1}^m t_i A_i \preceq I$, we have $\|\sum_{i=1}^m t_i A_i\| \leq 1$. By Lemma 3.7, we conclude that in the feasible region of the problem (28), the nuclear norm of $\sum_{i=1}^m t_i A_i$ is the convex envelop of the objective function of (28). As a result, we have

$$\operatorname{rank}\left(\sum_{i=1}^m t_i A_i\right) \geq \left\|\sum_{i=1}^m t_i A_i\right\|_* \tag{29}$$

for any $(t_1, \ldots, t_m)$ satisfying $0 \preceq \sum_{i=1}^m t_i A_i \preceq I$. By the positive semidefiniteness of $\sum_{i=1}^m t_i A_i$, we have

$$\left\|\sum_{i=1}^m t_i A_i\right\|_* = \operatorname{tr}\left(\sum_{i=1}^m t_i A_i\right) = \sum_{i=1}^m t_i \operatorname{tr}(A_i).$$

Thus, we may consider the following problem:

$$\eta^* = \max\left\{\sum_{i=1}^m t_i \operatorname{tr}(A_i) : \ \sum_{i=1}^m t_i A_i \succeq 0, \ \sum_{i=1}^m t_i A_i \preceq I\right\}, \tag{30}$$

which is an SDP problem with a finite optimal value $\eta^* \leq n$. By (29) and Lemma 3.7, the optimal objective value of (30) provides a lower bound for that of (28), i.e., $r^* \geq \lceil \eta^* \rceil$. The dual of (30) is given by

$$\min\{\langle I, X\rangle : \langle A_i, X\rangle - \langle A_i, Y\rangle = \mathrm{tr}(A_i),\ i = 1, \ldots, m,\ X \succeq 0,\ Y \succeq 0\}. \tag{31}$$

This problem is strictly feasible and has a finite optimal value. For instance, $(X, Y) = (2I, I)$ is a strictly feasible point. By the duality theory, there is no duality gap between (30) and (31), and hence we may solve either of them to get the optimal value η^*. An immediate consequence of the above analysis is the following result.

Theorem 3.8. *Let r^* be the maximum rank defined by (21), and let η^* be the optimal value of the SDP problem (30) or (31). When $r^* \leq n-2$, every nonzero solution X of the system (1) satisfies* $\mathrm{rank}(X) \leq n - \lceil \eta^* \rceil$.

This result provides an upper bound for the rank of nonzero solutions of (1), and the bound $n - \lceil \eta^* \rceil$ can efficiently be obtained by solving (30) or (31).

Remark 3.9. Consider the problem of finding $X \in S^n$ that satisfies

$$\langle A_i, X\rangle = b_j,\ j = 1, \ldots, m,\ X \succeq 0.$$

If the above system has a solution, then it has a solution X such that

$$\mathrm{rank}(X) \leq \left\lfloor \frac{\sqrt{8m+1}-1}{2} \right\rfloor, \tag{32}$$

which is called Barvinok-Pataki's bound [4, 40]. For a homogeneous system (i.e., $b_j = 0, j = 1, \ldots, m$), a solution satisfying the bound (32) can be only the trivial solution $X = 0$, and any nontrivial solution may not satisfy this bound. In other words, *the Barvinok-Pataki's bound is not necessarily valid for nontrivial solutions of a homogeneous system.* For example, let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The right-hand side of (32) is equal to 1 (since $m = 2$). However, as we mentioned earlier, all nontrivial solutions to the system $\langle A_1, X\rangle = 0$, $\langle A_2, X\rangle = 0$, $X \succeq 0$ have rank 2. Thus, Barvinok-Pataki's bound (32) cannot directly apply to nontrivial low-rank solutions of a homogeneous system like (1). In order to apply this bound to the homogenous system (1), we may introduce an extra equation, for instance, $\langle I, X\rangle = 1$, and consider the following system:

$$\langle A_i, X\rangle = 0,\ j = 1, \ldots, m,\ \langle I, X\rangle = 1,\ X \succeq 0. \tag{33}$$

Note that $\mathrm{rank}(tX) = \mathrm{rank}(X)$ for any $t \neq 0$. By the homogeneity of (1), we see that any nonzero solution (if exists) of (1) can be scaled so that it satisfies (33). Thus, if the system (1) has a nonzero solution, then the minimum rank of nonzero solutions of (1) and (33) are the same. Therefore, applying (32) to (33), we can conclude from Theorem 3.6 that when $r^* \leq n-2$, the system (1) has a solution $X \neq 0$ satisfying

$$\begin{aligned} 1 \leq \mathrm{rank}(X) &\leq \min\left\{ n - r^*, \left\lfloor \frac{\sqrt{8(m+1)+1}-1}{2} \right\rfloor \right\} \\ &\leq \min\left\{ n - \lceil \eta^* \rceil, \left\lfloor \frac{\sqrt{8(m+1)+1}-1}{2} \right\rfloor \right\}. \end{aligned}$$

So when r^* is relatively large, e.g., $n-2, n-3$, and so on, the rank of a nontrivial solution to (1) will be low. In such cases, Barvinok-Pataki's bound might be too loose (especially when m is relatively large). The upper bound given by $n-r^*$ or even $n-\lceil \eta^* \rceil$ for the rank of nonzero solutions can be much tighter than Barvinok-Pataki's bound in these situations.

4 Conditions for (1) to have no rank-one solution

The following necessary condition for the system (1) to have no rank-one solution has actually been shown in section 2 (see Lemmas 2.1 and 2.4, or Corollary 2.5).

Corollary 4.1. *When $m \le n-1$, if the system (1) does not have a rank-one solution, then there exists a vector x such that*

$$x^T A_i x = \frac{x_i}{x_n}, \ i=1,\ldots,m, \ \|x\|=1, \ x_n \in (-1,0).$$

Although the condition $\sum_{i=1}^m t_i A_i \succ 0$ ensures that the system (1) has no rank-one solution, this sufficient condition is too restrictive. In fact, by Lemma 3.2, it implies that the system (1) cannot have any solution with $\mathrm{rank}(X) \ge 1$. The purpose of this section is to show that another sufficient condition for the system (1) to have no rank-one solution can be developed from a homotopy invariance point of view. Note that the following three statements are equivalent: (i) The system (1) has no rank-one solution; (ii) $x=0$ is the only solution to the system (2); and (iii)

$$\max_{1\le i\le m} |x^T A_i x| > 0 \ \text{ for any } x \ne 0. \tag{34}$$

First, we formulate these equivalent statements as a nonlinear equation.

Lemma 4.2. *$x=0$ is the only solution to the system (2) if and only if there exists a constant $\beta>0$ such that for any $\mu \in (0,\beta]$, we have*

$$\{x : \|x\|=1\} = \{x : G_\mu(x)=0\},$$

where $G_\mu : R^n \to R^2$ is defined by

$$G_\mu(x) = \begin{pmatrix} \left|\sum_{i=1}^m |x^T A_i x| - \mu\right| - \left(\sum_{i=1}^m |x^T A_i x| - \mu\right) \\ x^T x - 1 \end{pmatrix}, \tag{35}$$

i.e., the set $\{x : \|x\|=1\}$ coincides with the solution set of the equation $G_\mu(x)=0$ for any $\mu \in (0,\beta]$.

Proof. Assume that $x=0$ is the only solution of the system (2). Thus, by (34), we have $\sum_{i=1}^m |x^T A_i x| > 0$ for any x such that $x^T x = 1$. By continuity, there exists a positive number $\beta>0$ (for instance, we can take $\beta = \min\left\{\sum_{i=1}^m |x^T A_i x| : \ \|x\|=1\right\}$ which is positive) such that

$$\sum_{i=1}^m |x^T A_i x| \ge \mu \tag{36}$$

holds for any x with $x^T x = 1$ and for any $\mu \in (0,\beta]$. Note that any inequality $h(x) \ge 0$ can be represented as the equation $|h(x)| - h(x) = 0$. So (36) can be rewritten as

$$\left|\sum_{i=1}^m |x^T A_i x| - \mu\right| - \left(\sum_{i=1}^m |x^T A_i x| - \mu\right) = 0.$$

This implies that any x satisfying $x^T x = 1$ is a solution to the equation

$$G_\mu(x) = 0$$

for any $\mu \in (0, \beta]$. Thus, the set $\{x : \|x\| = 1\}$ is contained in the solution set of $G_\mu(x) = 0$ for any $\mu \in (0, \beta]$. Since $G_\mu(x) = 0$ implies $\|x\| = 1$, the set $\{x : \|x\| = 1\}$ is exactly the solution set of $G_\mu(x) = 0$ for any given $\mu \in (0, \beta]$.

Conversely, let us assume that there exists a positive number $\beta > 0$ such that for any given constant $\mu \in (0, \beta]$, the solution set of the equation $G_\mu(x) = 0$ is equal to the set $\{x : \|x\| = 1\}$. We now prove that $x = 0$ is the only solution of (2). Assume the contrary that the system (2) has a solution $x \neq 0$. Then, $\widehat{x} = x/\|x\|$ is also a solution of the system (2), i.e., $\widehat{x}^T A_i \widehat{x} = 0,\ i = 1, \ldots, m$. By assumption, any unit vector is a solution to $G_\mu(x) = 0$ for any given $\mu \in (0, \beta]$. Thus, we have

$$\left|\sum_{i=1}^m |\widehat{x}^T A_i \widehat{x}| - \mu\right| - \left(\sum_{i=1}^m |\widehat{x}^T A_i \widehat{x}| - \mu\right) = 0.$$

Since $\widehat{x}^T A_i \widehat{x} = 0$ for all $i = 1, \ldots, m$, the above equality reduces to

$$0 = 2\mu,$$

which is a contradiction since $\mu \in (0, \beta]$. $\square$

We assume $m \leq n - 2$ in the remainder of this section. Again, by using Lemma 2.2, we have the following technical result.

Lemma 4.3. *Assume that $m \leq n - 2$. Let $\mu > 0$ be a given constant and G_μ be defined by (35). If $G_\mu(x^*) \neq 0$ for some x^* with $\|x^*\| = 1$, then there exists a sequence $\{x^k\}$ satisfying the following conditions: $\|x^k\| \to \infty$ as $k \to \infty$ and*

$$\begin{aligned} (x^k)^T A_1 x^k - (x^*)^T A_1 x^* &= -\gamma^k x_1^k, && (37) \\ &\vdots \\ (x^k)^T A_m x^k - (x^*)^T A_m x^* &= -\gamma^k x_m^k, && (38) \end{aligned}$$

$$\left|\sum_{i=1}^m |(x^k)^T A_i x^k| - \mu\right| - \left(\sum_{i=1}^m |(x^k)^T A_i x^k| - \mu\right) = -\gamma^k x_{n-1}^k, \quad (39)$$

$$(x^k - x^*)^T (x^k - x^*) = -\gamma^k x_n^k, \quad (40)$$

where $\gamma^k \in (0, \infty)$ for all k.

Proof. If $m < n - 2$, we may set $A_{m+1} = 0, \ldots, A_{n-2} = 0$, and consider the systems (1) and (2) with $A_i, i = 1, \ldots, m\,(= n - 2)$. Thus, without loss of generality, we assume $m = n - 2$. Since $G_\mu(x^*) \neq 0$ and $\|x^*\| = 1$, we have

$$\left|\sum_{i=1}^m |(x^*)^T A_i x^*| - \mu\right| - \left(\sum_{i=1}^m |(x^*)^T A_i x^*| - \mu\right) \neq 0.$$

Thus the following equation has no solution:

$$\Theta_\mu(x) = \begin{pmatrix} x^T A_1 x - (x^*)^T A_1 x^* \\ \vdots \\ x^T A_m x - (x^*)^T A_m x^* \\ \left|\sum_{i=1}^m |x^T A_i x| - \mu\right| - \left(\sum_{i=1}^m |x^T A_i x| - \mu\right) \\ (x - x^*)^T (x - x^*) \end{pmatrix} = 0. \tag{41}$$

Consider the homotopy between the identity mapping and $\Theta_\mu(x)$, i.e.,

$$H_\mu(x, t) = tx + (1 - t)\Theta_\mu(x),$$

and let

$$\mathcal{T}_\mu = \{x \in R^n : \; H_\mu(x, t) = 0 \text{ for some } t \in [0, 1]\}.$$

A similar proof to that of Lemma 2.3 can be used to show that $\mathcal{T}_\mu$ is unbounded. Here we include the proof for completeness. Assume to the contrary that $\mathcal{T}_\mu$ is bounded. Then there exists an open bounded ball D that is large enough to satisfy $\mathcal{T}_\mu \subset D$ and $\partial D \bigcap \mathcal{T}_\mu = \emptyset$. Thus $0 \notin \{H_\mu(x, t) : \; x \in \partial D, \; 0 \leq t \leq 1\}$, which implies that $\deg(I, D, 0)$ and $\deg(\Theta_\mu, D, 0)$ are well defined. By Lemma 2.2, we have

$$\deg(\Theta_\mu, D, 0) = \deg(I, D, 0) \neq 0,$$

which means that the equation $\Theta_\mu(x) = 0$ has a solution. This is a contradiction. Thus the set $\mathcal{T}_\mu$ is indeed unbounded, and hence there is an unbounded sequence $\{x^k\}$ in $\mathcal{T}_\mu$. Without loss of generality, let $x^k \neq 0$ for all k. By the definition of $\mathcal{T}_\mu$, there is a sequence $\{t^k\} \subseteq [0, 1]$ such that $H_\mu(x^k, t^k) = 0$, i.e.,

$$t^k x_1^k + (1 - t^k)[(x^k)^T A_1 x^k - (x^*)^T A_1 x^*] = 0,$$
$$\vdots$$
$$t^k x_m^k + (1 - t^k)[(x^k)^T A_m x^k - (x^*)^T A_m x^*] = 0,$$

$$t^k x_{n-1}^k + (1 - t^k)\left[\left|\sum_{i=1}^m |(x^k)^T A_i x^k| - \mu\right| - \sum_{i=1}^m |(x^k)^T A_i x^k| + \mu\right] = 0,$$

$$t^k x_n^k + (1 - t^k)(x^k - x^*)^T (x^k - x^*) = 0.$$

Since $\|x^k\| \to \infty$ as $k \to \infty$ and $\Theta_\mu(x^k) \neq 0$ for all $k \geq 1$, we see that $t^k \neq 1$ and $t^k \neq 0$ for all $k \geq 1$. Thus, $t^k \in (0, 1)$ for all $k \geq 1$. By setting $\gamma^k = \frac{t^k}{1-t^k} \in (0, \infty)$, the above system can be rewritten as

$$\Theta_\mu(x^k) = -\frac{t^k}{1 - t^k} x^k = -\gamma^k x^k, \quad k \geq 1,$$

which along with the definition of Θ_μ implies that

$$\begin{aligned} (x^k)^T A_1 x^k - (x^*)^T A_1 x^* &= -\gamma^k x_1^k, \\ &\vdots \\ (x^k)^T A_m x^k - (x^*)^T A_m x^* &= -\gamma^k x_m^k, \end{aligned}$$

$$\left|\sum_{i=1}^{m} |(x^k)^T A_i x^k| - \mu\right| - \left(\sum_{i=1}^{m} |(x^k)^T A_i x^k| - \mu\right) = -\gamma^k x_{n-1}^k,$$

$$(x^k - x^*)^T (x^k - x^*) = -\gamma^k x_n^k,$$

as desired. □

Based on Lemma 4.3, we can prove the following result.

Lemma 4.4. *Let $m \le n-2$ and $\mu > 0$ be a given constant. If $G_\mu(x^*) \neq 0$ for some x^* with $\|x^*\| = 1$, then there exists a vector $\widehat{x}$ satisfying the following conditions:*

$$\widehat{x}^T A_i \widehat{x} = \frac{\widehat{x}_i}{\widehat{x}_n}, \ i = 1, \ldots, m, \ \widehat{x}_{n-1} = 0 \ \widehat{x}_n \in [-1, 0), \ \|\widehat{x}\| = 1. \tag{42}$$

Proof. Without loss of generality, we still assume that $m = n-2$. Let $\{x^k\}$ be the sequence specified in Lemma 4.3. Since the left-hand sides of (39) and (40) are nonnegative, we see that $x_n^k < 0$ for all $k \ge 1$. Let $\widehat{x}$ be an accumulation point of $x^k/\|x^k\|$. By dividing (37)–(40) by $\|x^k\|^2$ and passing to a subsequence if necessary, we can prove that $x_n^k/\|x^k\| \to \widehat{x}_n \neq 0$. In fact, if $\widehat{x}_n = 0$, (40) implies $\gamma^k/\|x^k\| \to \infty$. Then it follows from (37)–(39) that $\widehat{x}_i = 0$ for all $i = 1, \ldots, n-1$. This contradicts the fact $\|\widehat{x}\| = 1$. Therefore, we conclude that $\widehat{x}_n \in [-1, 0)$. It then follows from (40) that $\gamma^k/\|x^k\| \to \widehat{\gamma} \in (0, \infty)$, where $\widehat{\gamma} = -1/\widehat{x}_n$. Normalizing the system (37)–(40) by $\|x^k\|^2$ and letting $k \to \infty$ yield

$$\widehat{x}^T A_i \widehat{x} = -\widehat{\gamma}\widehat{x}_i, \ i = 1, \ldots, m, \ 0 = -\widehat{\gamma}\widehat{x}_{n-1}, \ 1 = -\widehat{\gamma}\widehat{x}_n, \ \widehat{x}_n \in [-1, 0).$$

Eliminating $\hat{\gamma}$ from the above system yields the desired result. □

Basically, the next result shows that what an extra condition can make the necessary condition in Corollary 4.1 sufficient.

Theorem 4.5. *Let $m \le n-2$ and $A_i \in S^n$, $i = 1, \ldots, m$. Suppose that $e_n = (0, \ldots, 0, 1)^T \in R^n$ is not a solution of the system (2), and suppose that there is a vector x satisfying the following condition:*

$$x^T A_i x = \frac{x_i}{x_n}, \ i = 1, \ldots, m, \ x_n \in (-1, 0), \ \|x\| = 1. \tag{43}$$

If $x_{n-1} \neq 0$ for any x satisfying (43), then $x = 0$ is the only solution to the system (2), i.e., the system (1) has no rank-one solution.

Proof. Under the condition of this theorem, we prove that there is a $\beta > 0$ such that

$$\{x : \|x\| = 1\} = \{x : G_\mu(x) = 0\} \quad \text{for any } \mu \in (0, \beta],$$

where G_μ is defined by (35), and thus by Lemma 4.2, $x = 0$ is the only solution to the system (2). We prove this by contradiction. Assume that such a β does not exist, i.e., for any given $\beta > 0$ (no matter how small it is), there always exists a $\mu \in (0, \beta]$ such that $\{x : \|x\| = 1\} \neq \{x : G_\mu(x) = 0\}$. In other words, there exists an x^* with $\|x^*\| = 1$ such that $G_\mu(x^*) \neq 0$. Then, by Lemma 4.4, we conclude that there exists an $\widehat{x}$ satisfying (42). Since $e_n = (0, ..., 0, 1)^T \in R^n$ is not a solution of the system (2), we deduce that $\widehat{x}_n \neq -1$ in (42), and hence $\widehat{x}$ satisfies (43) and $\widehat{x}_{n-1} = 0$, contradicting the assumption of the theorem. □

Remark 4.6. The above theorem provides a sufficient condition for the system (1) to have no rank-one solution. This is equivalent to saying that when the system (1) has a rank-one solution, such a sufficient condition must fail. For example, let us consider the following system:

$$A_1 = \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

For this example, e_5 is not a solution of the system (1), and the condition (43) can be written as

$$\begin{cases} x_5(x_1^2 - x_4^2) = x_1 \\ x_5(x_1^2 - x_3^2 - x_4^2 - x_5^2) = x_2 \\ x_5(x_1 x_2) = x_3 \\ x_5 \in (-1, 0), \; \|x\| = 1. \end{cases}$$

If we set $x_4 = 0$, then the above condition imply that $x_1 = x_3 = 0$, $x_2 = -x_5^3$ and $x_2^2 + x_5^2 = 1$. Therefore, the point $x = (0, -t^3, 0, 0, t)^T$ satisfies the above condition with

$$t = -\left(\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{27}} \right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{27}} \right)^{1/3} \right)^{1/2} \in (-1, 0).$$

So for this example the sufficient condition in Theorem 4.5 does not hold, and it is easy to see that the system (1) with matrices given by (44) has a rank-one solution, for instance, $X = xx^T$ where $x = (0, 1, 0, 0, 0)^T$.

Remark 4.7. From the analysis in this paper, the condition (43) is intrinsically hidden behind the condition "there is no rank-one solution to the system (1)", or equivalently, "$x = 0$ is the only solution to the system (2)". Theorem 4.5 shows that this type of necessary condition together with some other conditions can be sufficient for the nonexistence of a rank-one solution to the system (1). However, the relationship between the new sufficient conditions in this paper and the known condition "$\sum_{i=1}^m t_i A_i \succ 0$" is not clear at present. These two types of conditions seem independent to each other. Corollary 3.3 and Theorem 3.4 indicate that only a small gap exists between the existence and nonexistence of a rank-one solution to the system (1). If $r^* = n$, then $X = 0$ is the only solution of the system (1). However, a small perturbation of the system such that $r^* = n - 1$ will guarantee the existence of a rank-one solution to the system. Thus, the development of a new sufficient condition weaker than $\sum_{i=1}^m t_i A_i \succ 0$ becomes subtle, and there might be no easy and simple way to state such a sufficient condition. As we have shown in section 3, checking whether or not the system (1) has a rank-one solution is equivalent to solving an SDP problem with rank constraints which is a difficult problem. This indicates that the conditions developed in sections 2 and 4 of this paper are not easy to check directly. However, these conditions make it possible for us to understand the problem from the nonlinear analysis perspective.

5 Conclusion

Some sufficient and/or necessary conditions for the existence of a rank-one solution to the system of HLME over the positive semidefinite cone have been developed. These conditions have been derived from two different perspectives: degree theory and rank optimization. The result out of the former shows that the $P_\emptyset$ property of the function defined by quadratic transformations can ensure the existence of a rank-one solution (e.g., Theorems 2.6 and 2.9). From the latter, it turns out that the maximum rank r^*, defined by (21), plays a key role in the existence of a rank-one solution. For instance, $r^* = n - 1$ can ensure the system of HLME has a rank-one solution (see Corollary 3.3), and the number of indefinite matrices in the system can be also related to the existence of a rank-one solution (see Theorem 3.4). Finally a sufficient condition for the nonexistence of a rank-one solution to the system of HLME was also given (see Theorem 4.5).

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