

SYLVESTER EQUATIONS FOR LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS ON THE REAL LINE

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ABSTRACT: Matrix Sylvester differential equations are introduced in the study of Laguerre-Hahn orthogonal polynomials. Matrix Sylvester differential systems are shown to yield representations for the Laguerre-Hahn orthogonal polynomials. Lax pairs are given, formed from the differential system and the recurrence relation, that yield discrete non-linear equations for the three term recurrence relation coefficients of the Laguerre-Hahn orthogonal polynomials.

KEYWORDS: Orthogonal polynomials on the real line, Stieltjes functions, Riccati differential equations, matrix Sylvester differential equations.

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1. Introduction

Laguerre-Hahn orthogonal polynomials arise in several subjects of mathematics, such as probability theory [9], differential equations [8, 14, 17], measure perturbation theory [6], constructive theory of approximation [11].

From an analytical point of view, Laguerre-Hahn orthogonal polynomials can be regarded as a generalization of semi-classical orthogonal polynomials, since they are related to Stieltjes functions that are solutions of Riccati differential equations

$$AS' = BS^2 + CS + D, \quad (1)$$

where $A - D$ are polynomials (note that in the semi-classical case there holds $B \equiv 0$) [1, 7, 15].

In this paper we present a study of the Laguerre-Hahn families of orthogonal polynomials in terms of matrix Sylvester differential equations. Such a connection is established by means of the equivalence between (1) and the matrix Sylvester differential equations

$$AY'_n = \mathcal{B}_n Y_n - Y_n \mathcal{C}, \quad n \geq 0, \quad (2)$$

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where $Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}$, with $\{P_n\}$ the sequence of monic orthogonal polynomials related to S and $\{P_n^{(1)}\}$ the corresponding sequence of associated polynomials (also known as the numerator polynomials), and $\mathcal{B}_n, \mathcal{C}$ matrices with polynomial entries. The theory of matrix Sylvester differential equations [16] provides a representation for the solutions of (2) in terms of the solutions of two first order linear systems, which we will relate to the semi-classical orthogonal polynomials.

Our first main result is contained in Theorem 1, where we show the equivalence between (1), (2),

$$AQ'_n = (\mathcal{B}_n + (BS + C/2)I)Q_n, \quad Q_n = \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}, \quad n \geq 0, \quad (3)$$

and

$$AA'_n = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}, \quad \mathcal{A}_n = \begin{bmatrix} x - \beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1, \quad (4)$$

with q_n the functions of the second kind, I the identity matrix, and where β_n, γ_n are the recurrence relation coefficients of $\{P_n\}$.

As a consequence of the previous equivalence, we obtain that a necessary and sufficient condition for a sequence of monic polynomials $\{P_n\}$, orthogonal with respect to a weight w , to be semi-classical is that the following differential system holds (cf. Theorem 2):

$$A\tilde{Y}'_n = (\mathcal{B}_n - C/2I)\tilde{Y}_n, \quad \tilde{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}, \quad n \geq 1,$$

where C is a polynomial (similar differential systems had been studied, for example, in [13, Section 3]). Such a characterization for semi-classical orthogonal polynomials allows us to deduce a representation for the solutions of (2) as $Y_n = \tilde{\mathcal{P}}_n \mathcal{L}^{-1}$, where \mathcal{L} satisfies $A\mathcal{L}' = \mathcal{C}\mathcal{L}$ and $\tilde{\mathcal{P}}_n$ is defined in terms of a semi-classical family, say $\{\tilde{P}_n\}$ (cf. Lemma 3 and Theorem 4). Furthermore, it is shown that the Stieltjes functions related to $\{P_n\}$ and $\{\tilde{P}_n\}$ are a fractional linear transformation of each other (cf. Lemma 3).

The above referred results will be applied to the study of the Laguerre-Hahn family of class zero, i.e., $\max\{\deg(A), \deg(B)\} \leq 2$ and $\deg(C) = 1$ in (1): in the Theorem 5 we show the solutions of the equation (4) in closed form expressions for the recurrence relation coefficients; in the Theorem 6

we show a representation of a given sequence of Laguerre-Hahn orthogonal polynomials.

Let us emphasize that the equations (4), enclosing nonlinear difference equations for the recurrence coefficients β_n and γ_n , are given by the Lax pair

$$Y_n = \mathcal{A}_n Y_{n-1}, \quad AY'_n = \mathcal{B}_n Y_n - Y_n \mathcal{C}.$$

The equations (4) can be viewed as comparable to the so-called Laguerre-Freud's equations that hold for semi-classical families [12]. There are studies on the recurrence coefficients of (semi-classical) orthogonal polynomials showing its relevance in the theory of integrable systems and Painlevé equations, and we refer the reader to [13] and the references therein (see also [4]).

This paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we establish the equivalence between (1), (2), (3), (4), and we deduce the characterization for semi-classical orthogonal polynomials in terms of first order differential systems. In Section 4 we study the representation of Laguerre-Hahn orthogonal polynomials. The Section 5 is devoted to the study of Laguerre-Hahn orthogonal polynomials of class zero.

2. Preliminary Results

Let $\mathbb{P} = \text{span}\{z^k : k \in \mathbb{N}_0\}$ be the space of polynomials with complex coefficients, and let \mathbb{P}' be its algebraic dual space. We will denote by $\langle u, f \rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$.

Given the moments of u , $u_n = \langle u, x^n \rangle$, $n \geq 0$, where we take $u_0 = 1$, the principal minors of the corresponding Hankel matrix are defined by $H_n = \det((u_{i+j})_{i,j=0}^n)$, $n \geq 0$, where, by convention, $H_{-1} = 1$. u is said to be *quasi-definite* (respectively, *positive-definite*) if $H_n \neq 0$ (respectively, $H_n > 0$), for all $n \geq 0$.

Definition 1. (see [18]) Let $u \in \mathbb{P}'$ and let $\{P_n\}_{n \geq 0}$ be a sequence of polynomials such that $\deg(P_n) = n$, $n \geq 0$. $\{P_n\}$ is said to be a sequence of orthogonal polynomials with respect to u if

$$\langle u, P_n P_m \rangle = h_n \delta_{n,m}, \quad h_n = \langle u, P_n^2 \rangle \neq 0, \quad n, m \geq 0. \quad (5)$$

Throughout the paper we shall take each P_n monic, that is, $P_n(z) = z^n +$ lower degree terms, and we will denote $\{P_n\}$ by SMOP.

The equivalence between the quasi-definiteness of $u \in \mathbb{P}'$ and the existence of a SMOP with respect to u is well-known in the literature of orthogonal

polynomials (see [5, 18]). Furthermore, if u is positive-definite, then it has an integral representation in terms of a positive Borel measure, μ , supported on an infinite set of points of the real line, I , such that

$$\langle u, x^n \rangle = \int_I x^n d\mu(x), \quad n \geq 0, \quad (6)$$

and the orthogonality condition (5) becomes

$$\int_I P_n(x)P_m(x)d\mu(x) = h_n\delta_{n,m}, \quad h_n > 0, \quad n, m \geq 0.$$

Further, if μ is defined in terms of a weight w , $d\mu(x) = w(x)dx$, then we will also say that $\{P_n\}$ is orthogonal with respect to w .

Monic orthogonal polynomials satisfy a three term recurrence relation [18]

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n = 1, 2, \dots \quad (7)$$

with $P_0(x) = 1$, $P_1(x) = x - \beta_0$ and $\gamma_n \neq 0$, $n \geq 1$, $\gamma_0 = 1$.

Definition 2. Let $\{P_n\}$ be the SMOP with respect to a linear functional u . The sequence of *associated polynomials* is defined by

$$P_n^{(1)}(x) = \langle u_t, \frac{P_{n+1}(x) - P_{n+1}(t)}{x - t} \rangle, \quad n \geq 0,$$

where u_t denotes the action of u on the variable t .

Definition 3. The *Stieltjes function* of $u \in \mathbb{P}'$ is defined in terms of the moments, (u_n) , of u by $S(x) = -\sum_{n=0}^{+\infty} \frac{u_n}{x^{n+1}}$.

S has an expansion in terms of a continued fraction, given by

$$S(x) = \frac{1}{x - \beta_0 - \frac{\gamma_1}{x - \beta_1 - \frac{\gamma_2}{\ddots}}} \quad (8)$$

where the γ 's and the β 's are the three term recurrence relation coefficients of the corresponding SMOP. Note that if u is positive-definite, defined by (6),

then S is given by $S(x) = \int_I \frac{d\mu(t)}{x - t}$, $x \in \mathbb{C} \setminus I$.

The sequence of *functions of the second kind* corresponding to $\{P_n\}$ is defined as follows:

$$q_{n+1} = P_{n+1}S - P_n^{(1)}, \quad n \geq 0, \quad q_0 = S.$$

Definition 4. (see [15]) A Stieltjes function, S , is said to be *Laguerre-Hahn* if there exist polynomials A, B, C, D , with $A \neq 0$, such that it satisfies a Riccati differential equation

$$AS' = BS^2 + CS + D. \quad (9)$$

The corresponding sequence of orthogonal polynomials is called *Laguerre-Hahn*. If $B = 0$, then S is said to be *Laguerre-Hahn affine* or *semi-classical*.

If S is related to a positive-definite linear functional defined in terms of a weight, w , then the semi-classical character of S means $w'/w = C/A$, with A, C polynomials, and such a differential equation is equivalent to $AS' = CS + D$, where D is a polynomial given in terms of A, C (see [15]).

Note that (9) is equivalent to the distributional equation [7, 15]

$$\mathcal{D}(Au) = \psi u + B(x^{-1}u^2), \quad \psi = A' + C.$$

In the sequel we will use the following matrices:

$$Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}, \quad \tilde{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}, \quad Q_n = \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}, \quad n \geq 0. \quad (10)$$

Lemma 1. Let $\{P_n\}$ be a SMOP and let β_n, γ_n be the coefficients of the three term recurrence relation (7). Let $\{Y_n\}, \{\tilde{Y}_n\}, \{Q_n\}$ be the sequences defined in (10). Then,

(a) Y_n and \tilde{Y}_n satisfy the difference equation

$$X_n = \mathcal{A}_n X_{n-1}, \quad \mathcal{A}_n = \begin{bmatrix} x - \beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1, \quad (11)$$

with initial conditions $Y_0 = \begin{bmatrix} x - \beta_0 & 1 \\ 1 & 0 \end{bmatrix}$, $\tilde{Y}_0 = \begin{bmatrix} x - \beta_0 & q_1/w \\ 1 & q_0/w \end{bmatrix}$;

(b) Q_n satisfies

$$Q_n = \mathcal{A}_n Q_{n-1}, \quad n \geq 1, \quad (12)$$

with \mathcal{A}_n given in (11) and initial conditions $Q_0 = \begin{bmatrix} (x - \beta_0)S - 1 \\ S \end{bmatrix}$.

Throughout the paper I denotes the 2×2 identity matrix. The (i, j) entry of a matrix X will be denoted by $X^{(i,j)}$.

3. Characterization in terms of matrix Sylvester differential equations

Theorem 1. *Let S be a Stieltjes function, let $\{Y_n\}$ and $\{Q_n\}$ be the corresponding sequences defined in (10), and let β_n, γ_n be the corresponding recurrence relation coefficients. The following statements are equivalent:*

(a) S satisfies

$$AS' = BS^2 + CS + D, \quad A, B, C, D \in \mathbb{P};$$

(b) Y_n satisfies the matrix Sylvester equation

$$AY'_n = \mathcal{B}_n Y_n - Y_n \mathcal{C}, \quad n \geq 0, \quad (13)$$

where

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & l_{n-1} + (x - \beta_n)\Theta_{n-1}/\gamma_n \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}$$

with l_n, Θ_n polynomials of uniformly bounded degrees, satisfying the initial conditions

$$A = (l_0 - C/2)(x - \beta_0) - B + \Theta_0, \quad 0 = D(x - \beta_0) + l_0 + C/2, \\ \Theta_{-1} = D, \quad l_{-1} = C/2; \quad (14)$$

(c) the matrices defined in (11), $\mathcal{A}_n = \begin{bmatrix} x - \beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}$, satisfy

$$A\mathcal{A}'_n = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}, \quad n \geq 1; \quad (15)$$

(d) Q_n satisfies

$$AQ'_n = (\mathcal{B}_n + (BS + C/2)I)Q_n, \quad n \geq 0.$$

Proof: Taking into account [3, Theorem 2] we only need to prove (b) \Leftrightarrow (c). (b) \Rightarrow (c). To obtain (15) we take derivatives on $Y_n = \mathcal{A}_n Y_{n-1}$ (cf. (11)), and substitute it in (13), thus obtaining

$$A\mathcal{A}'_n Y_{n-1} + A\mathcal{A}_n Y'_{n-1} = \mathcal{B}_n Y_n - Y_n \mathcal{C}.$$

Using (13) for $n - 1$ in the previous equation we get

$$A\mathcal{A}'_n Y_{n-1} + \mathcal{A}_n (\mathcal{B}_{n-1} Y_{n-1} - Y_{n-1} \mathcal{C}) = \mathcal{B}_n Y_n - Y_n \mathcal{C}.$$

Using the recurrence relation (11) for Y_n we obtain

$$A\mathcal{A}'_n Y_{n-1} + \mathcal{A}_n (\mathcal{B}_{n-1} Y_{n-1} - Y_{n-1} \mathcal{C}) = \mathcal{B}_n \mathcal{A}_n Y_{n-1} - \mathcal{A}_n Y_{n-1} \mathcal{C},$$

that is,

$$A\mathcal{A}'_n Y_{n-1} = (\mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}) Y_{n-1}.$$

Since Y_n is nonsingular, for all $n \geq 0$, there follows (15).

(c) \Rightarrow (b). If we multiply (15) by Y_{n-1} , we obtain

$$A(\mathcal{A}_n Y_{n-1})' - A\mathcal{A}_n Y'_{n-1} = \mathcal{B}_n \mathcal{A}_n Y_{n-1} - \mathcal{A}_n \mathcal{B}_{n-1} Y_{n-1}.$$

Taking into account the recurrence relation (11) for Y_n we get

$$AY'_n - \mathcal{B}_n Y_n = \mathcal{A}_n (AY'_{n-1} - \mathcal{B}_{n-1} Y_{n-1}),$$

thus

$$AY'_n - \mathcal{B}_n Y_n = \mathcal{A}_n \cdots \mathcal{A}_2 (AY'_1 - \mathcal{B}_1 Y_1).$$

The use of $\mathcal{A}_n \cdots \mathcal{A}_2 = Y_n Y_1^{-1}$ in the preceding equation yields an equation for Y_n of the Sylvester type (13), $AY'_n = \mathcal{B}_n Y_n - Y_n \tilde{C}$, with $\tilde{C} = -Y_1^{-1} (AY'_1 - \mathcal{B}_1 Y_1)$. \blacksquare

Corollary 1. *The following relations hold:*

$$\operatorname{tr}(\mathcal{B}_n) = 0, \quad n \geq 0, \quad (16)$$

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_0) + A \sum_{k=1}^n \frac{\Theta_{k-1}}{\gamma_k}, \quad n \geq 1, \quad (17)$$

and $\det(\mathcal{B}_0) = D(A + B) - (C/2)^2$.

Proof: The formulas for the trace and determinant of the matrices \mathcal{B}_n involved in equations of the same type as (15) were deduced in [2, Lemma 2.5]. The use of the initial conditions (14) yields (16) and (17). \blacksquare

Remark 1. Since $\operatorname{tr}(\mathcal{B}_n) = 0$, henceforth we parameterize the matrix \mathcal{B}_n in terms of the two functions l_n and Θ_n as $\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix}$.

Remark 2. Equation (15) reads as

$$\begin{cases} (x - \beta_n)(l_n - l_{n-1}) = A - \Theta_n + \frac{\gamma_n}{\gamma_{n-1}} \Theta_{n-2} \\ l_n - l_{n-2} = -\frac{(x - \beta_n)}{\gamma_n} \Theta_{n-1} + \frac{(x - \beta_{n-1})}{\gamma_{n-1}} \Theta_{n-2}, \quad n \geq 1, \end{cases}$$

which are comparable with the so-called Laguerre-Freud's equations [12, 13].

3.1. Characterization of semi-classical orthogonal polynomials. As a consequence of the previous theorem we deduce the characterization that follows.

Theorem 2. *Let $\{P_n\}$ be a SMOP with respect to a weight w , and let $\{q_n\}$ be the corresponding sequence of functions of the second kind. The weight w is semi-classical and satisfies $w'/w = C/A$ if, and only if, $\tilde{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}$ satisfies the matrix differential equation*

$$A\tilde{Y}'_n = \left(\mathcal{B}_n - \frac{C}{2}I \right) \tilde{Y}_n, \quad n \geq 1, \quad (18)$$

where \mathcal{B}_n is the matrix associated with the equation $AS' = CS + D$ for the Stieltjes function of w .

Proof: Note that $w'/w = C/A$ implies $AS' = CS + D$, $D \in \mathbb{P}$, for the corresponding Stieltjes function (see [15]). Taking into account the Theorem 1 we get

$$\begin{cases} AP'_{n+1} = (l_n - C/2)P_{n+1} + \Theta_n P_n \\ Aq'_{n+1} = (l_n + C/2)q_{n+1} + \Theta_n q_n, \end{cases} \quad n \geq 0. \quad (19)$$

Therefore, using the three term recurrence relation (7) for $\{P_n\}$ we obtain

$$AP'_n = \left(l_{n-1} - C/2 + \frac{(x - \beta_n)\Theta_{n-1}}{\gamma_n} \right) P_n - \frac{\Theta_{n-1}}{\gamma_n} P_{n+1}, \quad n \geq 1. \quad (20)$$

From (19) there follows

$$A \left(\frac{q_{n+1}}{w} \right)' = \Theta_n \frac{q_n}{w} + (l_n - C/2) \frac{q_{n+1}}{w}, \quad (21)$$

where we used $w'/w = C/A$. Furthermore, using the three term recurrence relation for $\{q_n\}$ (cf. (12)) we obtain

$$A \left(\frac{q_n}{w} \right)' = \left(l_{n-1} - C/2 + \frac{(x - \beta_n)\Theta_{n-1}}{\gamma_n} \right) \frac{q_n}{w} - \frac{\Theta_{n-1}}{\gamma_n} \frac{q_{n+1}}{w}, \quad n \geq 1. \quad (22)$$

Equations. (19)-(22) yield

$$A\tilde{Y}'_n = \tilde{\mathcal{B}}_n \tilde{Y}_n, \quad \tilde{\mathcal{B}}_n = \begin{bmatrix} l_n - C/2 & \Theta_n \\ -\Theta_{n-1}/\gamma_n & l_{n-1} + (x - \beta_n)\Theta_{n-1}/\gamma_n - C/2 \end{bmatrix}, \quad n \geq 1,$$

thus we get (18).

Conversely, if \tilde{Y}_n satisfies (18), then

$$(\det(\tilde{Y}_n))' = \frac{\operatorname{tr}(\mathcal{B}_n - C/2I)}{A} \det(\tilde{Y}_n).$$

Since $\det(\tilde{Y}_n) = (\gamma_1 \dots \gamma_n)/w$ and $\operatorname{tr}(\mathcal{B}_n - C/2I) = -C$, there follows $w'/w = C/A$, thus w is semi-classical. \blacksquare

Remark 3. The analogue result for orthonormal polynomials was established in [13].

4. Matrix Sylvester equations and Radon's Lemma

The theorem that follows is a particular case of the result known, in the literature of matrix Riccati equations, as Radon's Lemma [10, 16].

Theorem 3. *Let A be a polynomial, let \mathcal{B}_n/A , $n \geq 1$, and C/A be matrices whose entries are integrable functions in a domain G of the complex plane, and let $x_0 \in G$. If the matrices \mathcal{P}_n and \mathcal{L} , \mathcal{L} nonsingular, satisfy*

$$\begin{cases} A\mathcal{L}' = C\mathcal{L}, \\ \mathcal{L}(x_0) = I, \end{cases} \quad (23)$$

and

$$\begin{cases} A\mathcal{P}'_n = \mathcal{B}_n\mathcal{P}_n, \quad n \geq 1, \\ \mathcal{P}_n(x_0) = Y_n(x_0), \end{cases} \quad (24)$$

then the solution of $AY'_n = \mathcal{B}_nY_n - Y_nC$, in G , is given by:

$$Y_n = \mathcal{P}_n\mathcal{L}^{-1}, \quad n \geq 1.$$

Our aim is to find a representation for Y_n satisfying (13), $AY'_n = \mathcal{B}_nY_n - Y_nC$, related to $AS' = BS^2 + CS + D$ (cf. Theorem 1). To that end we start with some remarks on the solution of the corresponding problem (24). Note that we are searching for matrices \mathcal{P}_n of order two satisfying

$$A\mathcal{P}'_n = \mathcal{B}_n\mathcal{P}_n. \quad (25)$$

Hereafter we will consider $x_1 \in \mathbb{C}$ and \tilde{C} a polynomial such that $\int_{x_1}^x \frac{\tilde{C}(t)}{2A(t)} dt$ is defined in suitable domains.

Lemma 2. *Let \mathcal{B}_n be the matrices given in (13), and let \tilde{C} be a polynomial. A matrix \tilde{Y}_n satisfies*

$$A\tilde{Y}'_n = (\mathcal{B}_n - \tilde{C}/2I)\tilde{Y}_n \quad (26)$$

if, and only if, $\mathcal{P}_n = e^{\int_{x_1}^x \frac{\tilde{C}}{2A} dt} \tilde{Y}_n$ satisfies (25).

Taking into account the previous Lemma, we will solve (25) by considering

$$\mathcal{P}_n = e^{\int_{x_1}^x \frac{\tilde{C}}{2A} dt} \tilde{Y}_n,$$

where \tilde{Y}_n satisfies (26). Furthermore, taking into account the Theorem 2, we will search for \tilde{Y}_n defined by

$$\tilde{Y}_n = \begin{bmatrix} \tilde{P}_{n+1} & \tilde{q}_{n+1}/\tilde{w} \\ \tilde{P}_n & \tilde{q}_n/\tilde{w} \end{bmatrix}, \quad (27)$$

with $\{\tilde{P}_n\}$ a SMOP with respect to a weight function \tilde{w} and $\{\tilde{q}_n\}$ the corresponding sequence of functions of the second kind.

Remark 4. Note that (26) implies $\det(\tilde{Y}_n)' = \frac{\text{tr}(\mathcal{B}_n - \tilde{C}/2I)}{A} \tilde{Y}_n$, which combined with (27) yields $\tilde{w}'/\tilde{w} = \tilde{C}/A$, thus $\tilde{w} = e^{\int \tilde{C}/A}$.

Lemma 3. *Let S be a Stieltjes function that satisfies the Riccati differential equation $AS' = BS^2 + CS + D$. Let $\{P_n\}$ be the corresponding SMOP such that the equations (13), $AY'_n = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, hold, and let (24) be the corresponding system $AP'_n = \mathcal{B}_n P_n$. Let the following assumption hold:*

$$\exists \tilde{C} \in \mathbb{P}, \exists n_0 \geq 1 : \mathcal{P}_n = e^{\int_{x_1}^x \frac{\tilde{C}(t)}{2A(t)} dt} \tilde{Y}_{n-n_0}, \quad n \geq n_0 + 1, \quad (28)$$

with the \tilde{Y}_n 's given as in (27), related to a SMOP $\{\tilde{P}_n\}$. Denote by \tilde{S} the Stieltjes functions related to $\{\tilde{P}_n\}$. Then, the following statements hold:

(a) \tilde{S} is a fractional linear transformation of S ,

$$\tilde{S} = \frac{a + bS}{c + dS}, \quad a, b, c, d \in \mathbb{P}; \quad (29)$$

(b) \tilde{S} satisfies

$$A\tilde{S}' = \tilde{C}\tilde{S} + \tilde{D}, \quad \tilde{D} \in \mathbb{P}, \quad (30)$$

the polynomials \tilde{C}, \tilde{D} being related to B, C, D by

$$(bc - ad)B = (bd' - b'd)A + bd\tilde{C} + d^2\tilde{D}, \quad (31)$$

$$(bc - ad)C = (bc' + ad' - b'c - a'd)A + (ad + bc)\tilde{C} + 2cd\tilde{D}, \quad (32)$$

$$(bc - ad)D = (ac' - a'c)A + ac\tilde{C} + c^2\tilde{D}. \quad (33)$$

Proof: (a). Let us denote by $\tilde{\mathcal{A}}_n$ the matrices of the recurrence relation of $\{\tilde{Y}_n\}$. From $\mathcal{P}_n = e^{\int_{x_1}^z \frac{\tilde{C}}{2A} dt} \tilde{Y}_{n-n_0}$ and $\tilde{Y}_{n-n_0} = \tilde{\mathcal{A}}_{n-n_0} \tilde{Y}_{n-n_0-1}$ there follows

$$\mathcal{P}_n = \tilde{\mathcal{A}}_{n-n_0} \mathcal{P}_{n-1}, \quad n \geq n_0 + 1. \quad (34)$$

The substitution of (34) into $AP'_n = \mathcal{B}_n \mathcal{P}_n$ yields

$$A(\tilde{\mathcal{A}}_{n-n_0} \mathcal{P}_{n-1})' = \mathcal{B}_n \tilde{\mathcal{A}}_{n-n_0} \mathcal{P}_{n-1}, \quad n \geq n_0 + 1,$$

from which we obtain

$$A\tilde{\mathcal{A}}'_{n-n_0} = \mathcal{B}_n \tilde{\mathcal{A}}_{n-n_0} - \tilde{\mathcal{A}}_{n-n_0} \mathcal{B}_{n-1}, \quad n \geq n_0 + 1. \quad (35)$$

Using the equations enclosed in the positions (2,1) and (2,2) of (35) we obtain

$$\tilde{\gamma}_{n-n_0} = \gamma_n, \quad \tilde{\beta}_{n-n_0} = \beta_n, \quad n \geq n_0 + 1.$$

Consequently, taking into account the representation of the Stieltjes function in terms of a continued fraction (8), (29) follows.

(b). From (28) we obtain that $\{\tilde{P}_n\}$ is semi-classical, and the corresponding \tilde{w} satisfies $\tilde{w}'/\tilde{w} = \tilde{C}/A$ (note the remark 4). Thus, the first order differential equation (30) for the corresponding Stieltjes function, \tilde{S} , follows.

By substituting \tilde{S} given by (29) in (30) we get

$$A(bc - ad)S' = B_1 S^2 + C_1 S + D_1 \quad (36)$$

with

$$\begin{aligned} B_1 &= (bd' - b'd)A + bd\tilde{C} + d^2\tilde{D}, \\ C_1 &= (bc' + ad' - b'c - a'd)A + (ad + bc)\tilde{C} + 2cd\tilde{D}, \\ D_1 &= (ac' - a'c)A + ac\tilde{C} + c^2\tilde{D}. \end{aligned}$$

Since S satisfies $AS' = BS^2 + CS + D$ as well as (36), there follows, if $A - D$ and $B_1 - D_1$ are non vanishing, that

$$\varrho = \frac{B}{B_1} = \frac{C}{C_1} = \frac{D}{D_1}, \quad \varrho = 1/(bc - ad).$$

Hence, we obtain (31)-(33). ■

Theorem 4. *Let S be a Stieltjes function satisfying $AS' = BS^2 + CS + D$. To $AS' = BS^2 + CS + D$ associate $AY'_n = \mathcal{B}_n Y_n - Y_n \mathcal{C}$ and the corresponding systems (23) and (24), $A\mathcal{L}' = \mathcal{C}\mathcal{L}$, $AP'_n = \mathcal{B}_n \mathcal{P}_n$. Let G be a domain in the complex plane such that the entries of the matrices \mathcal{B}_n/A and \mathcal{C}/A are integrable in G . Assume that the assumption (28) holds. Then, there exists a polynomial \tilde{C} (defined by (31)-(33)) and a weight function $\tilde{w} = e^{\int \frac{\tilde{C}}{A}}$ such the following representation holds in G :*

$$Y_n = \begin{bmatrix} \sqrt{\tilde{w}} \tilde{P}_{n-n_0+1} & \tilde{q}_{n-n_0+1}/\sqrt{\tilde{w}} \\ \sqrt{\tilde{w}} \tilde{P}_{n-n_0} & \tilde{q}_{n-n_0}/\sqrt{\tilde{w}} \end{bmatrix} E_n \mathcal{L}^{-1}, \quad n \geq n_0 + 1,$$

with $\{\tilde{P}_n\}$ the SMOP with respect to \tilde{w} , $\{\tilde{q}_n\}$ the corresponding sequence of functions of the second kind, and $E_n = \mathcal{P}_n(x_0)^{-1}Y_n(x_0)$, $x_0 \in G$.

Proof: The result follows from the Theorem 3 combined with the Lemma 3 (note the remark 4). \blacksquare

5. Laguerre-Hahn orthogonal polynomials of class zero

Laguerre-Hahn orthogonal polynomials of class zero are related to Stieltjes functions, S , satisfying

$$AS' = BS^2 + CS + D, \quad \deg(C) = 1, \quad \max\{\deg(A), \deg(B)\} \leq 2.$$

There are, up to a linear change of variable, three canonical cases for the triples of polynomials satisfying the above Riccati differential equation, according to the degree of A be zero, one or two [1].

In subsection 5.1 we will show the solutions of the corresponding difference equation (15) in the class zero: we give closed form expressions for the recurrence relation coefficients of the orthogonal polynomials. In subsection 5.2 we will use Radon's Lemma in order to obtain representations for a sequence of Laguerre-Hahn orthogonal polynomials.

5.1. The recurrence relation coefficients.

Theorem 5. *Let S be a Stieltjes function satisfying $AS' = BS^2 + CS + D$ with $\deg(C) = 1$, $\max\{\deg(A), \deg(B)\} \leq 2$. Let $A(x) = a_2x^2 + a_1x + a_0$, $B(x) = b_2x^2 + b_1x + b_0$, $C(x) = c_1x + c_0$. The recurrence relation coefficients of the SMOP $\{P_n\}$ related to S are given by*

$$\beta_n = \frac{(r_1^2 - a_2^2)\beta_1 + 2a_1(n-1)(r_1 - (n-1)a_2)}{(r_1 - (2n-1)a_2)(r_1 - (2n-3)a_2)}, \quad n \geq 2, \quad (37)$$

$$\gamma_{n+1} = \frac{r_1(r_1 - 2a_2)\gamma_2 + \sum_{k=2}^n A(\beta_k)(r_1 - 2(k-1)a_2)}{(r_1 - 2na_2)(r_1 - 2(n-1)a_2)}, \quad n \geq 2, \quad (38)$$

where

$$\beta_1 = \frac{a_1 + 2\beta_0 D - c_0}{r_1 - a_2}, \quad r_1 = \Theta_0/\gamma_1, \quad \Theta_0 = A + B + (x - \beta_0)^2 D + (x - \beta_0)C, \\ \gamma_1 = \frac{\Theta_0}{2D + c_1 - a_2}, \quad \gamma_2 = \frac{A(\beta_1) + \gamma_1 D}{r_1 - 2a_2}. \quad (39)$$

Proof: The corresponding equations (15) read as

$$\begin{cases} (x - \beta_n)(l_n - l_{n-1}) = A - \Theta_n + \frac{\gamma_n}{\gamma_{n-1}}\Theta_{n-2} \\ l_n - l_{n-2} = -\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}}\Theta_{n-2}, \quad n \geq 1, \end{cases} \quad (40)$$

where the initial conditions hold (cf. (14)):

$$\begin{aligned} l_{-1}(x) &= C(x)/2, \quad l_0(x) = -(x - \beta_0)D - C(x)/2, \\ \Theta_{-1} &= D, \quad \Theta_0 = A(x) + B(x) + (x - \beta_0)^2D + (x - \beta_0)C(x). \end{aligned}$$

Note that under the stated conditions on the degrees of A, B, C , one has

$$l_n(x) = \ell_{n,1}x + \ell_{n,0}, \quad \Theta_n, D \text{ constants.} \quad (41)$$

By substituting (41) into (40) and equating the coefficients of x^2, x, x^0 we obtain, for all $n \geq 1$,

$$\ell_{n,1} - \ell_{n-1,1} = a_2, \quad (42)$$

$$\ell_{n,0} - \ell_{n-1,0} = \beta_n(\ell_{n,1} - \ell_{n-1,1}) + a_1, \quad (43)$$

$$-\beta_n(\ell_{n,0} - \ell_{n-1,0}) = a_0 - \Theta_n + \frac{\gamma_n}{\gamma_{n-1}}\Theta_{n-2},$$

$$\ell_{n,1} - \ell_{n-2,1} = -\frac{\Theta_{n-1}}{\gamma_n} + \frac{\Theta_{n-2}}{\gamma_{n-1}}, \quad (44)$$

$$\ell_{n,0} - \ell_{n-2,0} = \beta_n \frac{\Theta_{n-1}}{\gamma_n} - \beta_{n-1} \frac{\Theta_{n-2}}{\gamma_{n-1}}. \quad (45)$$

From (42) and (43) there follows

$$\ell_{n,1} = \ell_{0,1} + na_2, \quad n \geq 1, \quad (46)$$

$$\ell_{n,0} = \ell_{0,0} + a_2 \sum_{k=1}^n \beta_k + na_1, \quad n \geq 1. \quad (47)$$

Note that (46) and (47) are also valid for $n = 0$, using the convention $\sum_{k=i}^j \cdot = 0$ whenever $i > j$.

The use of (46) for n and $n - 2$ in (44) yields

$$\frac{\Theta_{n-1}}{\gamma_n} = \frac{\Theta_{n-2}}{\gamma_{n-1}} - 2a_2, \quad n \geq 2,$$

thus

$$\frac{\Theta_{n-1}}{\gamma_n} = \frac{\Theta_0}{\gamma_1} - 2(n-1)a_2, \quad n \geq 2.$$

Note that the above equality also holds for $n = 1$, thus we get

$$\Theta_{n-1} = (r_1 - 2(n-1)a_2)\gamma_n, \quad r_1 = \frac{\Theta_0}{\gamma_1}, \quad n \geq 1. \quad (48)$$

The use of (47) for n and $n-2$ as well as (48) for n and $n-1$ in (45) yields $a_2(\beta_{n-1} + \beta_n) + 2a_1 = \beta_n(r_1 - 2(n-1)a_2) - \beta_{n-1}(r_1 - 2(n-2)a_2)$, $n \geq 2$, that is,

$$\beta_n(r_1 - (2n-1)a_2) = \beta_{n-1}(r_1 - (2n-5)a_2) + 2a_1, \quad n \geq 2.$$

If we multiply the above equation by $r_1 - (2n-3)a_2$ we get

$$\begin{aligned} L_{n+1} &= L_n + 2a_1(r_1 - (2n-3)a_2), \\ L_n &= \beta_{n-1}(r_1 - (2n-3)a_2)(r_1 - (2n-5)a_2). \end{aligned}$$

Therefore, we obtain

$$L_{n+1} = L_2 + 2a_1 \sum_{k=2}^n (r_1 - (2k-3)a_2), \quad n \geq 2,$$

thus for $n \geq 2$

$$\beta_n(r_1 - (2n-1)a_2)(r_1 - (2n-3)a_2) = \beta_1(r_1^2 - a_2^2) + 2a_1(n-1)(r_1 - (n-1)a_2),$$

thus (37) follows.

To obtain an equation for the γ 's we start by evaluating the first equation of (40) at β_n , thus obtaining

$$\Theta_n - \frac{\gamma_n}{\gamma_{n-1}}\Theta_{n-2} = A(\beta_n), \quad n \geq 1.$$

Using (48) for $n+1$ and $n-1$ in the above equation we get

$$\gamma_{n+1}(r_1 - 2na_2) - \gamma_n(r_1 - 2(n-2)a_2) = A(\beta_n), \quad n \geq 2.$$

If we multiply the above equation by $r_1 - 2(n-1)a_2$ we get for $n \geq 2$

$$T_{n+1} = T_n + A(\beta_n)(r_1 - 2(n-1)a_2), \quad T_n = \gamma_n(r_1 - 2(n-2)a_2)(r_1 - 2(n-1)a_2).$$

Therefore, we obtain

$$T_{n+1} = T_2 + \sum_{k=2}^n A(\beta_k)(r_1 - 2(k-1)a_2), \quad n \geq 2,$$

thus (38) follows.

(39) follows from the initial conditions. ■

5.2. Representations of Laguerre-Hahn orthogonal polynomials of class zero via Radon's Lemma: an example.

Lemma 4. *Let $\{P_n\}$ be a SMOP related to $AS' = BS^2 + CS + D$, $A(x) = a_2x^2 + a_1x + a_0$, $B(x) = b_2x^2 + b_1x + b_0$, $C(x) = c_1x + c_0$. Then, the matrices $\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix}$ involved in the corresponding equations (13) are defined by $l_n(x) = \ell_{n,1}x + \ell_{n,0}$ and Θ_n constant, where, for all $n \geq 1$,*

$$\ell_{n,1} = c_1/2 + (n+1)a_2 + b_2, \quad (49)$$

$$\ell_{n,0} = c_0/2 + (n+1)a_1 + b_1 + \beta_0(b_2 + a_2) + \alpha_n a_2, \quad (50)$$

$$\begin{aligned} \Theta_n = & (\ell_{n,0} + c_0/2 - (n-1)a_1)\alpha_n - (\ell_{n,1} + c_1/2 - (n-2)a_2)\nu_n \\ & - (\nu_n + \beta_0\alpha_n - \gamma_1)D + na_0, \end{aligned} \quad (51)$$

and

$$\begin{aligned} \ell_{0,1} = -D - c_1/2, \quad \ell_{0,0} = \beta_0 D - c_0/2, \\ \Theta_0 = A + B + (x - \beta_0)^2 D + (x - \beta_0)C, \quad \Theta_{-1} = D, \end{aligned} \quad (52)$$

with

$$D = -a_2 - c_1 - b_2, \quad \alpha_n = \sum_{k=1}^n \beta_k, \quad \nu_n = \sum_{1 \leq i < j \leq n} \beta_i \beta_j - \sum_{k=2}^n \gamma_k, \quad n \geq 1.$$

Proof: Write

$$\begin{aligned} P_n^{(1)}(x) &= x^n - \alpha_n x^{n-1} + \nu_n x^{n-2} + \dots \\ P_{n+1}(x) &= x^{n+1} - (\alpha_n + \beta_0)x^n + (\nu_n + \beta_0\alpha_n - \gamma_1)x^{n-1} + \dots \end{aligned}$$

and compare the coefficients of the corresponding monomials. To obtain $l_0(x)$, Θ_0 , and Θ_{-1} , use the initial conditions (14). \blacksquare

For later purposes we show some results concerning the Hermite polynomials.

Lemma 5. *Let $\{H_n\}$ denote the sequence of monic Hermite polynomials, and let $\{\tilde{q}_n\}$ be the corresponding sequence of functions of the second kind. The following statements hold:*

(a) $\{H_n\}$ is related to the Stieltjes function \tilde{S} satisfying

$$A\tilde{S}' = \tilde{C}\tilde{S} + \tilde{D}, \quad A = 1, \quad \tilde{C} = -2x, \quad \tilde{D} = 2; \quad (53)$$

(b) the recurrence relation coefficients of $\{H_n\}$ are given by

$$\tilde{\beta}_n = 0, \quad \tilde{\gamma}_{n+1} = (n+1)/2, \quad n \geq 0; \quad (54)$$

(c) $\tilde{H}_n = \begin{bmatrix} H_{n+1} & \tilde{q}_{n+1}/\tilde{w} \\ H_n & \tilde{q}_n/\tilde{w} \end{bmatrix}$, with $\tilde{w} = e^{-x^2}$, satisfies the differential system

$$\tilde{H}'_n = \left(\tilde{\mathcal{B}}_n - \frac{\tilde{C}}{2}I \right) \tilde{H}_n, \quad \tilde{\mathcal{B}}_n = \begin{bmatrix} -x & n+1 \\ -2 & x \end{bmatrix}, \quad n \geq 1. \quad (55)$$

Proof: (53) is given in [15]. (54) follows from (37) and (38). To obtain (55) we use the Theorem 2 (cf. (18)), where the entries of $\tilde{\mathcal{B}}_n$ can be obtained using (49)-(51). \blacksquare

Lemma 6. Let $\{P_n\}$ be the SMOP related to the Stieltjes function S satisfying $AS' = BS^2 + CS + D$, where

$$A = 1, \quad B = -2x^2 + 2\lambda x + \rho - 1, \quad C = 2x, \quad D = 0,$$

with $\rho = 2\gamma_1$, $\lambda = \beta_0$, and $\rho\lambda \neq 0$. Let $\{Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}\}$ be the sequence associated with $\{P_n\}$ defined in (10). The following statements hold:

(a) the recurrence relation coefficients of $\{P_n\}$ are given by

$$\beta_n = 0, \quad \gamma_{n+1} = n/2, \quad n \geq 1; \quad (56)$$

(b) Y_n satisfies the matrix Sylvester equations $Y'_n = \mathcal{B}_n Y_n - \mathcal{C} Y_n$, $n \geq 0$, with

$$\mathcal{B}_0 = \begin{bmatrix} -x & \rho \\ 0 & x \end{bmatrix}, \quad \mathcal{B}_n = \begin{bmatrix} -x & n \\ -2 & x \end{bmatrix}, \quad n \geq 1, \quad \mathcal{C} = \begin{bmatrix} x & 0 \\ B & -x \end{bmatrix}. \quad (57)$$

(c) let x_0 be an arbitrary point in the complex plane. The solution of the initial value problem

$$\begin{cases} \mathcal{L}'(x) = \mathcal{C}(x)\mathcal{L}(x), \\ \mathcal{L}(x_0) = I, \end{cases} \quad (58)$$

with \mathcal{C} given in (57), is

$$\mathcal{L}^{(1,1)}(x) = e^{-x_0^2/2} e^{x^2/2}, \quad (59)$$

$$\mathcal{L}^{(1,2)}(x) \equiv 0, \quad (60)$$

$$\mathcal{L}^{(2,1)}(x) = e^{-x_0^2/2} e^{-x^2/2} \int_{x_0}^x B(s) e^{s^2} ds, \quad (61)$$

$$\mathcal{L}^{(2,2)}(x) = e^{x_0^2/2} e^{-x^2/2}, \quad (62)$$

and the solution of the initial value problem

$$\begin{cases} \mathcal{P}'_n(x) = \mathcal{B}_n(x) \mathcal{P}_n(x), & n \geq 1, \\ \mathcal{P}_n(x_0) = Y_n(x_0), \end{cases} \quad (63)$$

with \mathcal{B}_n given in (57), is for $n \geq 2$

$$\mathcal{P}_n(x) = e^{-x^2/2} \tilde{H}_{n-1}(x) \mathcal{K}_n, \quad \mathcal{K}_n = e^{x_0^2/2} \left(\tilde{H}_{n-1}(x_0) \right)^{-1} Y_n(x_0). \quad (64)$$

Proof: (56) follows from (37) and (38). To obtain the entries of \mathcal{B}_n in (57) use (49)-(52).

Let us solve (63). Since $\tilde{H}_n = \begin{bmatrix} H_{n+1} & \tilde{q}_{n+1}/\tilde{w} \\ H_n & \tilde{q}_n/\tilde{w} \end{bmatrix}$ satisfies (55),

$$\tilde{H}'_n = \left(\tilde{\mathcal{B}}_n - \frac{\tilde{C}}{2} I \right) \tilde{Y}_n, \quad \tilde{\mathcal{B}}_n = \begin{bmatrix} -x & n+1 \\ -2 & x \end{bmatrix}, \quad \tilde{C} = -2x, \quad n \geq 1,$$

then, taking into account the Lemma 2,

$$\mathcal{P}_n = e^{-x^2/2} \tilde{H}_{n-1} \quad (65)$$

satisfies $\mathcal{P}'_n = \tilde{\mathcal{B}}_{n-1} \mathcal{P}_n$, $n \geq 2$, that is, $\mathcal{P}'_n = \begin{bmatrix} -x & n \\ -2 & x \end{bmatrix} \mathcal{P}_n$, $n \geq 2$. Hence, \mathcal{P}_n defined by (65) is a fundamental matrix of the differential system in (63). Thus, a solution of the initial value problem (63) is given by (64). ■

Remark 5. Taking into account (54) and (56), one has $P_n^{(1)} = H_n$, $n \geq 1$. Consequently, the matrix \mathcal{K}_n given in (64) is such that

$$\mathcal{K}_n^{(1,2)} = e^{x_0^2/2}, \quad \mathcal{K}_n^{(2,2)} = 0. \quad (66)$$

Furthermore, using $Y_{n+1}(x_0) = \mathcal{A}_{n+1}(x_0) Y_n(x_0)$ (cf. (11)), we obtain

$$\tilde{H}_{n-1}(x_0) \mathcal{K}_n = \mathcal{A}_{n+1}^{-1}(x_0) \tilde{H}_n(x_0) \mathcal{K}_{n+1}, \quad n \geq 2,$$

from which we get

$$\mathcal{K}_n^{(1,1)} = \mathcal{K}_{n+1}^{(1,1)}, \quad \mathcal{K}_n^{(2,1)} = \mathcal{K}_{n+1}^{(2,1)}, \quad n \geq 2. \quad (67)$$

Theorem 6. *Let the conditions and the notations of the two previous Lemmas hold. Then,*

(a) *the following representation holds, for all $n \geq 2$:*

$$P_{n+1}(x) = e^{-x^2} \left(e^{x_0^2/2} \mathcal{K}_n^{(1,1)} - \int_{x_0}^x B(s) e^{s^2} ds \right) H_n(x) + e^{x_0^2/2} \mathcal{K}_n^{(2,1)} \tilde{q}_n(x), \quad (68)$$

where x_0 is an arbitrary complex number;

(b) *the Stieltjes functions S and \tilde{S} are related through*

$$\tilde{S} = \frac{a + bS}{c + dS}, \quad (69)$$

where

$$a = -1/d, \quad b(x) = 1/dx - \lambda/d, \quad c = 0, \quad d = \pm\sqrt{\rho/2}. \quad (70)$$

Proof: (a). Taking into account the Theorem 3, one has $Y_n = \mathcal{P}_n \mathcal{L}^{-1}$, where $\mathcal{L}, \mathcal{P}_n$, the solutions of the corresponding initial value problems (58) and (63), are given by (59)-(62) and (64), respectively. Therefore, $Y_n = \mathcal{P}_n \mathcal{L}^{-1}$ yields, in the position (1, 1), (68), where we used (66). The position (2, 1) gives us (68) for $n - 1$, where we used (67). The positions (1, 2) and (2, 2) of $Y_n = \mathcal{P}_n \mathcal{L}^{-1}$ yield $P_n^{(1)} = H_n$ and $P_{n-1}^{(1)} = H_{n-1}$, respectively.

(b). Note that \tilde{S} satisfies (53),

$$A\tilde{S}' = \tilde{C}\tilde{S} + \tilde{D}, \quad A = 1, \quad \tilde{C} = -2x, \quad \tilde{D} = 2.$$

Taking into account the Lemma 3, there holds a relation of the type (69), where the polynomials $a - d$ and the polynomials A, \tilde{C}, \tilde{D} are related through the equations (31)-(33). Thus, (70) follows. \blacksquare

References

- [1] H. Bouakkaz and P. Maroni, *Description des polynomes orthogonaux de Laguerre-Hahn de classe zero*, in "Orthogonal Polynomials and their applications", (C. Brezinski, L. Gori and A. Ronveaux Eds.) J. C. Baltzer A. G. Basel IMACS Annals on Computing and Applied Mathematics, 1991, pp. 189-194.
- [2] A. Branquinho and M.N. Rebocho, *On the semiclassical character of orthogonal polynomials satisfying structure relations*, J. Difference Equ. Appl. 18 (1) (2012) 111-138.
- [3] A. Branquinho, A. Foulquié Moreno, A. Paiva, and M.N. Rebocho, *Second order differential equations in the Laguerre-Hahn class* (submitted).
- [4] Y. Chen and M. Ismail, *Jacobi polynomials from compatibility conditions*, Proc. Amer. Math. Soc. 133 (2004) 465-472.

- [5] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [6] J. S. Dehesa, F. Marcellán, A. Ronveaux, *On orthogonal polynomials with perturbed recurrence relations*, J. Comput. Appl. Math. 30 (1990) 203-212.
- [7] J. Dini, *Sur les formes linéaires et les polynômes orthogonaux de Laguerre-Hahn*, Thèse de doctorat, Univ. Pierre et Marie Curie, Paris, 1988 (in French).
- [8] W. Hahn, *On Differential Equations for Orthogonal Polynomials*, Funkcialaj Ek., 21 (1978) 1-9.
- [9] M.E. Ismail, J. Letessier, and G. Valent, *Linear birth and death models and associated Laguerre polynomials*, J. Approx. Theory, 56 (1988) 337-348.
- [10] G. Jank, *Matrix Riccati Differential Equations*, (A.P.Santana, J.S. Neves e M.P.Oliveira eds.), Textos de Matemática, no. 36, DMUC, 2005.
- [11] A.P. Magnus, *Riccati acceleration of the Jacobi continued fractions and Laguerre-Hahn polynomials*, in “Padé Approximation and its Applications, Proc., Bad Honnef 1883”, Lect. Notes in Math. 1071 (H. Werner e H. T. Bunge, eds.), Springer Verlag, Berlin, 1984, pp. 213-230.
- [12] A. Magnus, *On Freud’s equations for exponential weights*, J. Approx. Theory 46 (1986) 65-99.
- [13] A. Magnus, *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Comp. Appl. Math. 57 (1995) 215-237.
- [14] F. Marcellán, A. Ronveaux, *Co-recursive orthogonal polynomials and fourth order differential equation*, J. Comp. Appl. Math. 25 (1) (1989) 105-109.
- [15] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, in “Orthogonal Polynomials and their applications”, (C. Brezinski, L. Gori and A. Ronveaux Eds.) J. C. Baltzer A. G. Basel IMACS Annals on Computing and Applied Mathematics, 1991, pp. 95-130.
- [16] J. Radon, *Zum Problem von Lagrange*, Hamburger Math. Einzelschr., 6, 1928.
- [17] A. Ronveaux, *Fourth-order differential equations for the numerator polynomials*, J. Phys. A.: Math. Gen. 21 (1988), 749-753.
- [18] G. Szegő, *Orthogonal Polynomials*, fourth ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Providence Rhode Island, 1975.

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