# On orthogonal polynomials with respect to a class of differential operators. 

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#### Abstract

We consider orthogonal polynomials with respect to a linear differential operator $$
\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(z) \frac{d^{k}}{d z^{k}},
$$ where $\left\{\rho_{k}\right\}_{k=0}^{M}$ are complex polynomials such that $\operatorname{deg}\left[\rho_{k}\right] \leq k, 0 \leq k \leq M$, with equality for at least one index. We analyze the uniqueness and zero location of these polynomials. An interesting phenomenon occurring in this kind of orthogonality is the existence of operators for which the associated sequence of orthogonal polynomials reduces to a finite set. For a given operator, we find a classification of the measures for which it is possible to guarantee the existence of an infinite sequence of orthogonal polynomials, in terms of a linear system of difference equations with varying coefficients. Also, for the case of a first-order differential operator, we locate the zeros and establish the strong asymptotic behavior of these polynomials.


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## 1. Introduction

Orthogonality with respect to a linear homogeneous differential operator was introduced in 1] as a generalization of the notion of orthogonal polynomials. There, the authors show that the concept of Chebyshev system plays a fundamental role to solve the problem of the uniqueness of the sequence of the polynomials. A further study of some algebraic and analytic properties of this type of orthogonality is done in [3, 6] for some first and second-order linear homogeneous differential operators. Formally, orthogonality with respect to a linear homogeneous differential operator is defined as follows,
Definition 1. Assume that $\mu$ is a finite positive Borel measure on the real line and let $\left\{\rho_{k}\right\}_{k=0}^{M}$ be a set of functions such that,

$$
\int\left|x^{j} \rho_{k}(x)\right| d \mu(x)<\infty, \quad 0 \leq j<\infty
$$

for all $k=0, \ldots, M$. Denote

$$
\begin{equation*}
\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}, \tag{1}
\end{equation*}
$$

an operator acting over the space of polynomials $\mathbb{P}$.
We say that $\left\{Q_{n}\right\}_{n=0}^{\infty}$, is a sequence of orthogonal polynomials with respect to the pair $\left(\mathcal{L}^{(M)}, \mu\right)$ if $\operatorname{deg}\left[Q_{n}\right] \leq n$ and

$$
\begin{equation*}
\int \mathcal{L}^{(M)}\left[Q_{n}(x)\right] P(x) d \mu(x)=0 \tag{2}
\end{equation*}
$$

for any polynomial $P$ such that $\operatorname{deg}[P] \leq n-1$.
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We recall that in the definition of orthogonality with respect to a differential operator given in [1], $\rho_{M}$ is assumed to be equal to 1 , but we shall drop this assumption. The determination of the sequence of these polynomials can be reduced to the solution of a system of $n$ algebraic linear homogeneous equations on the $n+1$ coefficients of $Q_{n}$, thus the existence is guaranteed. Unlike systems of orthogonal polynomials, it is not possible to affirm uniqueness up to a constant factor and this turns out to be a difficult problem. We say that an index $n$ is normal if for this $n$ the solution is uniquely determined up to a constant multiplicative factor. For a fixed non-negative integer $n, Q_{n}$ will be referred to as the orthogonal polynomial with respect to the pair $\left(\mathcal{L}^{(M)}, \mu\right)$ associated with the index $n$, which in general is not necessarily unique.

In this manuscript, we show necessary and sufficient conditions for the normality of an index $n \in \mathbb{Z}_{+}$ for exactly solvable differential operators and study properties of the sequence of orthogonal polynomials. Exactly solvable operators arise from quantum mechanics and were introduced by Turbiner in [12, 13],

Definition 2. Let $\mathbb{P}_{n}$ be the space of all polynomials of degree at most $n$. A linear differential operator of $M$-th order $\mathcal{L}^{(M)}$ is called exactly-solvable if $\mathcal{L}^{(M)}\left[\mathbb{P}_{n}\right] \subseteq \mathbb{P}_{n}$, for all $n \geq 0$, with equality for at least one index $n$.

Notice that any exactly-solvable operator has the form

$$
\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}
$$

where $\rho_{k}(x)=\sum_{j=0}^{k} \rho_{k, j} x^{j}$ are polynomials which satisfy the condition $\operatorname{deg}\left[\rho_{k}\right] \leq k$ with equality for at least one index $k$.

Some of the techniques used here could be extended to some degree to the general case of linear homogeneous differential operators with polynomial coefficients including the class of Heine-Stieltjes operators as well as the lowering and raising operators with polynomial coefficients, but we will not dwell on it.

The paper is organized as follows. In Section 2 we present connections between this type of orthogonality and some inner products and classify the exactly solvable operators for which this concept of orthogonality reduces to an inner product. In Section 3 we give necessary and sufficient conditions for the normality of an index $n$. The analysis of the existence of infinite sequences of polynomials $\left\{Q_{n}\right\}_{n=m+1}^{\infty}$, for some positive $m$, with $\operatorname{deg}\left[Q_{n}\right]=n$, in terms of a linear system of difference equations with varying coefficients is done in section 4 . In Section 5. we study the location of the zeros for the polynomials $Q_{n}$, and in Section 6, for a first-order differential operator, we obtain a curve that contains the accumulation points of the zeros of the polynomials giving also the strong asymptotic behavior of the polynomials. Finally, in Section 7 we discuss some possible extensions of the results.

## 2. Applications and relation to some inner products

Let us see some examples where orthogonality with respect to a differential operator reduces in some sense to orthogonality with respect to an inner product.

1. When $M=0$, we obtain the classical construction of orthogonal polynomials with respect to a standard inner product

$$
\int Q_{n}(x) P(x) d \mu(x)=0, \quad \operatorname{deg}[P] \leq n-1
$$

2. Let $\zeta \in \mathbb{C}$ be fixed and consider the differential operator $\mathcal{L}_{\zeta}: W^{1,2}(\mu) \rightarrow L^{2}(\mu)$

$$
\mathcal{L}_{\zeta}[f](x)=f(x)+(x-\zeta) f^{\prime}(x),
$$

where $W^{1,2}(\mu)=\left\{f \in L^{2}(\mu): f^{\prime} \in L^{2}(\mu)\right\}$ is the Sobolev space of index 1 . Let us consider a positive Borel measure $\mu$ supported on a subset $\Delta \subset \mathbb{R}$. The polar polynomial associated to $\mu$, see [2], is defined as the polynomial $Q_{n}$ of degree $n$ orthogonal with respect to $\left(\mathcal{L}_{\zeta}, \mu\right)$. Let us consider

$$
\begin{aligned}
\Pi_{0, \zeta} & =1 \\
\Pi_{n+1, \zeta}(z) & =(z-\zeta) Q_{n}(z), \quad n \geq 0 .
\end{aligned}
$$

Then it is not difficult to see that the family of polynomials $\left\{\Pi_{n, \zeta}\right\}_{n=0}^{\infty}$ is orthogonal with respect to the Sobolev inner product

$$
\langle f, g\rangle_{\zeta}=\eta f(\zeta) g(\zeta)+\int_{\Delta} f^{\prime}(x) g^{\prime}(x) d \mu(x)
$$

for some $\eta>0$.
The authors in [3] give a detailed study of this family for the case in which $\mu=\mu_{\lambda}, \lambda>-\frac{1}{2}$, is the (classical) Gegenbauer or ultraspherical measure, i.e. $d \mu_{\lambda}(x)=\left(1-x^{2}\right)^{\lambda-\frac{1}{2}} d x$.

Before we state our first result we remind that a Bochner-Krall type operator is an operator for which the associated sequence of eigenpolynomials is orthogonal with respect to some finite positive Borel measure, see $[5,77,8]$. Given a finite positive Borel measure on the real line $\mu$, consider the moment functional $\sigma: \mathbb{P} \longrightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\left\langle\sigma, x^{n}\right\rangle=\int x^{n} d \mu(x) \tag{3}
\end{equation*}
$$

Following [9], we define the moment functional $\sigma^{\prime}$, the derivative of $\sigma$, and $\phi \sigma$, the multiplication of $\sigma$ times a polynomial $\phi$ as:

$$
\begin{align*}
\left\langle\sigma^{\prime}, \psi\right\rangle & =-\left\langle\sigma, \psi^{\prime}\right\rangle  \tag{4}\\
\langle\phi \sigma, \psi\rangle & =\langle\sigma, \phi \psi\rangle
\end{align*}
$$

for all $\psi \in \mathbb{P}$.
Consider now the bilinear form on $\mathbb{P}$

$$
\begin{equation*}
[Q, P]=\int \mathcal{L}^{(M)}[Q(x)] P(x) d \mu(x) \tag{5}
\end{equation*}
$$

In general, it is not possible to affirm that the bilinear form (5) defines an inner product. The following theorem characterizes the exactly solvable operators and the measures $\mu$ for which (5) defines an inner product.

Theorem 1. Let $\mathcal{L}^{(M)}$ be an exactly solvable and $\mu$ a positive Borel measure. Then, a necessary and sufficient condition for (5) to be an inner product is that:

- $\mathcal{L}^{(M)}$ is a Bochner-Krall operator and $\mu$ is the measure such that the polynomial eigenfunctions of the operator form a system of orthogonal polynomials.
- $\mathcal{L}^{(M)}$ has positive eigenvalues.

In such a case, the sequence of monic polynomials $\left\{Q_{n}\right\}_{n=0}^{\infty}$ orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$ coincide with the monic orthogonal polynomials $\left\{P_{n}\right\}_{n=0}^{\infty}$ with respect to the measure $\mu$.

Proof. Suppose that the relation (5) defines an inner product. We have then that (5) is symmetric, taking into account (3) and (4) we have,

$$
\begin{array}{r}
\int \mathcal{L}^{(M)}[Q](x) P(x) d \mu(x)=\left\langle\mathcal{L}^{(M)}[Q](x) \sigma, P(x)\right\rangle=[Q, P] \\
=[P, Q]=\int \mathcal{L}^{(M)}[P](x) Q(x) d \mu(x)=\left\langle\mathcal{L}^{(M)}[P] \sigma, Q\right\rangle
\end{array}
$$

that is,

$$
\left\langle\mathcal{L}^{(M)}[Q] \sigma, P\right\rangle=\left\langle\mathcal{L}^{(M)}[P] \sigma, Q\right\rangle,
$$

and from [9, Theorem 2.4 (ii) implies i)] we have that $\mathcal{L}^{(M)}$ is a Bochner-Krall operator and $\mu$ is the measure with respect to which the polynomial eigenfunctions of the operator form a system of orthogonal polynomials.

The second condition follows from the fact that [., .] must define a positive definite bilinear form.
The converse implication is straightforward.
The assertion that the sequence of monic polynomials $\left\{Q_{n}\right\}_{n=0}^{\infty}$ coincides with the monic orthogonal polynomials $\left\{P_{n}\right\}_{n=0}^{\infty}$ with respect to the measure $\mu$ follows from the fact that $\mathcal{L}^{(M)}$ is an exactly solvable operator with positive eigenvalues $\left\{\lambda_{n}\right\}_{n=0}^{\infty}$ which implies that condition (2) is equivalent to solving

$$
\mathcal{L}^{(M)}\left[Q_{n}\right]=\lambda_{n} P_{n},
$$

from this we deduce that $Q_{n}=P_{n}$.
We also mention that for the case of first and second order differential operators, the $n$-th orthogonal polynomial associated to an index $n$ can be interpreted as the the equilibrium points of a flux of a complex potential due to a system of fixed points, cf. [2, [6].

## 3. Necessary and sufficient conditions for the normality of an index

In this section we give necessary and sufficient conditions for the normality of an index $n$ for the class of exactly solvable operators. As $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ is an exactly solvable operator, following (2), it is not difficult to see that the monic orthogonal polynomials with respect to the pair $\left(\mathcal{L}^{(M)}, \mu\right)$ associated to an index $n$ are linear combinations of a monic polynomial solution of

$$
\begin{equation*}
\mathcal{L}^{(M)}[y]=\lambda_{n} P_{n}, \tag{6}
\end{equation*}
$$

and a monic polynomial solution of

$$
\begin{equation*}
\mathcal{L}^{(M)}[y]=0 \tag{7}
\end{equation*}
$$

here $P_{n}$ denotes the $n$-th monic orthogonal polynomials with respect to $\mu$ and $\lambda_{n}=\sum_{k=0}^{M} \rho_{k, k} \frac{n!}{(n-k)!}$ is the coefficient associated to the factor $x^{n}$ of the polynomial $\mathcal{L}^{(M)}\left[x^{n}\right]$. By convention, $\frac{n!}{(n-k)!}=0$ when $k>n$. In the sequel we shall assume that $\lambda_{n}$ will denote this coefficient.

Before we state the results of the section we show with an example that in general we do not have normality of an index for the class of operators that we consider.

Example 1. [Second order differential operator] Suppose that $M=2$ and define $\mathcal{L}[f]=f^{\prime \prime}-2 x f^{\prime}+2 f, f \in$ $\mathbb{P}$. Notice that the eigenfunctions of this operator are the Hermite polynomials $\left\{H_{n}\right\}_{n=0}^{\infty}$ with eigenvalues $\lambda_{n}=2(1-n)$ and that $\mathcal{L}[x]=0$. Consider the measure $d \mu(x)=\frac{e^{-x^{2}} d x}{x^{2}+1}$ supported on $\mathbb{R}$ and denote by $\left\{P_{n}\right\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to $\mu$. Notice that if $n>3$ the polynomial $P_{n}$ can be expanded in the basis $\left\{H_{k}\right\}_{k=0}^{n}$ as

$$
\begin{equation*}
P_{n}(x)=H_{n}(x)+\alpha_{n-1} H_{n-1}(x)+\alpha_{n-2} H_{n-2}(x), \quad \alpha_{n-k}=\frac{\int P_{n}(x) H_{n-k}(x) e^{-x^{2}} d x}{\sqrt{\int H_{n-k}^{2}(x) e^{-x^{2}} d x}} ; k=1,2 ; \tag{8}
\end{equation*}
$$

from where we deduce that the monic orthogonal polynomial $Q_{n}$ with respect to $(\mathcal{L}, \mu)$ for the index $n$, for $n>3$, can be described as

$$
Q_{n}(x)=\left\{\begin{array}{l}
H_{n}(x)+\frac{\lambda_{n} \alpha_{n-1}}{\lambda_{n-1}} H_{n-1}(x)+\frac{\lambda_{n} \alpha_{n-2}}{\lambda_{n-2}} H_{n-2}(x)+c x, \quad c \in \mathbb{C} \\
x
\end{array}\right.
$$

By a similar argument, by expanding $P_{0}, P_{1}, P_{2}, P_{3}$ in terms of $H_{0}, H_{1}, H_{2}, H_{3}$ we have that the solutions to (6) and (77) give that if $n \leq 3$ then $Q_{0}(x)=1, Q_{1}(x)=Q_{2}(x)=Q_{3}(x)=x$; therefore, we have normality only for $n \leq 3$.
Remark 1. We correct here [1, Ex. 1]. There, it is stated that any exactly solvable operator for which $\rho_{M} \equiv 1$ and $\rho_{k} \not \equiv 0,0 \leq k<M$ satisfies the conditions of [1, Th. 3].

For the operator of Example $\square 1$ of the present paper, $\mathcal{L}[f]=f^{\prime \prime}-2 x f^{\prime}+2 f$. The function $\rho_{0,1}$ defined in [1, Th. 3] (according with the notation employed in that paper) simplifies to $\rho_{0}+\rho_{1}=2-2 x$ and this function has a zero on supp $(\mu)$. This particular operator does not satisfy the conditions given in [1, Th. 3]; therefore, that theorem cannot be applied for exactly solvable operators in general.

In order to provide a necessary and sufficient condition for the normality of an index, we introduce some auxiliary notation and prove some preliminaries lemmas. In the sequel, we denote by $\Delta_{n}$ the determinant of the Hankel matrix defined by the moments $\mu_{0}, \ldots, \mu_{2 n}$ of the measure $\mu$. Define $\Delta_{0,0}=\mu_{0}$ and denote by $\Delta_{n, i}, 0 \leq i \leq n$, the determinant of the following matrix with column $i+1$ deleted

$$
\left(\begin{array}{ccc}
\mu_{0} & \cdots & \mu_{n} \\
& \vdots & \\
\mu_{n-1} & \cdots & \mu_{2 n-1}
\end{array}\right)
$$

Consider the infinite upper triangular matrix $A=\left(a_{i, j}\right)$ with entries

$$
\begin{equation*}
a_{i, j}=\sum_{k=j-i}^{\min (M, j-1)} \rho_{k, i+k-j} \frac{(j-1)!}{(j-1-k)!}, \quad i \leq j \tag{9}
\end{equation*}
$$

and set $A_{n}=\left(a_{i, j}\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$.
Let $q_{n}(x)=\sum_{j=0}^{n} \alpha_{n, j} x^{j}$ be a generic polynomial of degree $n$. By $\mathfrak{a}_{n+1}=\left(\alpha_{n, 0}, \ldots, \alpha_{n, n}\right)^{t}$ we denote the column vector of the coefficients of $q_{n}$.
Lemma 2. Let $\mu$ be a positive Borel measure on the real line, $\left\{P_{n}\right\}_{n=0}^{\infty}$ the associated sequence of monic orthogonal polynomials, and $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ an exactly solvable operator, where $\rho_{k}(x)=\sum_{j=0}^{k} \rho_{k, j} x^{j}$. Then, (6) can be expressed as

$$
\begin{equation*}
A_{n+1} \mathfrak{a}_{n+1}=\lambda_{n} \mathfrak{b}_{n+1} \tag{10}
\end{equation*}
$$

and (7) can be expressed as

$$
\begin{equation*}
A_{n+1} \mathfrak{a}_{n+1}=0 \tag{11}
\end{equation*}
$$

where $\mathfrak{b}_{n+1}=\left(\beta_{n, 0}, \ldots, \beta_{n, n}\right)^{t}, \beta_{n, i}=\Delta_{n, i} \Delta_{n, n}^{-1}, 0 \leq i \leq n$, is the column vector of the coefficients of $P_{n}$. Proof. Let $q_{n}(x)=\sum_{j=0}^{n} \alpha_{n, j} x^{j}$. We have that

$$
\begin{align*}
\mathcal{L}^{M}\left[\sum_{j=0}^{n} \alpha_{j} x^{j}\right] & =\sum_{k=0}^{M}\left(\sum_{u=0}^{k} \rho_{k, u} x^{u} \sum_{j=0}^{n} \alpha_{n, j} x^{j-k} \frac{j!}{(j-k)!}\right)  \tag{12}\\
& =\sum_{k=0}^{M} \sum_{j=k}^{\min (k, n)} \sum_{u=0}^{k} \rho_{k, u} \frac{j!}{(j-k)!} x^{u+j-k} \alpha_{n, j} \\
& =\sum_{k=0}^{M} \sum_{i=0}^{n} \sum_{j=\max (k, i)}^{\min (n, i+k)}\left(\rho_{k, i+k-j} \frac{j!}{(j-k)!} \alpha_{n, j}\right) x^{i} \\
& =\sum_{i=0}^{n} \sum_{j=i}^{n} \sum_{k=j-i}^{\min (M, j)}\left(\rho_{k, i+k-j} \frac{j!}{(j-k)!} \alpha_{n, j}\right) x^{i} .
\end{align*}
$$

From Heine's formula for the monic orthogonal polynomials, we have

$$
P_{n}(x)=\Delta_{n, n}^{-1}\left|\begin{array}{ccc}
\mu_{0} & \cdots & \mu_{n}  \tag{13}\\
& \vdots & \\
\mu_{n-1} & \cdots & \mu_{2 n-1} \\
1 & \cdots & x^{n}
\end{array}\right|
$$

Therefore, (6) or (7) can be expressed in matrix form as

$$
A_{n+1} \mathfrak{a}_{\mathfrak{n}+1}=\lambda_{n} \mathfrak{b}_{\mathfrak{n}+1}
$$

and

$$
A_{n+1} \mathfrak{a}_{\mathfrak{n}+1}=0
$$

respectively.
Lemma 3. Let $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ be an exactly solvable operator, $\rho_{k}(x)=\sum_{i=0}^{k} \rho_{k, i} x^{i}$, and $A_{n+1}$ the matrix whose entries are defined by (91). Then, $a_{j+1, j+1}$ is the coefficient of $x^{j}$ of the polynomial $\mathcal{L}^{(M)}\left[x^{j}\right]$.

## Proof. We have

$$
\begin{aligned}
\mathcal{L}^{(M)}\left[x^{j}\right] & =\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}} x^{j} \\
& =\sum_{k=0}^{M} \rho_{k}(x) \frac{j!}{(j-k)!} x^{j-k} \\
& =\sum_{k=0}^{\min (M, j)} \sum_{i=0}^{k} \rho_{k, i} \frac{j!}{(j-k)!} x^{i+j-k} .
\end{aligned}
$$

From this expression we obtain that if $i=k$ then the coefficient of $x^{j}$ in $\mathcal{L}^{(M)}\left[x^{j}\right]$ is

$$
\sum_{k=0}^{\min (M, j)} \rho_{k, k} \frac{j!}{(j-k)!},
$$

which corresponds to the coefficient $a_{j+1, j+1}$ in (19) of the matrix $A_{n+1}$.
From the preceding lemmas we deduce a necessary and sufficient condition for the normality of an index $n$.

Theorem 4. Let $\mu$ be a positive Borel measure on the real line, $\left\{P_{n}\right\}_{n=0}^{\infty}$ the associated sequence of monic orthogonal polynomials, and $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ an exactly solvable differential operator. Then, an index $n \in \mathbb{Z}_{+}$is normal if and only if either
i) $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{k}\right]\right]=k, \quad \forall k: \quad 0 \leq k \leq n$, or
ii) There exist indexes $n_{1}, \ldots, n_{k} ; 0 \leq n_{1} \leq \ldots \leq n_{k} \leq n$, such that $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{n_{j}}\right]\right]<n_{j}, 1 \leq j \leq k$,

1) if $k \geq 1$, then $\left\{\mathcal{L}^{(M)}[1], \mathcal{L}^{(M)}[x], \ldots, \mathcal{L}^{(M)}\left[x^{n_{k}}\right]\right\}$ has $n_{k}$ linearly independent vectors,
2) if $n_{k}<n$, then the moments of the measure $\mu$ satisfy the relation

$$
\begin{equation*}
\sum_{j=0}^{n-n_{k}-1} \gamma_{n-n_{k}-j} \Delta_{n, n-j} \neq-\Delta_{n, n_{k}} \tag{14}
\end{equation*}
$$

where $\left\{\gamma_{i}\right\}_{i=1}^{n-n_{k}}$ are such that

$$
\begin{equation*}
\left(\gamma_{1}, \ldots, \gamma_{n-n_{k}}\right) B=\left(-a_{n_{k}+1, n_{k}+2}, \ldots,-a_{n_{k}+1, n+1}\right), \tag{15}
\end{equation*}
$$

and $B$ is the matrix

$$
B=\left(\begin{array}{ccccc}
a_{n_{k}+2, n_{k}+2} & a_{n_{k}+2, n_{k}+3} & & & \vdots \\
& \vdots & & & \\
0 & & \cdots & a_{n, n} & a_{n, n+1} \\
0 & & \cdots & 0 & a_{n+1, n+1}
\end{array}\right)
$$

Proof. We assume that $n \geq 1$, otherwise we have that $n=0$ is normal and there is nothing to prove. Suppose that the index $n$ is normal. This is equivalent to saying that the following alternatives for (6) and (7) hold.
a) Equation (6) has a unique monic polynomial solution.
b) Equation (7) has a unique non zero monic polynomial solution and $\lambda_{n} \neq 0$.
c) Equation (7) has a unique non zero monic polynomial solution and $\lambda_{n}=0$.

If we have alternative $a$ ), then Lemma 2 gives that this statement is equivalent to saying that $\operatorname{Ker}\left[A_{n+1}\right]=$ $\{0\}$, hence the elements of the diagonal of the matrix $A_{n+1}$ are non null. By Lemma 3 we obtain that $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{k}\right]\right]=k, \forall k: 0 \leq k \leq n$, that is, we have $i$ ).

Suppose we have alternatives $b$ ) or $c$ ). We have that there exist indexes $n_{1}, \ldots, n_{k} ; 0 \leq n_{1} \leq \ldots \leq n_{k} \leq n$, such that $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{n_{j}}\right]\right]<n_{j}, 1 \leq j \leq k$. For $k>1$, we partition the matrix $A_{n+1}$ in blocks as

$$
A_{n+1}=\left(\widetilde{B}_{2}, \widetilde{B}_{1}\right)
$$

where $\widetilde{B}_{2}$ is the block of $A_{n+1}$ formed by its first $n_{k}+1$ columns, and $\widetilde{B}_{1}$ has the columns $n_{k}+2, \ldots, n+1$ of $A_{n+1}$. When $n_{k}=n$,

$$
A_{n+1}=\left(\widetilde{B}_{2}\right)
$$

Suppose we have alternative b). If $k \geq 1$ and $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{n_{j}}\right]\right]<n_{j}, \forall j=1, \ldots, k$, by Lemma 3 $a_{n_{j}+1, n_{j}+1}=0$; hence, $\operatorname{rank}\left[\widetilde{B}_{1}\right]=n-n_{k}$, notice that since $\lambda_{n} \neq 0$ then we have that $n_{k}<n$, therefore the block $\widetilde{B}_{1}$ is not empty. As (7) has a unique non null monic polynomial solution and Lemma 2 we deduce that $\operatorname{dim}\left[\operatorname{Ker}\left[A_{n+1}\right]\right]=1$; therefore, $\operatorname{rank}\left[A_{n+1}\right]=n$. By denoting by $v_{i}$ the $i$-th column of $A_{n+1}$ and by noting that $\operatorname{span}\left[\left\{v_{i}\right\}_{i=1}^{n_{k}+1}\right] \bigcap \operatorname{span}\left[\left\{v_{i}\right\}_{i=n_{k}+2}^{n+1}\right]$ reduces to the null element, one has

$$
\begin{aligned}
n & =\operatorname{rank}\left[\left\{v_{i}\right\}_{i=1}^{n_{k}+1} \bigcup\left\{v_{i}\right\}_{i=n_{k}+2}^{n+1}\right] \\
& =\operatorname{rank}\left[\left\{v_{i}\right\}_{i=1}^{n_{k}+1}\right]+\operatorname{rank}\left[\left\{v_{i}\right\}_{i=n_{k}+2}^{n+1}\right]=\operatorname{rank}\left[\left\{v_{i}\right\}_{i=1}^{n_{k}+1}\right]+n-n_{k},
\end{aligned}
$$

which implies that $\operatorname{rank}\left[\left\{v_{i}\right\}_{i=1}^{n_{k}+1}\right]=n_{k}$; which is equivalent to saying that $\left\{\mathcal{L}^{(M)}[1], \mathcal{L}^{(M)}[x], \ldots, \mathcal{L}^{(M)}\left[x^{n_{k}}\right]\right\}$ has $n_{k}$ linearly independent vectors, and we have 1) of $i i$ ).

Consider now the statement of $b$ ) that $\lambda_{n} \neq 0$, or equivalently, that $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{n}\right]\right]=n$. From Lemma 3 we have necessarily that $n_{k}<n$. Since equation (7) has a unique non zero monic polynomial solution and the index $n$ is normal by hypothesis, then (6) has no solution. By Lemma 2 we deduce that the system

$$
A_{n+1} \mathfrak{a}_{n+1}=\lambda_{n} \mathfrak{b}_{n+1}
$$

is necessarily incompatible. Let us multiply the above relation on both sides by the matrix

$$
\Gamma=\left(\right)
$$

where $I_{n_{k} \times n_{k}}, \theta_{\left(n+1-n_{k}\right) \times n_{k}}, \theta_{n_{k} \times\left(n+1-n_{k}\right)}$ are blocks corresponding to the identity and null matrices of sizes $n_{k} \times n_{k},\left(n+1-n_{k}\right) \times n_{k}, n_{k} \times\left(n+1-n_{k}\right)$ respectively and the $\left\{\gamma_{i}\right\}_{i=1}^{n-n_{k}}$ are as in (15). By noting that the row $n_{k}+1$ of the matrix $A_{n+1}$ is a linear combination of the rows $n_{k}+2, \ldots, n+1$ one has

$$
\begin{equation*}
\Gamma A_{n+1} \mathfrak{a}_{n+1}=\lambda_{n} \Gamma \mathfrak{b}_{n+1} . \tag{16}
\end{equation*}
$$

Hence, the system (16) is incompatible if and only if the component $n_{k}+1$ in vector $\mathfrak{b}_{n+1}$ satisfies that

$$
\sum_{j=0}^{n-n_{k}-1} \gamma_{n-n_{k}-j} \Delta_{n, n-j} \neq-\Delta_{n, n_{k}}
$$

and we obtain 2) of $i i$ ). Conversely, if 2) and 1) of $i i$ ) hold then we have again the statement $b$ ) which is equivalent to the normality of $n$.

Finally, suppose that alternative c) is the case. Lemma 2 gives that systems (10) and (11) are the same. If $k=1$ then we have that $\operatorname{dim}\left[\operatorname{Ker}\left[A_{n+1}\right]\right]=1$ and we are done. Assume that $k>1$, then $n_{k}=n$ and

$$
A_{n+1}=\left(\widetilde{B}_{2}\right)
$$

As the solution to (7) is non zero and unique we have that $\operatorname{dim}\left[\operatorname{Ker}\left[A_{n+1}\right]\right]=1$, therefore, $\operatorname{rank}\left[A_{n+1}\right]=$ $n$ which implies that $\operatorname{rank}\left[\widetilde{B}_{2}\right]=n_{k}$; that is, the number of linearly independent rows of the block $\widetilde{B}_{2}$ is $n_{k}$ and in virtue of Lemmas 2 and 3, this is equivalent to saying that $\left\{\mathcal{L}^{(M)}[1], \mathcal{L}^{(M)}[x], \ldots, \mathcal{L}^{(M)}\left[x^{n_{k}}\right]\right\}$ has $n_{k}$ linearly independent vectors and we have 1) of $i i$ ). The converse implication is straightforward.

It is not difficult to see that the condition $i$ ) obtained in Theorem 4 is equivalent to affirming that

$$
\begin{equation*}
\left\{\mathcal{L}^{(M)}[1], \ldots, \mathcal{L}^{(M)}\left[x^{n}\right]\right\} \tag{17}
\end{equation*}
$$

is linearly independent which is also equivalent to saying that this set is a Markov system. In 11, Th 1] it was proved that if (17) forms a Markov system then we have normality of an index for linear homogeneous differential operators in general.

It seems natural to conjecture that for a general homogeneous linear differential operator a necessary and sufficient condition could be that, either (17) is a Markov system or if $k \geq 1$ then

$$
\left\{\mathcal{L}^{(M)}[1], \mathcal{L}^{(M)}[x], \ldots, \mathcal{L}^{(M)}\left[x^{n_{k}}\right]\right\}
$$

has $n_{k}$ linearly independent functions on the support of the measure $\mu$, plus some additional conditions on the moments of the measure.

## 4. Existence and uniqueness of polynomial solutions of degree $n$

An interesting phenomenon that occurs in this type of orthogonality is the existence of operators and measures for which the associated sequence of orthogonal polynomials reduces to a finite set. A straightforward example can be constructed to illustrate this.

Example 2. [First order differential operator] Let $\mathcal{L}[f](x)=x f^{\prime}, f \in \mathbb{P}$, and consider any positive Borel measure $\mu$ supported on a compact subset of $\mathbb{R}_{+}$. According to (2), the orthogonal polynomial $Q_{n}$ with respect to $(\mathcal{L}, \mu)$ associated to the index $n$ is defined by

$$
\int x Q_{n}^{\prime}(x) x^{k} d \mu(x)=0, \quad \forall k \leq n-1
$$

But this is possible if and only if $Q_{n}^{\prime} \equiv 0$. Hence the sequence $\left\{Q_{n}\right\}_{n=0}^{\infty}$ reduces to a constant.
The preceding example shows that the sequence of polynomials $\left\{Q_{n}\right\}_{n=0}^{\infty}$ orthogonal with respect to the operator $\mathcal{L}[f](x)=x f^{\prime}, f \in \mathbb{P}$, and any positive Borel measure supported on a compact subset of $\mathbb{R}_{+}$ reduces to a constant. In this section we analyze necessary and sufficient conditions on the pair $\left(\mathcal{L}^{(M)}, \mu\right)$ for the existence and uniqueness of infinite sequences of orthogonal polynomials. We shall need the following preliminary lemma.
Lemma 5. Let $\left\{\widehat{P}_{n}\right\}_{n=0}^{\infty}$, deg $\left[\widehat{P}_{n}\right]=n$, be a sequence of monic polynomials and $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ an exactly solvable differential operator on $\mathbb{P}$. Then the following conditions are equivalent:
i) $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{n}\right]\right]=n, \quad \forall n \geq 0$.
ii) For every $n \in \mathbb{Z}_{+}$there exists a unique monic polynomial $\widehat{Q}_{n}$ such that

$$
\mathcal{L}^{(M)}\left[\widehat{Q}_{n}\right]=\lambda_{n} \widehat{P}_{n} .
$$

iii) $\operatorname{Ker}\left[\mathcal{L}^{(M)}\right]=\{0\}$.
iv) $\lambda_{n}=\sum_{j=0}^{M} \rho_{j, j} \frac{n!}{(n-j)!} \neq 0, \quad \forall n \geq 0$.

Proof. i) $\Leftrightarrow i$ i)
Suppose that $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{n}\right]\right]=n, \forall k \geq 0$. Then we have that for every fixed $n_{0} \geq 0,\left\{\mathcal{L}^{(M)}\left[x^{n}\right]\right\}_{n=0}^{n_{0}}$ is a basis of $\mathbb{P}_{n_{0}}$. Hence, it is possible to find $\left\{\alpha_{n}\right\}_{n=0}^{n_{0}}$ such that $\widehat{P}_{n_{0}}(x)=\sum_{k=0}^{n_{0}} \alpha_{n_{0}, k} \mathcal{L}^{(M)}\left[x^{k}\right]$ and thus, by construction we have that there exists a unique monic polynomial $\widehat{Q}_{n_{0}}(x)=\sum_{k=0}^{n_{0}} \alpha_{k} x^{k}$ such that $\mathcal{L}^{(M)}\left[\widehat{Q}_{n}\right]=$ $\lambda_{n} \widehat{P}_{n}$ holds and we get $i i$ ).

Suppose now that for some index $n_{0}$ we have that $\operatorname{deg}\left[\mathcal{L}^{(M)}\left[x^{n_{0}}\right]\right]<n_{0}$. From this fact and the hypothesis that $\mathcal{L}^{(M)}$ is exactly solvable, every polynomial $Q_{n_{0}}$ of degree less or equal to $n_{0}$ satisfies that $\mathcal{L}^{(M)}\left[\widehat{Q}_{n_{0}}\right]$ is a polynomial of degree less than $n_{0}$ and hence it cannot satisfy $\mathcal{L}^{(M)}\left[\widehat{Q}_{n}\right]=\lambda_{n} \widehat{P}_{n}$. That is $\left.i i\right) \Rightarrow i$; therefore, $i) \Leftrightarrow i i)$.
$i i) \Leftrightarrow i i i)$.
Assume that $i i$ ) holds. As $\widehat{Q}_{n}$ is unique, for every non negative integer we have that $\operatorname{Ker}\left[\mathcal{L}^{(M)}\right]=\{0\}$; that is, $i i) \Rightarrow i i i)$. The converse implication is straightforward.
$i) \Leftrightarrow i v)$.
This follows from the fact that the coefficient associated to the factor $x^{n}$ in $\mathcal{L}^{(M)}\left[x^{n}\right]$ is equal to

$$
\sum_{j=0}^{M} \rho_{j, j} \frac{k!}{(k-j)!}
$$

We characterize now the exactly solvable operators for which we can guarantee the existence and uniqueness of an infinite sequence of orthogonal polynomials $\left\{Q_{n}\right\}_{n=0}^{\infty}$, such that each polynomial $Q_{n}$ has degree equal to $n$.

Theorem 6. Let $\mu$ be a positive Borel measure on the real line, $\left\{P_{n}\right\}_{n=0}^{\infty}$ the associated sequence of monic orthogonal polynomials, and $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ an exactly solvable operator with $\rho_{k}(x)=\sum_{j=0}^{k} \rho_{k, j} x^{j}$. Then, there exists a unique sequence of monic polynomials $\left\{Q_{n}\right\}_{n=0}^{\infty}$, each polynomial $Q_{n}$ of degree equal to $n$ and orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$, if and only if any of the statements of Lemma 5 hold.

Proof. It follows from Lemma 5 by taking $\left\{\widehat{P}_{n}\right\}_{n=0}^{\infty}=\left\{P_{n}\right\}_{n=0}^{\infty}$ in ii).
A natural question then arises. What happens if any of the conditions of Lemma 5 does not hold? It is not difficult to see that from the expression of $\lambda_{n}$ as a polynomial in $n$ given in $i v$ ) of Lemma 5. only for a finite number of values this relation will not be valid. Let us denote by $S$ the set of such indexes. In this case, it is also possible to give necessary and sufficient conditions on the measure $\mu$ to have an infinite sequence $\left\{Q_{n}\right\}_{n \notin S}$ such that $\operatorname{deg}\left[Q_{n}\right]=n$. The following theorem characterizes such measures in terms of a finite set of difference equations with given initial conditions.

Theorem 7. Let $\mu$ be a positive Borel measure on the real line, $\left\{P_{n}\right\}_{n=0}^{\infty}$ the associated sequence of monic orthogonal polynomials, $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ an exactly solvable differential operator, and $\rho_{k}(x)=\sum_{j=0}^{k} \rho_{k, j} x^{j}$. Suppose that condition iv) of Lemma 5 is not satisfied and denote by $S=\left\{n_{1}, \ldots, n_{k}\right\}$ the set of indexes for which that condition does not hold. Then, there exists a sequence of monic polynomials $\left\{Q_{n}\right\}_{n \notin S}$ orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$ such that deg $\left[Q_{n}\right]=n$ if and only if the moments of the measure $\mu$ satisfy the system

$$
\begin{gather*}
\sum_{v=-M}^{n_{1}}\left(\sum_{k=\max (-v, 0)}^{M} \sum_{i=\max (0, v)}^{\min \left(n_{1}, v+k\right)}(-1)^{i+n_{1}} \frac{n!}{(n-k)!} \Delta_{n_{1}, i} \rho_{k, v-i+k}\right) \mu_{n+v}=0 \\
\vdots  \tag{18}\\
\sum_{v=-M}^{n_{k}}\left(\sum_{k=\max (-v, 0)}^{M} \sum_{i=\max (0, v)}^{\min \left(n_{k}, v+k\right)}(-1)^{i+n_{k}} \frac{n!}{(n-k)!} \Delta_{n_{k}, i} \rho_{k, v-i+k}\right) \mu_{n+v}=0,
\end{gather*}
$$

where $n_{k} \in S$ and $n \notin S$. Moreover, if $\mu_{0}, \ldots, \mu_{2 n_{k}-1}$ are the moments of some positive measure supported on a subset of $\mathbb{R}$ satisfying (18), then for $n>n_{k}$ the system (18) defines a linear system of difference equations with varying coefficient and initial conditions $\mu_{0}, \ldots, \mu_{2 n_{k}-1}$.

Proof. By definition, the set $\left\{Q_{k}\right\}_{k=0}^{n}, n \notin S$ exists if and only if for every $n \notin S$ it is possible to find coefficients $\left\{\alpha_{k}\right\}_{k=0}^{n}$ such that

$$
P_{n}(x)=\sum_{k \notin S} \alpha_{k} \mathcal{L}^{(M)}\left[x^{k}\right]
$$

As $\mathcal{L}^{(M)}$ is exactly solvable the preceding condition is equivalent to

$$
\begin{equation*}
\operatorname{span}\left[\left\{P_{k}\right\}_{k=0}^{n}\right]=\operatorname{span}\left[\left\{\mathcal{L}^{(M)}\left[x^{k}\right]\right\}_{k=0}^{n}\right], \quad k \notin S \tag{19}
\end{equation*}
$$

and (19) is equivalent to saying that there exist coefficients $\left\{\beta_{k}\right\}_{k=0}^{n}$ such that

$$
\mathcal{L}^{(M)}\left[x^{n}\right]=\sum_{k \notin S} \beta_{k} P_{k}(x), \quad n \notin S,
$$

and this condition is satisfied if and only if $\mu$ satisfies the finite system of equations

$$
\begin{align*}
\int \mathcal{L}^{(M)}\left[x^{n}\right] P_{n_{1}}(x) d \mu(x) & =0  \tag{20}\\
\vdots & \\
\int \mathcal{L}^{(M)}\left[x^{n}\right] P_{n_{k}}(x) d \mu(x) & =0
\end{align*}
$$

for all $n \notin S$ and $n_{j} \in S, j=1, \ldots, k$. By substituting in (20) Heine's formula (13) for the monic orthogonal polynomials we obtain

$$
\begin{aligned}
& \int \sum_{k=0}^{M} \frac{n!}{(n-k)!}\left|\begin{array}{ccc}
\mu_{0} & \cdots & \mu_{n_{1}} \\
\mu_{n_{1}-1} & \cdots & \\
\sum_{j=0}^{k} \rho_{k, j} x^{n+j-k} & \cdots & \sum_{j=0}^{k} \rho_{k, j} x^{n+j-k+n_{1}}
\end{array}\right| d \mu(x)=0, \\
& \int \sum_{k=0}^{M} \frac{n!}{(n-k)!}\left|\begin{array}{ccc}
\mu_{0} & \ldots & \mu_{n_{k}} \\
\vdots \\
\sum_{j=0}^{k} \rho_{k, j} x^{n+j-k} & \cdots & \sum_{j=0}^{k} \rho_{k, j} x^{n+j-k+n_{k}}
\end{array}\right| d \mu(x)=0 .
\end{aligned}
$$

By commuting the integral and the summation symbols, expanding the determinant by minors and doing some change of indexes we have

$$
\begin{aligned}
& \sum_{k=0}^{M} \sum_{i=0}^{n_{1}} \sum_{u=-k}^{0}(-1)^{i+n_{1}} \frac{n!}{(n-k)!} \Delta_{n_{1}, i} \rho_{k, u+k} \mu_{n+u+i}=0 \\
& \vdots \\
& \sum_{k=0}^{M} \sum_{i=0}^{n_{k}} \sum_{u=-k}^{0}(-1)^{i+n_{k}} \frac{n!}{(n-k)!} \Delta_{n_{k}, i} \rho_{k, u+k} \mu_{n+u+i}=0
\end{aligned}
$$

which is equivalent to (18).
Consider now that $\mu_{0}, \ldots, \mu_{2 n_{k}-1}$ are the moments of some positive measure supported on a subset of $\mathbb{R}$ satisfying (18). It is not difficult to see that for $n>n_{k}$, the system (18) defines a linear system of difference equations with varying coefficients and with initial conditions $\mu_{0}, \ldots, \mu_{2 n_{k}-1}$.

For a given operator $\mathcal{L}^{(M)}$, we denote the class of positive Borel measures with support contained in $\mathbb{R}$ which satisfy system (18) as $\Xi_{\mathcal{L}^{(M)}}$. Note that in general, $\Xi_{\mathcal{L}^{(M)}}$ does not necessarily reduce to the empty set, as show the following examples.
Example 3. Consider the first order linear differential operator $\mathcal{L}[f](x)=x f^{\prime}(x)-f(x), f \in \mathbb{P}$. Note that $\mathcal{L}\left[x^{n}\right]=(n-1) x^{n}$. Hence, the set of indexes $n$ for which iv) of Lemma 5 is not fulfilled reduces to $n=1$. Then, (18) reads

$$
\begin{aligned}
\mu_{0} & =c \in \mathbb{R}^{+} \\
\mu_{1} & =c \in \mathbb{R} \\
\mu_{0} \mu_{n+1}-\mu_{1} \mu_{n} & =0, \quad n>1
\end{aligned}
$$

Example 4. Consider the Euler-Cauchy operator $\mathcal{L}^{(M)}[f](x)=\sum_{k=1}^{M} a_{k} x^{k} f^{(k)}(x)$, where $a_{k} \in \mathbb{R}$ are such that the polynomial $p(n)=\sum_{k=1}^{M} \frac{n!}{(n-k)!} a_{k}$ does not have roots for $n>0$, notice that $\mathcal{L}^{M}\left[x^{n}\right]=p(n) x^{n}$. If $p(n)$ has no integer roots for $n>0$, then the system (18) reduces to,

$$
\mu_{n}=0, \quad n \geq 1
$$

which implies that $\mu \equiv 0$. Hence, the set $\Xi_{\mathcal{L}^{(M)}}$ is empty.

Example 5. Let $\mathcal{L}_{H}[f](x)=f^{\prime \prime}(x)-2 x f^{\prime}(x), f \in \mathbb{P}$ be the Hermite operator. Then (18) is

$$
\begin{aligned}
\mu_{0} & =c, \quad c \in \mathbb{R}^{+}, \\
\mu_{1} & =0, \\
2 \mu_{n}-(n-1) \mu_{n-2} & =0, \quad n \geq 2
\end{aligned}
$$

which is the difference equation that defines the measure $c \mu_{H}$, where $d \mu_{H}(x)=e^{-x^{2}} d x$ and $c \in \mathbb{R}^{+}$.
In a similar way, for the Laguerre and Jacobi operators $\mathcal{L}_{L}, \mathcal{L}_{(\alpha, \beta)}$, respectively, we obtain that $\Xi_{\mathcal{L}_{L}}=$ $\left\{c \mu_{L}\right\}_{c \in \mathbb{R}^{+}}, \Xi_{\mathcal{L}_{(\alpha, \beta)}}=\left\{c \mu_{\alpha, \beta}\right\}_{c \in \mathbb{R}^{+}}$, where $d \mu_{\alpha, \beta}(x)=(1-x)^{\alpha}(1+x)^{\beta} d x, d \mu_{L}(x)=x^{\alpha} e^{-x} d x$ are the Jacobi and Laguerre measures, respectively. As a consequence, we obtain the following corollary,
Corollary 1. Let $\mathcal{L}$ be a classical operator, i.e. Jacobi, Laguerre or Hermite and $\mu$ a positive Borel measure with support contained in $\mathbb{R}$. Then there exists an infinite sequence $\left\{Q_{n}\right\}_{n=0}^{\infty}$ of polynomials orthogonal with respect to $(\mathcal{L}, \mu)$, with deg $\left[Q_{n}\right]=n$ if and only if $\mu$ is one of the measures $c \mu_{\alpha, \beta}, c \mu_{L}, c \mu_{H} ; c \in \mathbb{R}^{+}$. In such case, all the sequences of monic orthogonal polynomials $\left\{Q_{n}\right\}_{n=0}^{\infty}$ with respect to the pair $\left(\mathcal{L}^{(M)}, \mu\right)$ such that deg $\left[Q_{n}\right]=n$ are of the form $\left\{P_{n}+k_{n}\right\}_{n=0}^{\infty}$ where $\left\{k_{n}\right\}_{n=0}^{\infty}, k_{0}=0$, is an arbitrary sequence of complex numbers and $\left\{P_{n}\right\}_{n=0}^{\infty}$ is the sequence of monic orthogonal polynomials with respect to $\mu$.
Proof. Let $\mathcal{L}$ be a fixed classical operator and $\left\{\lambda_{n}\right\}_{n=0}^{\infty}$ the associated sequence of eigenvalues. Then, we have that $\lambda_{n}=0$ if and only if $n=0$. Hence, the system (18) of Theorem 7 reduces to a unique equation. A simple calculation yields that the moments of the measure $\mu$ coincide with the moments of the measure of orthogonality of the sequence of eigenpolynomials of $\mathcal{L}$ multiplied by a real positive constant $c$ (see Example 5 and the comment below it). Since the moment problem for a classical measure is determinate, we obtain that $\mu$ is the measure of orthogonality of the sequence of eigenpolynomials of $\mathcal{L}$ multiplied by a real positive constant $c$.

From Theorem 7 we have that for $n \geq 1$ there exists an infinite sequence $\left\{Q_{n}\right\}_{n \geq 1}$ of polynomials orthogonal with respect to $(\mathcal{L}, \mu)$ such that $\operatorname{deg}\left[Q_{n}\right]=n$ if and only if $\mu$ is the measure of orthogonality of the sequence of eigenpolynomials of $\mathcal{L}$ multiplied by a real positive constant $c$. A simple calculation shows that for $n=0$, the polynomial $Q_{0}=1$ satisfies the condition of orthogonality (2) and the statement is valid also for the sequence $\left\{Q_{n}\right\}_{n \geq 0}$.

It is not difficult to see that from the solutions of equations (6) and (7) we obtain that all the sequences of monic orthogonal polynomials $\left\{Q_{n}\right\}_{n>0}$ with respect to the pair $\left(\mathcal{L}^{(M)}, \mu\right)$ such that deg $\left[Q_{n}\right]=n$ are of the form $\left\{P_{n}+k_{n}\right\}_{n=0}^{\infty}$ where $\left\{k_{n}\right\}_{n=0}^{\infty}, k_{0}=0$ is an arbitrary sequence of complex numbers and $\left\{P_{n}\right\}_{n=0}^{\infty}$ is the sequence of monic orthogonal polynomials with respect to $\mu$.

Nevertheless, it is possible to guarantee the existence of a sequence $\left\{Q_{n}\right\}_{n>m}$, for some $m \in \mathbb{N}$ of polynomials orthogonal with respect to a classical operator for a measure $\mu$ which satisfies the condition $d \mu^{*}(x)=\rho(x) d \mu(x)$ where $\mu^{*}$ denotes the Jacobi, Hermite or Laguerre measure and $\rho$ is a non negative polynomial on the support of $\mu^{*}$ of degree $m$, as will be shown

Lemma 8. Let $\mathcal{L}$ be a classical operator, $\mu$ a finite positive Borel measure on $\mathbb{R}$, and $n$ a fixed positive integer number. Then, the differential equation (16) has a unique, except an additive constant, monic polynomial solution $Q_{n}$ of degree $n$ if and only if

$$
\begin{equation*}
\int P_{n}(x) d \mu^{*}(x)=0 \tag{21}
\end{equation*}
$$

where $P_{n}$ is the nth monic orthogonal polynomials with respect to the measure $\mu$.

Proof. Suppose that there exists a polynomial $Q_{n}$ of degree $n$ such that $\mathcal{L}\left[Q_{n}\right]=\lambda_{n} P_{n}$. Let us denote by $\left\{L_{n}\right\}$ the sequence of orthogonal polynomials with respect to the measure $\mu^{*}$. We have then

$$
\begin{align*}
& Q_{n}(z)=L_{n}(z)+\sum_{k=0}^{n-1} a_{(n, k)} L_{k}(z),  \tag{22}\\
& P_{n}(z)=L_{n}(z)+\sum_{k=0}^{n-1} b_{(n, k)} L_{k}(z), \tag{23}
\end{align*}
$$

where $a_{(n, k)}=\frac{\left\langle Q_{n}, L_{k}\right\rangle}{\left\langle L_{k}, L_{k}\right\rangle}$ and $b_{(n, k)}=\frac{\left\langle L_{n}, P_{k}\right\rangle}{\left\langle L_{k}, L_{k}\right\rangle}$.
Replacing $Q_{n}$ and $P_{n}$ in (6) by the linear combinations (22) and (23), from the linearity of $\mathcal{L}[\cdot]$ and the condition that $\mathcal{L}\left[L_{n}\right]=\lambda_{n} L_{n}$ we get

$$
b_{(n, 0)}=\frac{\int L_{n}(x) d \mu^{*}(x)}{\int d \mu^{*}}=0
$$

Conversely, assume that $P_{n}$ is the $n$th monic orthogonal polynomial with respect to $\mu$ fulfilling (21). Let $Q_{n}$ be the polynomial of degree $n$ defined by

$$
Q_{n}(z)=L_{n}(z)+\sum_{k=0}^{n-1} a_{(n, k)} L_{k}(z)
$$

where $a_{(n, 0)}=\Lambda_{n}$ is an arbitrary constant and $a_{(n, k)}=\frac{\lambda_{n}}{\lambda_{k}} \frac{\left\langle L_{n}, P_{k}\right\rangle}{\left\langle L_{k}, L_{k}\right\rangle}$. From the linearity of $\mathcal{L}[\cdot]$ and the condition that $\mathcal{L}\left[L_{n}\right]=\lambda_{n} L_{n}$ we get that $\mathcal{L}\left[Q_{n}\right]=\lambda_{n} P_{n}$.

As a consequence, we have
Theorem 9. Let $\mathcal{L}$ be a classical operator and $\mu$ be a finite positive Borel measure on $\mathbb{R}$, such that $d \mu^{*}(x)=$ $\rho(x) d \mu(x)$, with $\rho \in L^{2}(\mu)$. Then, $m$ is the smallest natural number such that there exists an infinite sequence $\left\{Q_{n}\right\}_{n>m}$ of polynomials orthogonal with respect to $(\mathcal{L}, \mu)$, with $\operatorname{deg}\left[Q_{n}\right]=n$ if and only if $\rho$ is a polynomial of degree $m$.

Proof. Suppose that $m$ is the smallest natural number such that for each $n>m$ there exists a monic polynomial $Q_{n}$ of degree $n$, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. According to Lemma 8

$$
\int P_{n}(x) d \mu^{*}(x)=\int P_{n}(x) \rho(x) d \mu(x) \begin{cases}=0 & \text { if } n>m \\ \neq 0 & \text { if } n=m\end{cases}
$$

This is equivalent to saying that $\rho(x)=\sum_{k=0}^{m} c_{k} P_{k}(x)$ with $c_{m} \neq 0$. The converse is straightforward.
Unlike Theorem 6, Theorem 7 does not guarantee the uniqueness of the sequence. A result for the uniqueness can be obtained by fixing an adequate number of points in the complex plane, as will be shown in the next theorem. Let $\Pi=\left\{\pi_{m_{1}}, \ldots, \pi_{m_{n}}\right\}$ be a set of polynomials and $\mathcal{Z}=\left\{\nu_{1}, \ldots, \nu_{n}\right\}$ a multiset [4] (that is, a set that allows repeated elements) of points in the complex plane. We will say that $\Pi$ is an interpolating system for $\mathcal{Z}$ if the following relation holds

$$
\left|\begin{array}{ccc}
\pi_{m_{1}}\left(\nu_{1}\right) & \cdots & \pi_{m_{n}}\left(\nu_{1}\right) \\
& \vdots & \\
\pi_{m_{1}}\left(\nu_{j}\right) & \cdots & \pi_{m_{n}}\left(\nu_{j}\right) \\
& \vdots & \\
\pi_{m_{1}}^{\left(m_{j}-1\right)}\left(\nu_{j}\right) & \cdots & \pi_{m_{n}}^{\left(m_{j}-1\right)}\left(\nu_{j}\right) \\
& \vdots & \\
\pi_{m_{n}}\left(\nu_{n}\right) & \cdots & \pi_{m_{n}}\left(\nu_{n}\right)
\end{array}\right| \neq 0
$$

notice that if for some index $j$ we have $m_{j}$ points of the set $\mathcal{Z}$ repeated, we have completed the $j$-th and $\left(j+m_{j}-1\right)$-th rows by taking the derivatives up to order $m_{j}-1$. In particular, if $\mathcal{Z}$ consists of a single point repeated $n$ times, then the above determinant coincides with the Wronskian of the system $\Pi$.
Theorem 10. Assume that $\mu \in \Xi_{\mathcal{L}^{(M)}}$ is not empty, let $\mathcal{L}^{(M)}=\sum_{k=0}^{M} \rho_{k}(x) \frac{d^{k}}{d x^{k}}$ be an exactly solvable differential operator. Let the set $S$ be as defined in Theorem 7 and let us fix (allowing repeated elements) $\left\{\nu_{1, n}, \ldots, \nu_{j_{0}, n}\right\}$ points on the complex plane. Then, there exists a unique monic polynomial $R_{n-n_{j_{0}}}$ of degree $n-n_{j_{0}}$ such that

$$
Q_{n}(x)=\left(x-\nu_{1, n}\right) \cdots\left(x-\nu_{j_{0}, n}\right) R_{n-n_{j_{0}}}(x)
$$

is orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$ provided that $\left\{Q_{n_{j}}\right\}_{j=1}^{j_{0}}$ is an interpolating system for $\left\{\nu_{1, n}, \ldots, \nu_{j_{0}, n}\right\}$. Here $n \notin S,\left\{Q_{n_{j}}\right\}_{j=1}^{j_{0}}$ a basis of monic polynomial solutions to (7), and $j_{0}$ is the largest value for which $n_{j_{0}}<n, n_{j_{0}} \in S$.

Proof. According to Theorem 7 if $\Xi_{\mathcal{L}^{(M)}}$ does not reduce to the empty set, then there exists a sequence $\left\{Q_{n}\right\}_{j \notin S}$ of monic polynomials orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$ such that $\operatorname{deg}\left[Q_{n}\right]=n$. Let $S=$ $\left\{n_{1}, \ldots, n_{k}\right\}$, notice that if $n<n_{1}$ then by $i$ ) of Lemmant the index $n$ is normal, hence the monic polynomial $Q_{n}$ is unique, therefore we do not have necessarily that $Q_{n}$ vanishes at the points $\left\{\nu_{1, n}, \ldots, \nu_{j_{0}, n}\right\}$.

Assume that $n_{1}<n$, consider $\left\{Q_{n_{j}}\right\}_{j=1}^{j_{0}}$ a basis of monic polynomial solutions to (17), and assume that $\widehat{Q}_{n}$ is a monic polynomial solution of degree $n$ to (6). Then, for a given index $n$, there exist unique coefficients $\left\{\alpha_{j}\right\}_{j=1}^{j_{0}}$ such that any monic polynomial solution $Q_{n}$ of degree $n$ to equation (6) can be expressed as

$$
\begin{equation*}
Q_{n}(x)=\widehat{Q}_{n}(x)+\sum_{j=1}^{j_{0}} \alpha_{j} Q_{n_{j}}(x) \tag{24}
\end{equation*}
$$

Let us consider the multiset $\left\{\nu_{1, n}, \ldots, \nu_{j_{0}, n}\right\}$ of $j_{0}$ points on the complex plane. To prove the existence of a monic polynomial $R_{n-n_{j_{0}}}$ of degree $n-n_{j_{0}}$ such that

$$
\begin{equation*}
Q_{n}(x)=\left(x-\nu_{1, n}\right) \cdots\left(x-\nu_{j_{0}, n}\right) R_{n-n_{j_{0}}}(x) \tag{25}
\end{equation*}
$$

we evaluate the polynomial

$$
\widehat{Q}_{n}(x)+\beta_{1} Q_{n_{1}}(x)+\cdots+\beta_{j_{0}} Q_{n_{j_{0}}}(x)
$$

at the points $x=\nu_{j, n}$ and take derivatives up to order $m_{\nu_{j, n}}-1$, where $m_{\nu_{j, n}}$ is the number of times that the point $\nu_{j, n}$ appears in the multiset. We obtain that

$$
\begin{align*}
\widehat{Q}_{n}\left(\nu_{1}\right) & =\beta_{1} Q_{n_{1}}\left(\nu_{1}\right)+\cdots+\beta_{j_{0}} Q_{n_{j_{0}}}\left(\nu_{1}\right)  \tag{26}\\
& \vdots  \tag{27}\\
\widehat{Q}_{n}\left(\nu_{j_{0}}\right) & =\beta_{1} Q_{n_{1}}\left(\nu_{j_{0}}\right)+\cdots+\beta_{j_{0}} Q_{n_{j_{0}}}\left(\nu_{j_{0}}\right)
\end{align*}
$$

by defining $\alpha_{1}, \ldots, \alpha_{j_{0}}$ as the solution of the above system, we obtain the existence. The uniqueness follows immediately by condition that the $\left\{Q_{n_{j}}\right\}_{j=1}^{j_{0}}$ is an interpolating system for $\left\{\nu_{1, n}, \ldots, \nu_{j_{0}, n}\right\}$.

## 5. Zero location of the polynomials $Q_{n}$ for a subclass of exactly solvable operators

In this section we study the location of the zeros of orthogonal polynomials with respect to a certain subclass of differential operators. We start with a discussion of the class of operators which we shall consider.

Definition 3. Given $M \geq 1$, we say that the linear differential operator $\mathcal{L}^{(M)}$ of $M$-th order factorizes on $\mathbb{P}$ if there exist multi-indexes $\left(m_{1}, \ldots, m_{J}\right),\left(n_{1}, \ldots, n_{J}\right)$ and polynomials $\left\{\rho_{m_{j}}\right\}_{j=1}^{J}$ with $\operatorname{deg}\left[\rho_{m_{j}}\right]=m_{j}, j=$ $1, \ldots, J$, such that for each polynomial $\Pi_{n} \in \mathbb{P}$ we have

$$
\begin{equation*}
\mathcal{L}^{(M)}\left[\Pi_{n}\right](z)=\left[\rho_{m_{J}}(z) \cdots\left[\rho_{m_{2}}(z)\left[\rho_{m_{1}}(z) \Pi_{n}(z)\right]^{\left(n_{1}\right)}\right]^{\left(n_{2}\right)} \cdots\right]^{\left(n_{J}\right)} \tag{28}
\end{equation*}
$$

If $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$, we shall denote

$$
\begin{aligned}
\mathcal{L}_{1}^{\left(n_{1}\right)}[f](z) & :=\left(\rho_{m_{1}}(z) f(z)\right)^{\left(n_{1}\right)} \\
& \vdots \\
\mathcal{L}_{J}^{\left(n_{J}\right)}[f](z) & :=\left(\rho_{m_{J}}(z) f(z)\right)^{\left(n_{J}\right)},
\end{aligned}
$$

and then

$$
\mathcal{L}^{(M)}[f]=\mathcal{L}_{J}^{\left(n_{J}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}[f]
$$

We are interested in exactly solvable operators $\mathcal{L}^{(M)}$ which factorize on $\mathbb{P}$, for the case in which $\left\{\rho_{m_{j}}\right\}_{j=1}^{J}$ are polynomials with reals roots. According to Definition 2 of exactly solvable operator, we have necessarily that

$$
\begin{equation*}
\sum_{k=1}^{J} m_{k}=\sum_{k=1}^{J} n_{k}=M \tag{29}
\end{equation*}
$$

We denote by $C_{M}$ the convex hull of the zeros of $\prod_{i=1}^{J} \rho_{m_{i}}$.
If $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$, then it is not difficult to see that $i$ ) of Lemma 5 is equivalent to the condition,

$$
\begin{equation*}
\sum_{i=1}^{j}\left(m_{i}-n_{i}\right) \geq 0, \quad \forall j \leq J \tag{30}
\end{equation*}
$$

Hence, the class of operators that factorize on $\mathbb{P}$ for which there exists a unique infinite sequence $\left\{Q_{n}\right\}_{n=0}^{\infty}$, of monic polynomials such that $\operatorname{deg}\left[Q_{n}\right]=n$, and orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$, for every positive Borel measure $\mu$ supported on $\mathbb{R}$, are those which satisfy condition (30).

To locate the zeros of orthogonal polynomials with respect to operators that factorize on $\mathbb{P}$ we use an integral representation for these operators and then we apply known theorems for zero location of iterated integrals of polynomials. From the preceding discussions, it is already known that we have cases of operators for which the associated sequence of orthogonal polynomials is not unique. We will first analyze the class of operators defined by condition (30); that is, the class for which the existence of the full sequence $\left\{Q_{n}\right\}_{n=0}^{\infty}$ can be guaranteed. For these operators the following integral representation holds.
Lemma 11. Let $P_{n}$ be the $n$-th monic orthogonal polynomial with respect to $\mu, \mathcal{L}^{(M)}$ is such that factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)}=\mathcal{L}_{J}^{\left(n_{J}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}$ and satisfies (30). Then, the following representation holds

$$
Q_{n}=\lambda_{n} I_{1} \circ \cdots \circ I_{J}\left[P_{n}\right]
$$

where $I_{j}$ is the integration operator, given by

$$
I_{j}[f](z)=\frac{1}{\rho_{m_{j}}(z)} \int_{z_{n_{j}, j}}^{z} \int_{z_{n_{j}-1, j}}^{t_{n_{j}-1}} \cdots \int_{z_{1, j}}^{t_{1}} f(t) d t d t_{1} \cdots d t_{n_{j}-1}
$$

and $\left\{z_{i, j}\right\}_{\substack{i=1, \ldots, n_{j} \\ j=1, \ldots, J}} \subset C_{M}$.
Proof. As (30) holds, then by $i i$ ) of Lemma 5 we have $\mathcal{L}^{(M)}\left[Q_{n}\right]=\lambda_{n} P_{n}$ is solvable. Let us consider the function

$$
f(z):=\left[\rho_{m_{J}}(z) \mathcal{L}_{J-1}^{\left(n_{J-1}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}\left[Q_{n}\right](z)\right]^{\left(n_{J}-1\right)}
$$

Applying successively Rolle's Theorem, taking into account (30), and that the polynomials $\rho_{m_{j}}$ have their zeros on $C_{M}$ and are reals we obtain that $f$ has at least a zero $z_{1, J}$ in $C_{M}$. Hence, $f(z)=\lambda_{n} \int_{z_{1, J}}^{z} P_{n}(t) d t$.

By a similar argument we will have

$$
\begin{equation*}
\rho_{m_{J}}(z) \mathcal{L}_{J-1}^{\left(n_{J-1}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}\left[Q_{n}\right](z)=\lambda_{n} \int_{z_{n_{J}, J}}^{z} \int_{z_{n_{J}-1, J}}^{t_{n_{J}-1}} \cdots \int_{z_{1, J}}^{t_{1}} P_{n}(t) d t d t_{1} \cdots d t_{n_{J}-1} \tag{31}
\end{equation*}
$$

which implies that the polynomial $\int_{z_{n_{J}, J}}^{z} \int_{z_{n_{J}-1, J}}^{t_{n J}-1} \cdots \int_{z_{1, J}}^{t_{1}} P_{n}(t) d t d t_{1} \cdots d t_{n_{J}-1}$ is divisible by $\rho_{m_{J}}$. Therefore, after a finite number of steps we will have

$$
\begin{aligned}
Q_{n}(z)= & \lambda_{n} \frac{1}{\rho_{m_{1}}(z)} \int_{z_{n_{1}, 1}}^{z} \int_{z_{n_{1}-1,1}}^{t_{n_{1}-1}} \cdots \int_{z_{1,1}}^{t_{1}} \cdots \frac{1}{\rho_{m_{J}}(z)} \\
& {\left[\int_{z_{n_{J}, J}}^{z} \int_{z_{n_{J}-1, J}}^{t_{n_{J}-1}} \cdots \int_{z_{1, J}}^{t_{1}} P_{n}(t) d t d t_{1} \cdots d t_{n_{J}-1}\right] \cdots d t d t_{1} \cdots d t_{n_{1}-1} } \\
= & \lambda_{n} I_{1} \circ \cdots \circ I_{J}\left[P_{n}\right](z) .
\end{aligned}
$$

Consider now the class of operators which do not satisfy the condition (30). A representation similar to the one obtained in the preceding lemma can also be given. Let us prove some preliminary lemmas.
Lemma 12. Assume that $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)}=\mathcal{L}_{J}^{\left(n_{J}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}$. Then $\operatorname{Ker}\left[\mathcal{L}^{(M)}\right]=\{0\}$ if and only if $\mathcal{L}^{(M)}[1] \neq 0$.
Proof. The implication $\operatorname{Ker}\left[\mathcal{L}^{(M)}\right]=\{0\} \quad \Rightarrow \quad \mathcal{L}^{(M)}[1] \neq 0$ is straightforward. Assume that $\mathcal{L}^{(M)}[1] \neq 0$. Note that

$$
\operatorname{deg}\left[\mathcal{L}_{j}^{\left(n_{j}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}[1]\right]=\sum_{i=1}^{j}\left(m_{i}-n_{i}\right)
$$

hence, $\sum_{i=1}^{j}\left(m_{i}-n_{i}\right) \geq 0, \forall j \leq J$. Therefore, from (30) and $\left.i i i\right)$ of Lemma 5 we obtain that $\operatorname{Ker}\left[\mathcal{L}^{(M)}\right]=$ $\{0\}$.
Lemma 13. Let us have that $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)}=\mathcal{L}_{J}^{\left(n_{J}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}$ and denote by $j_{0}$ the largest index such that $\sum_{i=1}^{j_{0}}\left(m_{i}-n_{i}\right)<0$. Then, $\operatorname{Ker}\left[\mathcal{L}^{(M)}\right]=\left\{1, \ldots, x^{n_{j_{0}}^{\prime}}\right\}$, where $n_{j_{0}}^{\prime}=\sum_{i=1}^{j_{0}}\left(n_{i}-m_{i}\right)-1$.
Proof. Since $\mathcal{L}^{(M)}$ is a composition of operators, it is not difficult to see that if $1 \leq n \leq n_{j_{0}}^{\prime}$ then $\mathcal{L}^{(M)}\left[x^{n}\right]=0$. Hence, $\left\{1, \ldots, x^{n_{j_{0}}^{\prime}}\right\} \subset \operatorname{Ker}\left[\mathcal{L}^{(M)}\right]$. Suppose now that $n=n_{j_{0}}^{\prime}+m, m \geq 1$. Then, we have

$$
\operatorname{deg}\left[\mathcal{L}_{j_{0}}^{\left(n_{j_{0}}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}\left[x^{n}\right]\right]=m-1 \geq 0
$$

and thus $\mathcal{L}^{(M)}\left[x^{n}\right] \neq 0$.
An analogue of Lemma 11, for operators that do not satisfy condition (30), is

Lemma 14. Assume that $\mu \in \Xi_{\mathcal{L}^{(M)}} \neq \emptyset, \mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)}=\mathcal{L}_{J}^{\left(n_{J}\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}$ and suppose that condition (30) is not satisfied. Let us fix a multiset $\mathcal{Z}=\left\{\nu_{1, n}, \ldots, \nu_{n_{j_{0}^{\prime}}+1, n}\right\}$ of real numbers such that $\left\{1, \ldots, x^{n_{j_{0}}^{\prime}}\right\}$ is an interpolating system for $\mathcal{Z}$, where $n_{j_{0}}^{\prime}$ is as in Lemma 13, and let $Q_{n}$ be the monic orthogonal polynomial with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$ that vanishes at the points $\left\{\nu_{1, n}, \ldots, \nu_{n_{j_{0}}+1, n}\right\}$. Then, the following representation holds

$$
Q_{n}(x)=\lambda_{n} I_{1} \circ \cdots I_{J-1} \circ \widehat{I}_{J, n}\left[P_{n}^{\left(n_{j_{0}}\right)}\right](x), \quad n>n_{j_{0}}^{\prime}
$$

where $I_{j}$ is the integral operator given by

$$
\begin{aligned}
I_{j}[f](z) & =\frac{1}{\rho_{m_{j}}(z)} \int_{z_{n_{j}, j}}^{z} \int_{z_{n_{j}-1, j}}^{t_{n_{j}-1}} \cdots \int_{z_{1, j}}^{t_{1}} f(t) d t d t_{1} \cdots d t_{n_{j}-1} \\
\widehat{I}_{J, n}[f] & =I_{J} \circ I_{*, n}[f] \\
I_{*, n}[f] & =\int_{z_{n_{j_{0}}+1, J}^{*}}^{z} \int_{z_{n_{j_{0}}^{\prime}, J}^{*}}^{t_{n_{j_{0}}^{\prime}}} \cdots \int_{z_{1, J}^{*}}^{t_{1}} f(t) d t d t_{1} \cdots d t_{n_{j_{0}}^{\prime}}
\end{aligned}
$$

and $\left\{z_{i, J}^{*}\right\}_{i=1, \ldots, n_{j_{0}}^{\prime}+1},\left\{z_{i, j}\right\}_{\substack{i=1, \ldots, n_{j} \\ j=1, \ldots, J}} \subset C_{M}^{*}$, where $C_{M}^{*}$ is the convex hull of the zeros of

$$
\left(x-\nu_{1, n}\right) \cdots\left(x-\nu_{n_{j_{0}}, n}\right)\left(\prod_{i=1}^{J} \rho_{m_{i}}(x)\right)
$$

Proof. By Lemma 13 we have that $\operatorname{Ker}\left[\mathcal{L}^{(M)}\right]=\left\{1, \ldots, x^{n_{j_{0}}}\right\}$, where $n_{j_{0}}^{\prime}=\sum_{i=1}^{j_{0}}\left(n_{i}-m_{i}\right)-1$. Set $S=$ $\left\{0, \ldots, n_{j_{0}}^{\prime}\right\}$. Theorem 10 yields that there exists a unique monic polynomial $R_{n-n_{j_{0}}^{\prime}-1}$ such that if $n>n_{j_{0}}^{\prime}$

$$
\mathcal{L}^{(M)}\left[\Pi_{n_{j_{0}}^{\prime}+1} R_{n-n_{j_{0}}^{\prime}-1}\right](x)=\lambda_{n} P_{n}(x),
$$

where $\Pi_{n_{j_{0}}+1}(x)=\left(x-\nu_{1, n}\right) \cdots\left(x-\nu_{n_{j_{0}}^{\prime}+1, n}\right)$ and $P_{n}$ is the $n$th monic orthogonal polynomial with respect to the measure $\mu$. Taking derivatives up to order $n_{j_{0}}$ in the above expression, we obtain

$$
\mathcal{L}_{J}^{\left(n_{J}+n_{j_{0}}^{\prime}+1\right)} \circ \cdots \circ \mathcal{L}_{1}^{\left(n_{1}\right)}\left[\Pi_{n_{j_{0}}^{\prime}+1} R_{n-n_{j_{0}}^{\prime}-1}\right](x)=\lambda_{n} P_{n}^{\left(n_{j_{0}}^{\prime}+1\right)}(x)
$$

or, equivalently,

$$
\widehat{\mathcal{L}}\left[R_{n-n_{j_{0}}^{\prime}-1}\right](x)=\lambda_{n} P_{n}^{\left(n_{j_{0}}^{\prime}+1\right)}(x)
$$

where $\widehat{\mathcal{L}}=\widehat{\mathcal{L}}_{J}^{\left(n_{J}\right)} \circ \mathcal{L}_{J-1}^{\left(n_{J-1}\right)} \circ \cdots \circ \mathcal{L}_{2}^{\left(n_{2}\right)} \circ \widehat{\mathcal{L}}_{1}^{\left(n_{1}\right)}$,

$$
\begin{aligned}
\widehat{\mathcal{L}}_{J}^{\left(n_{J}\right)}[f] & =\mathcal{L}_{J}^{\left(n_{J}+n_{j_{0}}^{\prime}+1\right)}[f], \quad f \in \mathbb{P}, \\
\widehat{\mathcal{L}}_{1}^{\left(n_{1}\right)}[f] & =\mathcal{L}_{1}^{\left(n_{1}\right)}\left[\Pi_{n_{j_{0}}^{\prime}+1} f\right] .
\end{aligned}
$$

Since the polynomial $R_{n-n_{j_{0}}^{\prime}-1}$ is unique, we deduce that $\widehat{\mathcal{L}}[1] \neq 0$, hence Lemma 12 gives that $\operatorname{Ker}[\widehat{\mathcal{L}}]=$ $\{0\}$. Therefore, from the equivalence of $i i i)$ of Lemma 5 and the relation (30), $\widehat{\mathcal{L}}$ satisfies $\sum_{i=1}^{j}\left(m_{i}-n_{i}\right) \geq$ $0, \forall j \leq J$. By Lemma 11, we obtain

$$
R_{n-n_{j_{0}}^{\prime}-1}(x)=\lambda_{n} \widehat{I}_{1} \circ I_{2} \circ \cdots \circ I_{J-1} \circ \widehat{I}_{J}\left[P_{n}^{\left(n_{j_{0}}^{\prime}+1\right)}\right](x)
$$

or, equivalently,

$$
Q_{n}(x)=\lambda_{n} I_{1} \circ \cdots \circ I_{J-1} \circ \widehat{I}_{J}\left[P_{n}^{\left(n_{j_{0}}^{\prime}+1\right)}\right](x), \quad n>n_{j_{0}}^{\prime}
$$

where $I_{j}$ is the integral operator given by

$$
\begin{aligned}
I_{j}[f](z) & =\frac{1}{\rho_{m_{j}}(z)} \int_{z_{n_{j}}, j}^{z} \int_{z_{n_{j}-1}, j}^{t_{n_{j}-1}} \cdots \int_{z_{1, j}}^{t_{1}} f(t) d t d t_{1} \cdots d t_{n_{j}-1} \\
\widehat{I}_{1}[f] & =\frac{1}{\Pi_{n_{j_{0}}^{\prime}+1}^{\prime}(x)} I_{1}[f], \quad f \in \mathbb{P} \\
\widehat{I}_{J}[f] & =I_{J} \circ I_{*}[f], \\
I_{*}[f] & =\int_{z_{n_{j_{0}}+1, J}^{*}}^{z} \int_{z_{n_{j_{0}}^{\prime}, J}^{*}}^{t_{n_{j_{0}}^{\prime}}} \cdots \int_{z_{1, J}^{*}}^{t_{1}} f(t) d t d t_{1} \cdots d t_{n_{j_{0}}^{\prime}}
\end{aligned}
$$

and $\left\{z_{i, J}^{*}\right\}_{i=1, \ldots, n_{j_{0}}^{\prime}+1},\left\{z_{i, j}\right\}_{\substack{i=1, \ldots, n_{j} \\ j=1, \ldots, J}} \subset C_{M}^{*}$, being $C_{M}^{*}$ is the convex hull of the zeros of the polynomial

$$
\left(x-\nu_{1, n}\right) \cdots\left(x-\nu_{n_{j_{0}}^{\prime}+1, n}\right)\left(\prod_{i=1}^{J} \rho_{m_{i}}(x)\right)
$$

### 5.1. Zero location

Assume that the exactly solvable operator $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ and that there exists a unique infinite sequence $\left\{Q_{n}\right\}_{n=0}^{\infty}$, of monic polynomials, each polynomial $Q_{n}$ of degree equal to $n$, and orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$, for every positive Borel measure $\mu$ supported on $\mathbb{R}$. The following theorem, see 10, Exer. 20, pag. 74], and the results of the preceding section can be used now to locate the zeros of the family $\left\{Q_{n}\right\}$.
Theorem 15. If all the zeros of the $n$th degree polynomial $f$ lie in a convex region $K$ containing the point $a$, then all the zeros of $F(z)=\int_{a}^{z} f(t) d t$ lie in the domain bounded by the envelope of all circles passing through $a$ and having centers on the boundary of $K$.

The following lemma will be necessary for the zero location theorem.
Lemma 16. Let $I_{j}$ be the integral operator defined in Lemma 11. Assume that the set $\left\{z_{i, j}\right\}_{i=1}^{n_{j}}$ and the zeros of the $n$th degree polynomial $\Pi_{n}$ lie the circle $C(0, r)$ with center in the origin and radius $r$. Then, the zeros of $I_{j}\left[\Pi_{n}\right]$ lie in the circle $C\left(0,3^{n_{j}} r\right)$.

Proof. If $z_{i, j}$ and the zeros of the $n$th degree polynomial $\Pi_{n}$ lie in a circle $C(0, r)$ of radius $r$, by Theorem 15 the zeros of $\int_{z_{1, j}}^{z} \Pi_{n}(t) d t$ lie in the envelope of all the circles with center in the boundary of $C(0, r)$ and passing through $z_{1, j}$. It is not difficult to see that this envelope and the set $\left\{z_{i, j}\right\}_{i=2}^{n_{j}}$ are contained in the circle $C(0,3 r)$. Using the same argument, we obtain that the zeros of $I_{j}\left[\Pi_{n}\right]$ are located in the circle $C\left(0,3^{n_{j}} r\right)$.

Consider now the case of operators for which the full sequence of $\left\{Q_{n}\right\}_{n=0}^{\infty}$ exists, for every Borel measure $\mu$ supported on a subset of $\mathbb{R}$ or, equivalently, the operators for which this can be guaranteed are those which satisfy the condition (30). We have then,
Theorem 17. Let $\mathcal{L}^{(M)}$ be an exactly solvable operator that factorizes on $\mathbb{P}$ satisfying the condition (30) and $\mu$ a positive Borel measure supported in $[-1,1]$. Then, the zeros of the sequence $\left\{Q_{n}\right\}_{n=0}^{\infty}$, of monic polynomials orthogonal with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$ are located in a circle of radius $R$, where $R=3^{M} d$, with $d=\max \left\{1, \sup _{z \in C_{M}}|z|\right\}$.

Proof. As $\left\{P_{n}\right\}_{n=0}^{\infty}$ is the sequence of orthogonal with respect to $\mu$, their zeros are in $[-1,1]$. Notice that the interval $C_{M}$ and the zeros of the sequence $\left\{P_{n}\right\}_{n=0}^{\infty}$ are contained in a circle with center at the origin and radius $d=\max \left\{1, \sup _{z \in C_{M}}|z|\right\}$. From Lemma 11 we have that $Q_{n}$ can be represented as $Q_{n}(z)=$ $\lambda_{n} I_{1} \circ \cdots \circ I_{J}\left[P_{n}\right](z)$. Applying successively Lemma 16 we obtain that the zeros are located in a circle of radius $R$, where $R=3^{M} d$, with $d=\max \left\{1, \sup _{z \in C_{M}}|z|\right\}$.

Consider now the class of operators which do not satisfy the condition (30). In this case the associated sequence of orthogonal polynomials is not unique, nevertheless, in Theorem 10 it was shown that if we fix an adequate number of points we can define a unique infinite sequence of orthogonal polynomials. We have
Theorem 18. Let $\mathcal{L}^{(M)}$ be an exactly solvable operator that factorizes on $\mathbb{P}$ and assume that condition (30) is not satisfied, $\mu \in \Xi_{\mathcal{L}^{(M)}} \neq \emptyset$ such that $\operatorname{supp}(\mu) \subset[-1,1]$ and consider a sequence of multisets $\left\{\nu_{1, n}, \ldots, \nu_{n_{j_{0}}^{\prime}+1, n}\right\}$, satisfying the hypothesis of Lemma 13. Then, the zeros of the sequence $\left\{Q_{n}\right\}_{n=n_{0}+1}^{\infty}$ of monic orthogonal polynomials with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$ such that $Q_{n}\left(\nu_{j, n}\right)=0,1 \leq j \leq n_{j_{0}}^{\prime}+1$ are located in a circle of radius $R$, where $R=3^{M} d$, with $d=\max \left\{1, \sup _{z \in C_{M}^{*}}|z|\right\}, C_{M}^{*}=\sup _{n} \bigcup_{j=0}^{n} C_{M, j}^{*}$, being $C_{M, n}^{*}$ is the convex hull of the zeros of $\left(x-\nu_{1, n}\right) \cdots\left(x-\nu_{n_{j_{0}}^{\prime}}+1, n\right)\left(\prod_{i=1}^{J} \rho_{m_{i}}(x)\right)$.
Proof. By hypothesis, the zeros of the sequence $\left\{P_{n}\right\}_{n=0}^{\infty}$ are contained in $[-1,1]$. According to Theorem 10 there exists a unique sequence $\left\{Q_{n}\right\}_{n=n_{0}^{\prime}+1}^{\infty}$ of monic orthogonal polynomials with respect to $\left(\mathcal{L}^{(M)}, \mu\right)$. By Lemma 14 ,

$$
Q_{n}(x)=\lambda_{n} I_{1} \circ \cdots \circ \widehat{I}_{J}\left[P_{n}^{\left(n_{j_{0}}^{\prime}+1\right)}\right](x), \quad n>n_{j_{0}}^{\prime}
$$

where $I_{j}$ is the integral operator given by

$$
\begin{aligned}
I_{j}[f](z) & =\frac{1}{\rho_{m_{j}}(z)} \int_{z_{n_{j}, j}}^{z} \int_{z_{n_{j}-1, j}}^{t_{n_{j}-1}} \cdots \int_{z_{1, j}}^{t_{1}} f(t) d t d t_{1} \cdots d t_{n_{j}-1} \\
\widehat{I}_{J}[f] & =I_{J} \circ I_{*}[f], \\
I_{*}[f] & =\int_{z_{n_{j_{0}}^{*}+1, J}^{*}}^{z} \int_{z_{n_{j_{0}}^{\prime}, J}^{*}}^{t_{n_{j_{0}}^{\prime}}} \cdots \int_{z_{1, J}^{*}}^{t_{1}} f(t) d t d t_{1} \cdots d t_{n_{j_{0}}^{\prime}}
\end{aligned}
$$

and $\left\{z_{i, J}^{*}\right\}_{i=1, \ldots, n_{j_{0}}^{\prime}+1},\left\{z_{i, j}\right\}_{\substack{i=1, \ldots, n_{j} \\ j=1, \ldots, J}} \subset C_{M, n}^{*}$, being $C_{M, n}^{*}$ is the convex hull of the zeros of the polynomial

$$
\left(x-\nu_{1, n}\right) \cdots\left(x-\nu_{n_{j_{0}}^{\prime}}+1, n\right)\left(\prod_{i=1}^{J} \rho_{m_{i}}(x)\right)
$$

Note that the set $C_{M, n}^{*}$ and the zeros of $\left\{P_{n}\right\}_{n \geq 0}$ are contained in a circle with center at the origin and radius $d=\max \left\{1, \sup _{z \in C^{*}}|z|\right\}$. Applying successively Lemma 16 we obtain that the zeros of $Q_{n}$ are located in a circle of radius $R_{n}$, where $R_{n}=3^{M} d_{n}$, with $d_{n}=\max \left\{1, \sup _{z \in C_{M, n}^{*}}|z|\right\}$. Hence, the zeros of the full sequence are located in a circle of radius $R=3^{M} d$, with $d=\max \left\{1, \sup _{z \in C_{M}^{*}}|z|\right\}, C_{M}^{*}=\sup _{n} \bigcup_{j=0}^{n} C_{M, j}^{*}$.

## 6. The polar polynomials case

In this section we study analytic properties of the polar polynomials, already introduced in Section 2 Let us denote by $d \mu_{T}(x)=\frac{1}{\sqrt{1-x^{2}}} d x$ the first kind Chebyshev measure and by $T_{n}$ the $n$-th Chebyshev monic polynomial of the first kind. We shall study these polynomials for the class of finite positive Borel measures on $[-1,1]$ defined as $d \mu(x)=\frac{d \mu_{T}(x)}{\rho(x)}$ with $\rho(z)=r \prod_{i=1}^{m}\left(z-\nu_{i}\right)$ a non negative polynomial on $[-1,1]$. Denote by $\mathcal{P}_{m}\left(\mu_{T}\right)$ this class of measures. This complements the study carried out in [3] where the measure $\mu$ is the Gegenbauer measure. We obtain a curve which contains the accumulation points of the zeros of these polynomials and a formula for the strong asymptotic behavior of these polynomials in $\mathbb{C} \backslash[-1,1]$.

### 6.1. Strong asymptotic behavior and zero location

We recall that a measure supported on $[-1,1]$ is in the Szegö class $\mathfrak{S}$ if its absolutely continuous part $\mu^{\prime}$ satisfies

$$
\int_{-1}^{1} \frac{\log \mu^{\prime}(x)}{\sqrt{1-x^{2}}} d x>-\infty
$$

The asymptotic properties of orthogonal polynomials with respect to a measure supported on $[-1,1]$ in the Szegö class can be described by means of the Szegö function $D(\mu, z)$, cf. [11, §6.1].
Definition 4. Let $\mu \in \mathfrak{S}$, then the Szegö function $D(d \mu, z)$ is defined by

$$
D(\mu(x), z)=\exp \left[\frac{1}{4 \pi} \int_{-\pi}^{\pi} \log \mu^{\prime}(\cos (t)) \frac{1+z e^{-\imath t}}{1-z e^{-\imath t}} d t\right]
$$

for $|z|<1$.
Let $\varphi(z)=z+\sqrt{z^{2}-1}$, we take the branch of $\sqrt{z^{2}-1}$ for which $|\varphi(z)|>1$ whenever $z \in \mathbb{C} \backslash[-1,1]$.
It is well known that orthogonal polynomials with respect to a measure which belongs to the Szegö class have the following outer strong asymptotic behavior, cf. 11, $\S 6.1$ Lemma 18, page 67],
Lemma 19. Let $\mu$ be a positive Borel measure supported on $[-1,1], P_{n}$ the $n$th monic orthogonal polynomials associated to $\mu$. Then

$$
\frac{\delta_{n} P_{n}(z)}{\varphi(z)^{n}} \rightrightarrows \frac{1}{\sqrt{2 \pi}}\left(D\left(\sqrt{1-x^{2}} d \mu(x), \varphi(z)^{-1}\right)\right)^{-1}
$$

uniformly on compact subsets of $\overline{\mathbb{C}} \backslash[-1,1]$, where $\delta_{n}$ denotes the leading coefficient of the corresponding orthonormal polynomial of degree $n$.

The next lemmas are essential in the proof of the main theorem of this section
Lemma 20. Suppose that $\mu \in \mathcal{P}_{m}\left(\mu_{T}\right)$, then

$$
\begin{equation*}
P_{n}(z)=\sum_{k=0}^{m} b_{n, n-k} T_{n-k}(z), \quad b_{n, n-k}=\frac{\int_{-1}^{1} P_{n}(x) T_{n-k}(x) d \mu_{T}(x)}{\int_{-1}^{1} T_{n-k}^{2}(x) d \mu_{T}(x)} \tag{32}
\end{equation*}
$$

where $P_{n}, T_{n}$ are the monic orthogonal polynomials associated to the measures $\mu, \mu_{T}$, respectively, and the $b_{n-k, k}$ satisfy

$$
\begin{equation*}
\lim _{n \rightarrow \infty} b_{n, n-k}=2^{m-k} a_{m-k}, \quad 0 \leq k \leq m \tag{33}
\end{equation*}
$$

where $a_{k}=(-1)^{k} \sum_{1 \leq \nu_{1}<\cdots<\nu_{k} \leq m} u_{\nu_{1}}^{-1} \ldots u_{\nu_{k}}^{-1}, \quad u_{\nu_{k}}=\varphi\left(\nu_{k}\right)$.

Proof. If $\mu \in \mathcal{P}_{m}\left(\mu_{T}\right)$ then $d \mu_{T}(x)=\rho(x) d \mu(x)$, where $\rho(x)=\prod_{i=1}^{m}\left(x-\nu_{i}\right)$ is nonnegative on $[-1,1]$. Therefore

$$
P_{n}(z)=\sum_{k=0}^{m} b_{n, n-k} T_{n-k}(z), \quad b_{n, n-k}=\frac{\int_{-1}^{1} P_{n}(x) T_{n-k}(x) d \mu_{T}(x)}{\int_{-1}^{1} T_{n-k}^{2}(x) d \mu_{T}(x)}
$$

Hence, if $z=\frac{1}{2}\left(u+u^{-1}\right)$ then $T_{n-k}(z)=\frac{u^{n-k}+u^{k-n}}{2^{n-k}}$, and

$$
\begin{equation*}
\frac{2^{n} P_{n}(z)}{u^{n}}=\sum_{k=0}^{m} 2^{k} b_{n, n-k} u^{-k}+\frac{1}{u^{2 n}} \sum_{k=0}^{m} 2^{k} b_{n, n-k} u^{k} \tag{34}
\end{equation*}
$$

From [11, $\S 6.1$ theorem 26] and Definition [4, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \delta_{n} 2^{-n}=\frac{1}{\sqrt{2 \pi}} D(\rho(x), 0) \tag{35}
\end{equation*}
$$

From Lemma 19 and (35), we obtain

$$
\begin{equation*}
\frac{2^{n} P_{n}(z)}{u^{n}} \rightrightarrows(D(\rho(x), 0))^{-1}\left(D\left(\frac{1}{\rho(x)}, \varphi(z)^{-1}\right)\right)^{-1}=(D(\rho(x), 0))^{-1} D\left(\rho(x), \varphi(z)^{-1}\right) \tag{36}
\end{equation*}
$$

uniformly on closed subsets of $\overline{\mathbb{C}} \backslash[-1,1]$. By [11, $\S 6.1$ Lemma 19] and Definition 4

$$
\begin{aligned}
D\left(\rho, \varphi(z)^{-1}\right) & =2^{m} \exp \left(\frac{1}{2 \pi} \int_{-1}^{1} \frac{\log (\rho(t))}{\sqrt{1-t^{2}}} d t\right) \prod_{k=1}^{m} \frac{z-\nu_{k}}{\varphi(z)-\varphi\left(\nu_{k}\right)} \\
D(\rho, 0) & =\exp \left(\frac{1}{2 \pi} \int_{-1}^{1} \frac{\log (\rho(t))}{\sqrt{1-t^{2}}} d t\right)
\end{aligned}
$$

Hence, if $z=\frac{1}{2}\left(u+u^{-1}\right)$, using Vieta's formula, we have that the following identity holds

$$
\begin{equation*}
2^{m} \prod_{k=1}^{m} \frac{z-\nu_{k}}{\varphi(z)-\varphi\left(\nu_{k}\right)}=2^{m} \prod_{k=1}^{m}\left(1-\frac{1}{u u_{\nu_{k}}}\right)=\sum_{k=0}^{m} 2^{m} a_{k} u^{k-m} \tag{37}
\end{equation*}
$$

where $a_{k}=(-1)^{k} \sum_{1 \leq \nu_{1}<\cdots<\nu_{k} \leq m} u_{\nu_{1}}^{-1} \ldots u_{\nu_{k}}^{-1}, u_{\nu_{k}}=\varphi\left(\nu_{k}\right)$.
From (34), (36), and (37),

$$
\sum_{k=0}^{m}\left(2^{k} b_{n, n-k}-2^{m} a_{m-k}\right) u^{-k}+\frac{1}{u^{2 n}} \sum_{k=0}^{m} 2^{k} b_{n, n-k} u^{k} \rightrightarrows 0
$$

uniformly on compact subsets of $\overline{\mathbb{C}} \backslash[-1,1]$. Therefore, $\lim _{n \rightarrow \infty} b_{n, n-k}=2^{m-k} a_{m-k}, \quad 0 \leq k \leq m$.
Lemma 21. Suppose that $\mu \in \mathcal{P}_{m}\left(\mu_{T}\right)$. If $K$ is a compact subset of $\overline{\mathbb{C}} \backslash[-1,1]$ and $\zeta \in \mathbb{C} \backslash[-1,1]$ then

$$
\begin{aligned}
(z-\zeta) Q_{n}(z)= & \frac{u^{n+1}}{2^{n+1}(n+1)} \Psi_{n}(u)-\frac{u_{\zeta}^{n+1}}{2^{n+1}(n+1)} \Psi_{n}\left(u_{\zeta}\right), \quad z=\frac{1}{2}\left(u+u^{-1}\right), \quad|u|>1 \\
& \Psi_{n}(u) \rightrightarrows\left(1-\frac{1}{u^{2}}\right)(D(\rho, 0))^{-1} D\left(\rho, u^{-1}\right), \quad u_{\zeta}=\varphi(\zeta)
\end{aligned}
$$

Proof. From the definition of the polynomials $Q_{n}$, we have

$$
\begin{equation*}
(z-\zeta) Q_{n}(z)=\int_{\zeta}^{z} P_{n}(t) d t \tag{38}
\end{equation*}
$$

From (38) and (32), it follows that

$$
(z-\zeta) Q_{n}(z)=\int_{\zeta}^{z} P_{n}(t) d t=\int_{\zeta}^{z} \sum_{k=0}^{m} b_{n, n-k} T_{n-k}(t) d t
$$

Making the change of variables $t=\frac{u+u^{-1}}{2}$, we obtain

$$
\begin{equation*}
(z-\zeta) Q_{n}(z)=\int_{\zeta}^{z} P_{n}(t) d t=\int_{\varphi(\zeta)}^{\varphi(z)} \sum_{k=0}^{m} b_{n, n-k} T_{n-k}\left(\frac{u+u^{-1}}{2}\right)\left(\frac{1}{2}-\frac{1}{2 u^{2}}\right) d u \tag{39}
\end{equation*}
$$

Taking into account that for $n>m+1$

$$
\begin{gather*}
\int T_{n-k}\left(\frac{u+u^{-1}}{2}\right)\left(\frac{1}{2}-\frac{1}{2 u^{2}}\right) d u=\int\left(\frac{u^{n-k}+u^{-n+k}}{2^{n-k}}\right)\left(\frac{1}{2}-\frac{1}{2 u^{2}}\right) d u= \\
\frac{u^{n+1}}{2^{n+1}(n+1)}\left(g_{n-k}(u)-g_{n-2-k}(u)\right)+C \tag{40}
\end{gather*}
$$

where

$$
g_{n-k}(u)=\frac{\frac{1}{2^{n-k}}\left(\frac{(n+1)}{(n-k+1)} u^{-k+n}+\frac{(n+1)}{(-n+k+1)} u^{k-n}\right)}{\left(\frac{u}{2}\right)^{n}}
$$

Hence, if we denote $\Psi_{n}(u)=\sum_{k=0}^{m} b_{n, n-k}\left(g_{n-k}(u)-g_{n-2-k}(u)\right)$, from (39) and (40), we obtain

$$
(z-\zeta) Q_{n}(z)=\left.\frac{u^{n+1}}{2^{n+1}(n+1)} \Psi_{n}(u)\right|_{\varphi(\zeta)} ^{\varphi(z)}
$$

Using Lemma 32, (36) of Lemma 20, and taking into account that $\lim _{n \rightarrow \infty} \frac{(n+1)}{(n-k-1)}=1,0 \leq k \leq m$, we obtain that

$$
\Psi_{n}(u) \rightrightarrows\left(1-\frac{1}{u^{2}}\right)(D(\rho(x), 0))^{-1} D\left(\rho(x), u^{-1}\right), \quad|u|>1
$$

Theorem 22. Suppose that $\mu \in \mathcal{P}_{m}\left(\mu_{T}\right)$, where $\mu_{T}$ is the first kind Chebyshev measure. Then the accumulation points of zeros of $\left\{Q_{n}\right\}_{n=0}^{\infty}$ are located on the set $E=\mathcal{E}(\zeta) \bigcup[-1,1]$, where $\mathcal{E}(\zeta)$ is the ellipse

$$
\begin{equation*}
\mathcal{E}(\zeta):=\left\{z \in \mathbb{C}: z=\cosh \left(\eta_{\zeta}+i \theta\right), 0 \leq \theta<2 \pi\right\} \tag{41}
\end{equation*}
$$

and $\eta_{\zeta}:=\ln |\varphi(\zeta)|=\ln \left|\zeta+\sqrt{\zeta^{2}-1}\right|$. If $\delta(\zeta)>2$ then $E=\mathcal{E}(\zeta)$.

Proof. From Lemma 21, the zeros of $Q_{n}$ satisfy that

$$
\begin{equation*}
\left|\Psi_{n}(u) \frac{u^{n+1}}{2^{n+1}(n+1)}\right|^{\frac{1}{n}}=\left|\Psi_{n}\left(u_{\zeta}\right) \frac{u_{\zeta}^{n+1}}{2^{n+1}(n+1)}\right|^{\frac{1}{n}}, \quad z=\frac{1}{2}\left(u+u^{-1}\right), \quad|u|>1 \tag{42}
\end{equation*}
$$

and from the definition of the function $\Psi_{n}$, we have that

$$
\lim _{n \rightarrow \infty}\left|\Psi_{n}(u)\right|^{\frac{1}{n}}=1, \quad|u|>1
$$

Therefore, taking limits on both sides of (42), we deduce that the zeros of $Q_{n}$ can not accumulate outside the set

$$
\left\{z \in \mathbb{C}:\left|z+\sqrt{z^{2}-1}\right|=e^{\eta_{\zeta}}\right\} \bigcup[-1,1] \bigcup\{\zeta\}
$$

Hence, if $z$ is an accumulation point of zeros of the polynomials $Q_{n}$, we have that $z+\sqrt{z^{2}-1}=e^{\eta_{\zeta}+i \theta}$ and $z-\sqrt{z^{2}-1}=e^{-\left(\eta_{\zeta}+i \theta\right)}$ for $0 \leq \theta<2 \pi$, and $2 z=e^{\eta_{\zeta}+i \theta}+e^{-\left(\eta_{\zeta}+i \theta\right)}$.

## 7. Concluding remarks

Theorem 4 can be extended to operators with polynomial coefficients in general, giving results for example, for the case of Heine-Stieltjes operators. For this, we can use the same technique of considering the expression of the matrix $A_{n+1}$ for these operators, which is not difficult to construct. It would be interesting to obtain results similar to Theorems 7 and 10 to classify the measures for which it is possible to ensure the existence and uniqueness, in some sense, of orthogonal polynomials with respect to $(\mathcal{L}, \mu)$ for $n>m, m \in \mathbb{Z}_{+}$. It would also be of interest to obtain results on the zero location and asymptotic behavior for more general classes of operators as well as to consider more general relations of orthogonality, for instance,

$$
\int \mathcal{L}_{0}^{(M)}\left[Q_{n}(x)\right] P(x) d \mu(x)+\cdots+\int \mathcal{L}_{k}^{(M+k)}\left[Q_{n}(x)\right] P^{(k)}(x) d \mu_{k}(x)=0
$$

for any polynomial $P$ such that $\operatorname{deg}[P] \leq n-1$, where the $\left\{\mathcal{L}_{j}^{(M+k)}\right\}_{j=0}^{k}$ are linear homogeneous differential differential operators with coefficients satisfying conditions analogous to (2).

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