# Anticipated backward doubly stochastic differential equations 

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#### Abstract

In this paper, we deal with a new type of differential equations called anticipated backward doubly stochastic differential equations (anticipated BDSDEs). The coefficients of these BDSDEs depend on the future value of the solution $(Y, Z)$. We obtain the existence and uniqueness theorem and a comparison theorem for the solutions of these equations. Besides, as an application, we also establish a duality between the anticipated BDSDEs and the delayed doubly stochastic differential equations (delayed DSDEs).


Keywords: anticipated backward doubly stochastic differential equation, comparison theorem, duality

## 1 Introduction

Backward stochastic differential equation (BSDE) was considered the general form the first time by Pardoux-Peng [10] in 1990. In the last twenty years, the theory of BSDEs has been studied with great interest due to its applications in the pricing/hedging problem (see e.g. [4, 5]), in the stochastic control and game theory (see e.g. [5, [6]), and in the theory of partial differential equations (see e.g. [2, 3, 11]).

In order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs), Pardoux-Peng [12] first studied the backward doubly stochastic differential equations (BDSDEs) of the general form

$$
\begin{equation*}
Y_{t}=\xi+\int_{t}^{T} f\left(s, Y_{s}, Z_{s}\right) d s+\int_{t}^{T} g\left(s, Y_{s}, Z_{s}\right) d \overleftarrow{B}_{s}-\int_{t}^{T} Z_{s} d W_{s}, \quad t \in[0, T] \tag{1.1}
\end{equation*}
$$

[^0]where the integral with respect to $\left\{B_{t}\right\}$ is a "backward Itô integral", and the integral with respect to $\left\{W_{t}\right\}$ is a standard forward integral. Note that these two types of integrals are particular cases of the Itô-Skorohod integral, see NualartPardoux [9]. Pardoux-Peng [12] proved that under Lipschitz condition on the coefficients, BDSDE (1.1) has a unique solution. Since then, the theory of BDSDEs has been developed rapidly by many researchers. Bally-Matoussi [1 gave the probabilistic representation of the solutions in Sobolev space of semilinear SPDEs in terms of BDSDEs. Matoussi-Scheutzow 8] studied BDSDEs and their applications in SPDEs. Shi et al. [14] proved a comparison theorem for BDSDEs with Lipschitz condition on the coefficients. Lin [7] obtained a generalized comparison theorem and a generalized existence theorem of BDSDEs.

On the other hand, recently, Peng-Yang [13] (see also [16]) introduced the socalled anticipated BSDEs (ABSDEs) of the following form:

$$
\left\{\begin{aligned}
-d Y_{t} & =f\left(t, Y_{t}, Z_{t}, Y_{t+\delta(t)}, Z_{t+\zeta(t)}\right) d t-Z_{t} d W_{t}, & & t \in[0, T] \\
Y_{t} & =\xi_{t}, & & t \in[T, T+K] \\
Z_{t} & =\eta_{t}, & & t \in[T, T+K]
\end{aligned}\right.
$$

where $\delta(\cdot):[0, T] \rightarrow \mathbb{R}^{+} \backslash\{0\}$ and $\zeta(\cdot):[0, T] \rightarrow \mathbb{R}^{+} \backslash\{0\}$ are continuous functions satisfying
(a1) there exists a constant $K \geq 0$ such that for each $t \in[0, T]$,

$$
t+\delta(t) \leq T+K, \quad t+\zeta(t) \leq T+K
$$

(a2) there exists a constant $M \geq 0$ such that for each $t \in[0, T]$ and each nonnegative integrable function $g(\cdot)$,

$$
\int_{t}^{T} g(s+\delta(s)) d s \leq M \int_{t}^{T+K} g(s) d s, \quad \int_{t}^{T} g(s+\zeta(s)) d s \leq M \int_{t}^{T+K} g(s) d s
$$

Peng-Yang [13] proved the existence and uniqueness of the solution to the above equation, and studied the duality between anticipated BSDEs and delayed SDEs.

In this paper, we are interested in the following BDSDEs with coefficients depending on the future value of the solution $(Y, Z)$ :

$$
\left\{\begin{array}{rlrl}
-d Y_{t}= & f\left(t, Y_{t}, Z_{t}, Y_{t+\delta(t)}, Z_{t+\zeta(t)}\right) d t &  \tag{1.2}\\
& +g\left(t, Y_{t}, Z_{t}, Y_{t+\delta(t)}, Z_{t+\zeta(t)}\right) d \overleftarrow{B}_{t}-Z_{t} d W_{t},, & & t \in[0, T] \\
Y_{t}= & \xi_{t}, & & t \in[T, T+K] \\
Z_{t}= & \eta_{t}, & & t \in[T, T+K]
\end{array}\right.
$$

where $\delta>0$ and $\zeta>0$ satisfy (a1)-(a2).

We prove that under proper assumptions, the solution of the above anticipated BDSDE (ABDSDE) exists uniquely, and a comparison theorem is given for the 1-dimensional anticipated BDSDEs. It may be mentioned here that, to deal with (1.2), the most important thing for us is to establish the similar conclusions as in [12] and [14] for BDSDE (1.1) with $\xi$ belonging to a larger space. Besides, as an application, we study a duality between the anticipated BDSDE and delayed DSDE.

The paper is organized as follows: in Section 2, we make some preliminaries. In Section 3, we mainly study the existence and uniqueness of the solutions of anticipated BDSDEs, and in Section 4, a comparison result is given. As an application, in Section 5, we establish a duality between an anticipated BDSDE and a delayed DSDE. Finally in Section 6, the conclusion and future work are presented.

## 2 Preliminaries

Let $T>0$ be fixed throughout this paper. Let $\left\{W_{t}\right\}_{t \in[0, T]}$ and $\left\{B_{t}\right\}_{t \in[0, T]}$ be two mutually independent standard Brownian motion processes, with values respectively in $\mathbb{R}^{d}$ and $\mathbb{R}^{l}$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of $P$-null sets of $\mathcal{F}$. We define

$$
\mathcal{F}_{t}:=\mathcal{F}_{0, t}^{W} \vee \mathcal{F}_{t, T}^{B}, t \in[0, T] ; \quad \mathcal{G}_{s}:=\mathcal{F}_{0, s}^{W} \vee \mathcal{F}_{s, T+K}^{B}, s \in[0, T+K]
$$

where for any processes $\left\{\varphi_{t}\right\}, \mathcal{F}_{s, t}^{\varphi}=\sigma\left\{\varphi_{r}-\varphi_{s}, s \leq r \leq t\right\} \vee \mathcal{N}$. We will use the following notations:

- $L^{2}\left(\mathcal{G}_{T} ; \mathbb{R}^{m}\right):=\left\{\xi \in \mathbb{R}^{m} \mid \xi\right.$ is a $\mathcal{G}_{T}$-measurable random variable such that $\left.E|\xi|^{2}<+\infty\right\} ;$
- $L_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m}\right):=\left\{\varphi: \Omega \times[0, T] \rightarrow \mathbb{R}^{m} \mid \varphi\right.$ is a $\mathcal{G}_{t}$-progressively measurable process such that $\left.E \int_{0}^{T}\left|\varphi_{t}\right|^{2} d t<+\infty\right\}$;
- $S_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m}\right):=\left\{\varphi: \Omega \times[0, T] \rightarrow \mathbb{R}^{m} \mid \varphi\right.$ is a continuous and $\mathcal{G}_{t}$-progressively measurable process such that $\left.E\left[\sup _{0 \leq t \leq T}\left|\varphi_{t}\right|^{2}\right]<+\infty\right\}$.

Remark 2.1 It should be mentioned here that, the existing result about BDS$D E s$ are established almost under the condition that the terminal value $\xi$ is $\mathcal{F}_{T^{-}}$ measurable (see [12], [14], etc.). In this paper, we will first treat the case when $\xi$ is $\mathcal{G}_{T}$-measurable.

For each $t \in[0, T]$, let
$f(t, \cdot, \cdot, \cdot, \cdot): \Omega \times[0, T] \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d} \times L_{\mathcal{G}}^{2}\left(t, T+K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(t, T+K ; \mathbb{R}^{m \times d}\right) \rightarrow L^{2}\left(\mathcal{G}_{t} ; \mathbb{R}^{m}\right)$,
$g(t, \cdot, \cdot, \cdot, \cdot): \Omega \times[0, T] \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d} \times L_{\mathcal{G}}^{2}\left(t, T+K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(t, T+K ; \mathbb{R}^{m \times d}\right) \rightarrow L^{2}\left(\mathcal{G}_{t} ; \mathbb{R}^{m \times l}\right)$.
We make the following hypotheses:
(H1) There exists a constant $c>0$ such that for any $r, \bar{r} \in[t, T+K]$, $(t, y, z, \theta, \phi),\left(t, y^{\prime}, z^{\prime}, \theta^{\prime}, \phi^{\prime}\right) \in[0, T] \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d} \times L_{\mathcal{G}}^{2}\left(t, T+K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}(t, T+$ $\left.K ; \mathbb{R}^{m \times d}\right)$,
$\left|f\left(t, y, z, \theta_{r}, \phi_{\bar{r}}\right)-f\left(t, y^{\prime}, z^{\prime}, \theta_{r}^{\prime}, \phi_{\bar{r}}^{\prime}\right)\right|^{2} \leq c\left(\left|y-y^{\prime}\right|^{2}+\left|z-z^{\prime}\right|^{2}+E^{\mathcal{F}_{t}}\left[\left|\theta_{r}-\theta_{r}^{\prime}\right|^{2}+\left|\phi_{\bar{r}}-\phi_{\bar{r}}^{\prime}\right|^{2}\right]\right)$.
(H2) $E\left[\int_{0}^{T}|f(s, 0,0,0,0)|^{2} d s\right]<+\infty$.
(H3) There exist constants $c>0,0<\alpha_{1}<1,0 \leq \alpha_{2}<\frac{1}{M}$, satisfying $0<\alpha_{1}+\alpha_{2} M<1$, such that for any $r, \bar{r} \in[t, T+K],(t, y, z, \theta, \phi),\left(t, y^{\prime}, z^{\prime}, \theta^{\prime}, \phi^{\prime}\right) \in$ $[0, T] \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d} \times L_{\mathcal{G}}^{2}\left(t, T+K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(t, T+K ; \mathbb{R}^{m \times d}\right)$,
$\left|g\left(t, y, z, \theta_{r}, \phi_{\bar{r}}\right)-g\left(t, y^{\prime}, z^{\prime}, \theta_{r}^{\prime}, \phi_{\bar{r}}^{\prime}\right)\right|^{2} \leq c\left(\left|y-y^{\prime}\right|^{2}+E^{\mathcal{F}_{t}}\left|\theta_{r}-\theta_{r}^{\prime}\right|^{2}\right)+\alpha_{1}\left|z-z^{\prime}\right|^{2}+\alpha_{2} E^{\mathcal{F}_{t}}\left|\phi_{\bar{r}}-\phi_{\bar{r}}^{\prime}\right|^{2}$.
$(\mathbf{H} 4) E\left[\int_{0}^{T}|g(s, y, z, \theta, \phi)|^{2} d s\right]<+\infty$, for any $(y, z, \theta, \phi)$.

## 3 Existence and uniqueness theorem

In this section, we will mainly study the existence and uniqueness of the solution to anticipated BDSDE (1.2). For this purpose, we first consider a simple case when the coefficients $f$ and $g$ do not depend on the value or the future value of $(Y, Z)$ :

$$
\begin{equation*}
Y_{t}=\xi_{T}+\int_{t}^{T} f(s) d s+\int_{t}^{T} g(s) d \overleftarrow{B}_{s}-\int_{t}^{T} Z_{s} d W_{s}, \quad t \in[0, T] \tag{3.1}
\end{equation*}
$$

where $f \in L_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m}\right), g \in L_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m \times l}\right)$ and $\xi_{T} \in L^{2}\left(\mathcal{G}_{T} ; \mathbb{R}^{m}\right)$.
Theorem 3.1 Given $\xi_{T} \in L^{2}\left(\mathcal{G}_{T} ; \mathbb{R}^{m}\right)$, BDSDE (3.1) has a unique solution $(Y, Z) \in L_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m \times d}\right)$.

Proof. To prove the existence, we define a filtration by

$$
\mathcal{H}_{t}:=\mathcal{F}_{0, t}^{W} \vee \mathcal{F}_{0, T+K}^{B}, t \in[0, T+K]
$$

and a $\mathcal{H}_{t}$-square integrable martingale

$$
M_{t}:=E^{\mathcal{H}_{t}}\left[\xi_{T}+\int_{0}^{T} f(s) d s+\int_{0}^{T} g(s) d \overleftarrow{B}_{s}\right], \quad t \in[0, T]
$$

Thanks to Itô's martingale representation theorem, there exists a process $Z \in$ $L_{\mathcal{H}}^{2}\left(0, T ; \mathbb{R}^{m \times d}\right)$ such that

$$
M_{t}=M_{0}+\int_{0}^{t} Z_{s} d W_{s}, \quad t \in[0, T]
$$

which implies

$$
M_{t}=M_{T}-\int_{t}^{T} Z_{s} d W_{s}, \quad t \in[0, T] .
$$

Hence
$E^{\mathcal{H}_{t}}\left[\xi_{T}+\int_{0}^{T} f(s) d s+\int_{0}^{T} g(s) d \overleftarrow{B}_{s}\right]=\xi_{T}+\int_{0}^{T} f(s) d s+\int_{0}^{T} g(s) d \overleftarrow{B}_{s}-\int_{t}^{T} Z_{s} d W_{s}$
Subtract $\int_{0}^{t} f(s) d s+\int_{0}^{t} g(s) d \overleftarrow{B}_{s}$ from both sides, then we have

$$
Y_{t}=\xi_{T}+\int_{t}^{T} f(s) d s+\int_{t}^{T} g(s) d \overleftarrow{B}_{s}-\int_{t}^{T} Z_{s} d W_{s}
$$

where

$$
Y_{t}:=E^{\mathcal{H}_{t}}\left[\xi_{T}+\int_{t}^{T} f(s) d s+\int_{t}^{T} g(s) d \overleftarrow{B}_{s}\right]
$$

Next we show that $(Y, Z)$ are in fact $\mathcal{G}_{t}$-adapted. In fact, it is obvious that

$$
Y_{t}=E\left[\Theta \mid \mathcal{G}_{t} \vee \mathcal{F}_{0, t}^{B}\right]
$$

where $\Theta:=\xi_{T}+\int_{t}^{T} f(s) d s+\int_{t}^{T} g(s) d \overleftarrow{B}_{s}$ is $\mathcal{F}_{0, T}^{W} \vee \mathcal{F}_{t, T+K}^{B}$ measurable. Note that $\mathcal{F}_{0, t}^{B}$ is independent of $\mathcal{G}_{t} \vee \sigma(\Theta)$, then we know

$$
Y_{t}=E^{\mathcal{G}_{t}}[\Theta]
$$

Now

$$
\int_{t}^{T} Z_{s} d W_{s}=\xi_{T}+\int_{t}^{T} f(s) d s+\int_{t}^{T} g(s) d \overleftarrow{B}_{s}-Y_{t}
$$

and the right side is $\mathcal{F}_{0, T}^{W} \vee \mathcal{F}_{t, T+K}^{B}$ measurable. Then from Itô's martingale representation theorem, $\left(Z_{s}\right)_{s \in[t, T]}$ is $\mathcal{F}_{0, s}^{W} \vee \mathcal{F}_{t, T+K}^{B}$ adapted, which implies $Z_{s}$ is $\mathcal{F}_{0, s}^{W} \vee \mathcal{F}_{t, T+K}^{B}$ measurable for any $t \leq s$. Thus $Z_{s}$ is $\mathcal{F}_{0, s}^{W} \vee \mathcal{F}_{s, T+K}^{B}$ measurable.

To show the uniqueness. We suppose that $(\bar{Y}, \bar{Z})$ is the difference of two solutions. Then

$$
\bar{Y}_{t}+\int_{t}^{T} \bar{Z}_{s} d W_{s}=0, \quad t \in[0, T] .
$$

Hence

$$
E\left|\bar{Y}_{t}\right|^{2}+E \int_{t}^{T}\left|\bar{Z}_{s}\right|^{2} d s=0
$$

which implies $\bar{Y}_{t} \equiv 0$, a.s. and $\bar{Z}_{t} \equiv 0$ a.s., a.e..
Now we establish the main result of this part.
Theorem 3.2 Assume that (a1)-(a2) and (H1)-(H4) hold. Then for given $(\xi, \eta) \in$ $S_{\mathcal{G}}^{2}\left(T, T+K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(T, T+K ; \mathbb{R}^{m \times d}\right)$, the anticipated BDSDE (1.2) has a unique solution $(Y, Z) \in S_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{m \times d}\right)$.

Proof. Denote by $\mathcal{S}$ the space of $(Y, Z) \in L_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{m \times d}\right)$ such that $\left(Y_{t}, Z_{t}\right)_{t \in[T, T+K]}=\left(\xi_{t}, \eta_{t}\right)_{t \in[T, T+K]}$. Given $(y, z) \in \mathcal{S}$, we consider the following equation:

$$
\left\{\begin{array}{rlrl}
-d Y_{t}= & f\left(t, y_{t}, z_{t}, y_{t+\delta(t)}, z_{t+\zeta(t)}\right) d t &  \tag{3.2}\\
& +g\left(t, y_{t}, z_{t}, y_{t+\delta(t)}, z_{t+\zeta(t)}\right) d \overleftarrow{B}_{t}-Z_{t} d W_{t}, & & t \in[0, T] ; \\
Y_{t}= & \xi_{t}, & & t \in[T, T+K] ; \\
Z_{t}= & \eta_{t}, & & t \in[T, T+K]
\end{array}\right.
$$

It is obvious that the above equation is equivalent to the BDSDE

$$
\left\{\begin{aligned}
-d \tilde{Y}_{t}= & f\left(t, y_{t}, z_{t}, y_{t+\delta(t)}, z_{t+\zeta(t)}\right) d t \\
& +g\left(t, y_{t}, z_{t}, y_{t+\delta(t)}, z_{t+\zeta(t)}\right) d \overleftarrow{B}_{t}-\tilde{Z}_{t} d W_{t}, \quad t \in[0, T] \\
\tilde{Y}_{T}= & \xi_{T} \in \mathcal{G}_{T}
\end{aligned}\right.
$$

which admits a unique solution in the space $S_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{m \times d}\right)$ according to Theorem 3.1. Thus BDSDE (3.2) has a unique solution in $\mathcal{S}$. Define a mapping $I$ from $\mathcal{S}$ into itself by $(Y, Z)=I(y, z)$, then $(Y, Z)$ is the unique solution of BDSDE (3.2).

Let $\left(y^{\prime}, z^{\prime}\right)$ be another element of $\mathcal{S}$, and $\left(Y^{\prime}, Z^{\prime}\right)=I\left(y^{\prime}, z^{\prime}\right)$. We make the following notations:

$$
\begin{aligned}
& \bar{y}=y-y^{\prime}, \bar{z}=z-z^{\prime}, \bar{Y}=Y-Y^{\prime}, \bar{Z}=Z-Z^{\prime}, \\
& \bar{f}_{t}=f\left(t, y_{t}, z_{t}, y_{t+\delta(t)}, z_{t+\zeta(t)}\right)-f\left(t, y_{t}^{\prime}, z_{t}^{\prime}, y_{t+\delta(t)}^{\prime}, z_{t+\zeta(t)}^{\prime}\right), \\
& \bar{g}_{t}=g\left(t, y_{t}, z_{t}, y_{t+\delta(t)}, z_{t+\zeta(t)}\right)-g\left(t, y_{t}^{\prime}, z_{t}^{\prime}, y_{t+\delta(t)}^{\prime}, z_{t+\zeta(t)}^{\prime}\right) .
\end{aligned}
$$

For any $\beta>0$, apply Itô's formula to $e^{\beta t}\left|\bar{Y}_{t}\right|^{2}$,

$$
\begin{aligned}
& e^{\beta t}\left|\bar{Y}_{t}\right|^{2}+\int_{t}^{T} e^{\beta s}\left[\beta\left|\bar{Y}_{s}\right|^{2}+\left|\bar{Z}_{s}\right|^{2}\right] d s \\
& =2 \int_{t}^{T} e^{\beta s} \bar{Y}_{s} \bar{f}_{s} d s+\int_{t}^{T} e^{\beta s}\left|\bar{g}_{s}\right|^{2} d s+2 \int_{t}^{T} e^{\beta s} \bar{Y}_{s} \bar{g}_{s} d \overleftarrow{B}_{s}-2 \int_{t}^{T} e^{\beta s} \bar{Y}_{s} \bar{Z}_{s} d W_{s}
\end{aligned}
$$

Take mathematical expectation on both sides, then we have

$$
e^{\beta t} E\left|\bar{Y}_{t}\right|^{2}+E \int_{t}^{T} e^{\beta s}\left[\beta\left|\bar{Y}_{s}\right|^{2}+\left|\bar{Z}_{s}\right|^{2}\right] d s=2 E \int_{t}^{T} e^{\beta s} \bar{Y}_{s} \bar{f}_{s} d s+E \int_{t}^{T} e^{\beta s}\left|\bar{g}_{s}\right|^{2} d s
$$

Hence from $(A 1),(A 2)$ and the inequality $2 a b \leq \lambda a^{2}+\frac{1}{\lambda} b^{2}$,

$$
\begin{aligned}
& e^{\beta t} E\left|\bar{Y}_{t}\right|^{2}+E \int_{t}^{T} e^{\beta s}\left[\beta\left|\bar{Y}_{s}\right|^{2}+\left|\bar{Z}_{s}\right|^{2}\right] d s \\
& \leq E \int_{t}^{T} e^{\beta s}\left[\lambda\left|\bar{Y}_{s}\right|^{2}+\frac{1}{\lambda}\left|\bar{f}_{s}\right|^{2}\right] d s+E \int_{t}^{T} e^{\beta s}\left|\bar{g}_{s}\right|^{2} d s \\
& \leq E \int_{t}^{T} e^{\beta s}\left[\lambda\left|\bar{Y}_{s}\right|^{2}+\left(\frac{c}{\lambda}+c\right)\left(\left|\bar{y}_{s}\right|^{2}+\left|\bar{y}_{s+\delta(s)}\right|^{2}\right)+\left(\frac{c}{\lambda}+\alpha_{1}\right)\left|\bar{z}_{s}\right|^{2}+\left(\frac{c}{\lambda}+\alpha_{2}\right)\left|\bar{z}_{s+\zeta(s)}\right|^{2}\right] d s \\
& \leq E \int_{t}^{T+K} e^{\beta s}\left[\lambda\left|\bar{Y}_{s}\right|^{2}+\left(\frac{c}{\lambda}+c\right)(1+M)\left|\bar{y}_{s}\right|^{2}+\left(\frac{c}{\lambda}(1+M)+\alpha_{1}+\alpha_{2} M\right)\left|\bar{z}_{s}\right|^{2}\right] d s,
\end{aligned}
$$

which implies

$$
\begin{aligned}
& E \int_{t}^{T+K} e^{\beta s}\left[(\beta-\lambda)\left|\bar{Y}_{s}\right|^{2}+\left|\bar{Z}_{s}\right|^{2}\right] d s \\
& \leq E \int_{t}^{T+K} e^{\beta s}\left[\left(\frac{c}{\lambda}+c\right)(1+M)\left|\bar{y}_{s}\right|^{2}+\left(\frac{c}{\lambda}(1+M)+\alpha_{1}+\alpha_{2} M\right)\left|\bar{z}_{s}\right|^{2}\right] d s \\
& =\left(\frac{c}{\lambda}(1+M)+\alpha_{1}+\alpha_{2} M\right) E \int_{t}^{T+K} e^{\beta s}\left[\frac{c(1+\lambda)(1+M)}{c(1+M)+\lambda\left(\alpha_{1}+\alpha_{2} M\right)}\left|\bar{y}_{s}\right|^{2}+\left|\bar{z}_{s}\right|^{2}\right] d s
\end{aligned}
$$

Hence if we choose $\lambda=\lambda_{0}$ satisfying $\bar{c}:=\frac{c}{\lambda_{0}}(1+M)+\alpha_{1}+\alpha_{2} M<1$, choose $\beta=\lambda_{0}+\frac{c\left(1+\lambda_{0}\right)(1+M)}{c(1+M)+\lambda_{0}\left(\alpha_{1}+\alpha_{2} M\right)}$, and denote $\gamma:=\frac{c\left(1+\lambda_{0}\right)(1+M)}{c(1+M)+\lambda_{0}\left(\alpha_{1}+\alpha_{2} M\right)}$, then we deduce

$$
E \int_{t}^{T+K} e^{\beta s}\left[\gamma\left|\bar{Y}_{s}\right|^{2}+\left|\bar{Z}_{s}\right|^{2}\right] d s \leq \bar{c} E \int_{t}^{T+K} e^{\beta s}\left[\gamma\left|\bar{y}_{s}\right|^{2}+\left|\bar{z}_{s}\right|^{2}\right] d s
$$

Thus $I$ is a strict contraction on $\mathcal{S}$ and it has a unique fixed point $(Y, Z) \in \mathcal{S}$. Now due to Burkholder-Davis-Gundy inequality, it is easy to check that $Y \in$ $S_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{m}\right)$. The proof is complete.

Remark 3.3 In the proof of Theorem 3.2, we use the norm

$$
|(Y, Z)|_{(\beta, \gamma)} \equiv\left\{E \int_{0}^{T+K} e^{\beta_{s}}\left(\gamma\left|Y_{s}\right|^{2}+\left|Z_{s}\right|^{2}\right) d s\right\}^{\frac{1}{2}}
$$

which is very convenient for us to establish a strict contraction mapping. In fact, it is obvious that this new norm is equivalent to both norms $|(Y, Z)|_{(\beta, 1)}$ and $|(Y, Z)|_{(0,1)}$, and the latter is just the general norm defined on the space $L_{\mathcal{G}}^{2}(0, T+$ $\left.K ; \mathbb{R}^{m}\right) \times L_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{m \times d}\right)$.

## 4 Comparison theorem

In this part, we are concerned with the following 1-dimensional anticipated BDSDEs:

$$
\left\{\begin{aligned}
-d Y_{t}^{j} & =f^{j}\left(t, Y_{t}^{j}, Z_{t}^{j}, Y_{t+\delta(t)}^{j}, Z_{t+\zeta(t)}^{j}\right) d t+g\left(t, Y_{t}^{j}, Z_{t}^{j}\right) d \overleftarrow{B}_{t}-Z_{t}^{j} d W_{t}, & & t \in[0, T] ; \\
Y_{t}^{j} & =\xi_{t}^{j}, & & t \in[T, T+K] \\
Z_{t}^{j} & =\eta_{t}^{j}, & & t \in[T, T+K]
\end{aligned}\right.
$$

where $j=1,2$, and $(a 1)-(a 2),(H 1)-(H 4)$ hold. Then by Theorem 3.2, (4.1) has a unique solution.

Our objective is to obtain a comparison result for these two equations. For this purpose, we first consider a simple case when the coefficients $f^{j}$ and $g$ do not depend on the future value of $\left(Y^{j}, Z^{j}\right)$ :

$$
\begin{equation*}
Y_{t}^{j}=\xi_{T}^{j}+\int_{t}^{T} f^{j}\left(s, Y_{s}^{j}, Z_{s}^{j}\right) d s+\int_{t}^{T} g\left(s, Y_{s}^{j}, Z_{s}^{j}\right) d \overleftarrow{B}_{s}-\int_{t}^{T} Z_{s}^{j} d W_{s}, \quad t \in[0, T] \tag{4.2}
\end{equation*}
$$

Theorem 4.1 Let $\left(Y^{j}, Z^{j}\right) \in S_{\mathcal{G}}^{2}(0, T ; \mathbb{R}) \times L_{\mathcal{G}}^{2}\left(0, T ; \mathbb{R}^{d}\right)(j=1,2)$ be the unique solutions to BDSDEs (4.2) respectively. If $\xi_{T}^{1} \geq \xi_{T}^{2}$, a.s., and for any $(t, y, z) \in$ $[0, T] \times \mathbb{R} \times \mathbb{R}^{d}, f^{1}(t, y, z) \geq f^{2}(t, y, z)$, a.s., then $Y_{t}^{1} \geq Y_{t}^{2}$, a.s., for all $t \in[0, T]$.

Proof. Denote

$$
\bar{Y}_{t}:=Y_{t}^{2}-Y_{t}^{1}, \quad \bar{Z}_{t}:=Z_{t}^{2}-Z_{t}^{1}, \quad \bar{\xi}_{T}:=\xi_{T}^{2}-\xi_{T}^{1}
$$

then $(\bar{Y}, \bar{Z})$ satisfies

$$
\bar{Y}_{t}=\bar{\xi}_{T}+\int_{t}^{T}\left[f^{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f^{1}\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right] d s+\int_{t}^{T}\left[g\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-g\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right] d \overleftarrow{B}_{s}-\int_{t}^{T} \bar{Z}_{s} d W_{s}
$$

Applying Itô's formula to $\left|\bar{Y}_{t}^{+}\right|^{2}$, we have

$$
\begin{aligned}
\left|\bar{Y}_{t}^{+}\right|^{2}= & \left|\bar{\xi}_{T}^{+}\right|^{2}+2 \int_{t}^{T} \bar{Y}_{s}^{+}\left[f^{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f^{1}\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right] d s \\
& +2 \int_{t}^{T} \bar{Y}_{s}^{+}\left[g\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-g\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right] d \overleftarrow{B}_{s}-2 \int_{t}^{T} \bar{Y}_{s}^{+} \bar{Z}_{s} d W_{s} \\
& -\int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left|\bar{Z}_{s}\right|^{2} d s+\int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left|g\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-g\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right|^{2} d s .
\end{aligned}
$$

Taking expectation on both sides and noting that $\xi_{T}^{1} \geq \xi_{T}^{2}$, we get

$$
\begin{aligned}
E\left|\bar{Y}_{t}^{+}\right|^{2}+E-\int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left|\bar{Z}_{s}\right|^{2} d s= & 2 E \int_{t}^{T} \bar{Y}_{s}^{+}\left[f^{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f^{1}\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right] d s \\
& +E \int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left|g\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-g\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right|^{2} d s
\end{aligned}
$$

While,

$$
\begin{aligned}
& 2 E \int_{t}^{T} \bar{Y}_{s}^{+}\left[f^{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f^{1}\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right] d s \\
& =2 E \int_{t}^{T} \bar{Y}_{s}^{+}\left[f^{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f^{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)+f^{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f^{1}\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right] d s \\
& \leq 2 E \int_{t}^{T} \bar{Y}_{s}^{+}\left|f^{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f^{1}\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right| d s \leq 2 \sqrt{c} E \int_{t}^{T} \bar{Y}_{s}^{+}\left[\left|\bar{Y}_{s}\right|+\left|\bar{Z}_{s}\right|\right] d s \\
& \leq\left(2 \sqrt{c}+\frac{c}{1-\alpha_{1}}\right) E \int_{t}^{T}\left|\bar{Y}_{s}^{+}\right|^{2} d s+\left(1-\alpha_{1}\right) \int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left|\bar{Z}_{s}\right|^{2} d s
\end{aligned}
$$

and

$$
\begin{aligned}
& E \int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left|g\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-g\left(s, Y_{s}^{1}, Z_{s}^{1}\right)\right|^{2} d s \\
& \leq E \int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left[c\left|\bar{Y}_{s}\right|^{2}+\alpha_{1}\left|\bar{Z}_{s}\right|^{2}\right] d s \\
& \leq c E \int_{t}^{T}\left|\bar{Y}_{s}^{+}\right|^{2} d s+\alpha_{1} \int_{t}^{T} 1_{\left\{Y_{s}^{2}>Y_{s}^{1}\right\}}\left|\bar{Z}_{s}\right|^{2} d s
\end{aligned}
$$

Then, thanks to the above inequalities, we obtain

$$
E\left|\bar{Y}_{t}^{+}\right|^{2} \leq\left(c+2 \sqrt{c}+\frac{c}{1-\alpha_{1}}\right) E \int_{t}^{T}\left|\bar{Y}_{s}^{+}\right|^{2} d s
$$

which implies

$$
E\left|\bar{Y}_{t}^{+}\right|^{2}=0, \quad \text { for all } t \in[0, T] .
$$

Therefore $Y_{t}^{1} \geq Y_{t}^{2}$, a.s., for all $t \in[0, T]$.
From now on, we consider the anticipated BDSDEs (4.1). We give the following result. For the proof, the reader is referred to [15].

Proposition 4.2 Putting $t_{0}=T$, we define by iteration
$t_{i}:=\min \left\{t \in[0, T]: \min \{s+\delta(s), s+\zeta(s)\} \geq t_{i-1}\right.$, for all $\left.s \in[t, T]\right\}, \quad i \geq 1$.
Set $N:=\max \left\{i: t_{i-1}>0\right\}$. Then $N$ is finite, $t_{N}=0$ and

$$
[0, T]=\left[0, t_{N-1}\right] \cup\left[t_{N-1}, t_{N-2}\right] \cup \cdots \cup\left[t_{2}, t_{1}\right] \cup\left[t_{1}, T\right] .
$$

Proposition 4.3 For $j=1,2$, suppose that $\left(Y^{j}, Z^{j}\right)$ is the unique solution to the anticipated $B D S D E$ (4.1). Then for fixed $i \in\{1,2, \ldots, N\}$, over time interval $\left[t_{i}, t_{i-1}\right]$, 4.1) is equivalent to

$$
\left\{\begin{align*}
-d \bar{Y}_{t}^{j} & =f^{j}\left(t, \bar{Y}_{t}^{j}, \bar{Z}_{t}^{j}, \bar{Y}_{t+\delta(t)}^{j}, \bar{Z}_{t+\zeta(t)}^{j}\right) d t+g\left(t, \bar{Y}_{t}^{j}, \bar{Z}_{t}^{j}\right) d \overleftarrow{B}_{t}-\bar{Z}_{t}^{j} d W_{t}, & & t \in\left[t_{i}, t_{i-1}\right]  \tag{4.3}\\
\bar{Y}_{t}^{j} & =Y_{t}^{j}, & & t \in\left[t_{i-1}, T+K\right] \\
\bar{Z}_{t}^{j} & =Z_{t}^{j}, & & t \in\left[t_{i-1}, T+K\right]
\end{align*}\right.
$$

which is also equivalent to the following BDSDE with terminal condition $Y_{t_{i-1}}^{j}$ :
$\tilde{Y}_{t}^{j}=Y_{t_{i-1}}^{j}+\int_{t}^{t_{i-1}} f^{j}\left(s, \tilde{Y}_{s}^{j}, \tilde{Z}_{s}^{j}, Y_{s+\delta(s)}^{j}, Z_{s+\zeta(s)}^{j}\right) d s+\int_{t}^{t_{i-1}} g\left(s, \tilde{Y}_{s}^{j}, \tilde{Z}_{s}^{j}\right) d \overleftarrow{B}_{s}-\int_{t}^{t_{i-1}} \tilde{Z}_{s}^{j} d W_{s}$.
That is to say,

$$
Y_{t}^{j}=\bar{Y}_{t}^{j}=\tilde{Y}_{t}^{j}, Z_{t}^{j}=\bar{Z}_{t}^{j}=\tilde{Z}_{t}^{j}=\frac{d\left\langle\tilde{Y}^{j}, W\right\rangle_{t}}{d t}, t \in\left[t_{i}, t_{i-1}\right], j=1,2
$$

where $\left\langle\tilde{Y}^{j}, W\right\rangle$ is the variation process generated by $\tilde{Y}^{j}$ and the Brownian motion $W$.

The main result of this part is
Theorem 4.4 Let $\left(Y^{j}, Z^{j}\right) \in S_{\mathcal{G}}^{2}(0, T+K ; \mathbb{R}) \times L_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{d}\right)(j=1,2)$ be the unique solutions to anticipated BDSDEs (4.1) respectively. If
(i) $\xi_{s}^{1} \geq \xi_{s}^{2}, s \in[T, T+K]$, a.e., a.s.;
(ii) for all $t \in[0, T],(y, z) \in \mathbb{R} \times \mathbb{R}^{d}, \theta^{j} \in S_{\mathcal{G}}^{2}(t, T+K ; \mathbb{R})(j=1,2)$ such that $\theta^{1} \geq$ $\theta^{2},\left\{\theta_{r}^{j}\right\}_{r \in[t, T]}$ is a continuous semimartingale and $\left(\theta_{r}^{j}\right)_{r \in[T, T+K]}=\left(\xi_{r}^{j}\right)_{r \in[T, T+K]}$,

$$
\begin{equation*}
f^{1}\left(t, y, z, \theta_{t+\delta(t)}^{1}, \eta_{t+\zeta(t)}^{1}\right) \geq f^{2}\left(t, y, z, \theta_{t+\delta(t)}^{2}, \eta_{t+\zeta(t)}^{2}\right) \text {, a.e., a.s., } \tag{4.5}
\end{equation*}
$$

$f^{1}\left(t, y, z, \theta_{t+\delta(t)}^{1},\left.\frac{d\left\langle\theta^{1}, W\right\rangle_{r}}{d r}\right|_{r=t+\zeta(t)}\right) \geq f^{2}\left(t, y, z, \theta_{t+\delta(t)}^{2},\left.\frac{d\left\langle\theta^{2}, W\right\rangle_{r}}{d r}\right|_{r=t+\zeta(t)}\right)$, a.e., a.s.,
$f^{1}\left(t, y, z, \xi_{t+\delta(t)}^{1},\left.\frac{d\left\langle\theta^{1}, W\right\rangle_{r}}{d r}\right|_{r=t+\zeta(t)}\right) \geq f^{2}\left(t, y, z, \xi_{t+\delta(t)}^{2},\left.\frac{d\left\langle\theta^{2}, W\right\rangle_{r}}{d r}\right|_{r=t+\zeta(t)}\right)$, a.e., a.s.,
then $Y_{t}^{1} \geq Y_{t}^{2}$, a.e., a.s..
Proof. Consider the anticipated BDSDE (4.1) one time interval by one time interval. For the first step, we consider the case when $t \in\left[t_{1}, T\right]$. According to Proposition 4.3, we can equivalently consider

$$
\tilde{Y}_{t}^{j}=\xi_{T}^{j}+\int_{t}^{T} f^{j}\left(s, \tilde{Y}_{s}^{j}, \tilde{Z}_{s}^{j}, \xi_{s+\delta(s)}^{j}, \eta_{s+\zeta(s)}^{j}\right) d s+\int_{t}^{T} g\left(s, \tilde{Y}_{s}^{j}, \tilde{Z}_{s}^{j}\right) d \overleftarrow{B}_{s}-\int_{t}^{T} \tilde{Z}_{s}^{j} d W_{s}
$$

from which we have

$$
\begin{equation*}
Z_{t}^{j}=\tilde{Z}_{t}^{j}=\frac{d\left\langle\tilde{Y}^{j}, W\right\rangle_{t}}{d t}, \quad t \in\left[t_{1}, T\right] \tag{4.8}
\end{equation*}
$$

Noticing that $\xi^{j} \in S_{\mathcal{G}}^{2}(T, T+K ; \mathbb{R})(j=1,2)$ and $\xi^{1} \geq \xi^{2}$, from (4.5) in (ii), we can get, for $s \in\left[t_{1}, T\right], y \in \mathbb{R}, z \in \mathbb{R}^{d}$,

$$
f^{1}\left(s, y, z, \xi_{s+\delta(s)}^{1}, \eta_{s+\zeta(s)}^{1}\right) \geq f^{2}\left(s, y, z, \xi_{s+\delta(s)}^{2}, \eta_{s+\zeta(s)}^{2}\right)
$$

According to Theorem 4.1, we can get

$$
\tilde{Y}_{t}^{1} \geq \tilde{Y}_{t}^{2}, \quad t \in\left[t_{1}, T\right], \quad \text { a.e., a.s. }
$$

which implies

$$
\begin{equation*}
Y_{t}^{(1)} \geq Y_{t}^{(2)}, \quad t \in\left[t_{1}, T+K\right], \quad \text { a.e., a.s.. } \tag{4.9}
\end{equation*}
$$

For the second step, we consider the case when $t \in\left[t_{2}, t_{1}\right]$. Similarly, according to Proposition 4.3, we can consider the following BSDE equivalently: $\tilde{\tilde{Y}}_{t}^{j}=Y_{t_{1}}^{j}+\int_{t}^{t_{1}} f^{j}\left(s, \tilde{\tilde{Y}}_{s}^{j}, \tilde{\tilde{Z}}_{s}^{j}, Y_{s+\delta(s)}^{j}, Z_{s+\zeta(s)}^{j}\right) d s+\int_{t}^{t_{1}} g\left(s, \tilde{\tilde{Y}}_{s}^{j}, \tilde{Z}_{s}^{j}\right) d \overleftarrow{B}_{s}-\int_{t}^{t_{1}} \tilde{\tilde{Z}}_{s}^{j} d W_{s}$,
from which we have $Z_{t}^{j}=\tilde{\tilde{Z}}_{t}^{j}=\frac{d\left\langle\tilde{\tilde{Y}}^{j}, W\right\rangle_{t}}{d t}$ for $t \in\left[t_{2}, t_{1}\right]$. Noticing (4.8) and (4.9), according to (ii), we have, for $s \in\left[t_{2}, t_{1}\right], y \in \mathbb{R}, z \in \mathbb{R}^{d}$,

$$
f^{1}\left(s, y, z, Y_{s+\delta(s)}^{1}, Z_{s+\zeta(s)}^{1}\right) \geq f^{2}\left(s, y, z, Y_{s+\delta(s)}^{2}, Z_{s+\zeta(s)}^{2}\right)
$$

Applying Theorem 4.1 again, we can finally get

$$
Y_{t}^{1} \geq Y_{t}^{2}, \quad t \in\left[t_{2}, t_{1}\right], \quad \text { a.e., a.s.. }
$$

Similarly to the above steps, we can give the proofs for the other cases when $t \in\left[t_{3}, t_{2}\right],\left[t_{4}, t_{3}\right], \cdots,\left[t_{N}, t_{N-1}\right]$.

Example 4.5 Now suppose that we are facing with the following two ABDSDEs:

$$
\begin{aligned}
\left\{\begin{array}{rlrl}
-d Y_{t}^{1}= & E^{\mathcal{F}_{t}}\left[Y_{t+\delta(t)}^{1}+\sin \left(2 Y_{t+\delta(t)}^{1}\right)+\left|Z_{t+\zeta(t)}^{1}\right|+2\right] d t \\
& +\left[Y_{t}^{1}+\frac{1}{\sqrt{3}}\left|Z_{t}^{1}\right|\right] d \overleftarrow{B}_{t}-Z_{t}^{1} d W_{t}, & & \\
\left\{\begin{aligned}
Y_{t}^{1}= & \xi_{t}^{1},
\end{aligned}\right. & t \in[0, T] ; \\
Z_{t}^{1}= & \eta_{t}^{1}, & & t \in[T, T+T+K],
\end{array}\right. \\
\left\{\begin{array}{rlrl}
-d Y_{t}^{2}= & E^{\mathcal{F}_{t}}\left[Y_{t+\delta(t)}^{2}+2\left|\cos Y_{t+\delta(t)}^{2}\right|+\sin Z_{t+\zeta(t)}^{2}-2\right] d t & \\
& +\left[Y_{t}^{2}+\frac{1}{\sqrt{3}}\left|Z_{t}^{2}\right|\right] d \overleftarrow{B}_{t}-Z_{t}^{2} d W_{t}, & & t \in[0, T] ; \\
Y_{t}^{2}= & \xi_{t}^{2}, & t \in[T, T+K], \\
Z_{t}^{2}= & \eta_{t}^{2}, & t \in[T, T+K],
\end{array}\right.
\end{aligned}
$$

where $\xi_{t}^{(1)} \geq \xi_{t}^{(2)}, t \in[T, T+K]$.
It is obvious that
$x+\sin (2 x)+|u|+2 \geq y+2|\cos y|+\sin v-2$, for all $x \geq y, x, y \in \mathbb{R}, u, v \in \mathbb{R}^{d}$, which implies (4.5)-(4.7), then according to Theorem4.4, we get $Y_{t}^{1} \geq Y_{t}^{2}$, a.e., a.s..

Remark 4.6 By the same way, for the case when $\delta=\zeta$, 4.5)- (4.7) can be replaced by (4.6) together with

$$
f^{1}\left(t, y, z, \xi_{t+\delta(t)}^{1}, \eta_{t+\zeta(t)}^{1}\right) \geq f^{2}\left(t, y, z, \xi_{t+\delta(t)}^{2}, \eta_{t+\zeta(t)}^{2}\right) \text {, a.e., a.s.. }
$$

For a special case when $f^{1}$ and $f^{2}$ are independent of the anticipated term $Z$, we easily get the following comparison result.

Theorem 4.7 $\operatorname{Let}\left(Y^{j}, Z^{j}\right) \in S_{\mathcal{G}}^{2}(0, T+K ; \mathbb{R}) \times L_{\mathcal{G}}^{2}\left(0, T+K ; \mathbb{R}^{d}\right)(j=1,2)$ be the unique solutions to the following $A B D S D E s$ respectively:
$\left\{\begin{aligned}-d Y_{t}^{j} & =f^{j}\left(t, Y_{t}^{j}, Z_{t}^{j}, Y_{t+\delta(t)}^{j}\right) d t+g\left(t, Y_{t}^{j}, Z_{t}^{j}\right) d \overleftarrow{B}_{t}-Z_{t}^{j} d W_{t}, & & t \in[0, T] ; \\ Y_{t}^{j} & =\xi_{t}^{j}, & & t \in[T, T+K] .\end{aligned}\right.$
If
(i) $\xi_{s}^{1} \geq \xi_{s}^{2}, s \in[T, T+K]$, a.e., a.s.;
(ii) for all $t \in[0, T],(y, z) \in \mathbb{R} \times \mathbb{R}^{d}, \theta^{j} \in S_{\mathcal{G}}^{2}(t, T+K ; \mathbb{R})(j=1,2)$ such that $\theta^{1} \geq \theta^{2}$ and $\left(\theta_{r}^{j}\right)_{r \in[T, T+K]}=\left(\xi_{r}^{j}\right)_{r \in[T, T+K]}$,

$$
\begin{equation*}
f^{1}\left(t, y, z, \theta_{t+\delta(t)}^{1}\right) \geq f^{2}\left(t, y, z, \theta_{t+\delta(t)}^{2}\right), \text { a.e., a.s. } \tag{4.10}
\end{equation*}
$$

then $Y_{t}^{1} \geq Y_{t}^{2}$, a.e., a.s..
Remark 4.8 The coefficients $f^{1}$ and $f^{2}$ will satisfy (4.10), if for any $(t, y, z) \in$ $[0, T] \times \mathbb{R} \times \mathbb{R}^{d}, \theta \in L_{\mathcal{G}}^{2}(t, T+K ; \mathbb{R}), r \in[t, T+K], f^{1}\left(t, y, z, \theta_{r}\right) \geq f^{2}\left(t, y, z, \theta_{r}\right)$, together with one of the following:
(i) for any $(t, y, z) \in[0, T] \times \mathbb{R} \times \mathbb{R}^{d}, f^{1}(t, y, z, \cdot)$ is increasing, i.e., $f^{1}\left(t, y, z, \theta_{r}\right) \geq$ $f^{1}\left(t, y, z, \theta_{r}^{\prime}\right)$, if $\theta \geq \theta^{\prime}, \theta, \theta^{\prime} \in L_{\mathcal{G}}^{2}(t, T+K ; \mathbb{R}), r \in[t, T+K]$;
(ii) for any $(t, y, z) \in[0, T] \times \mathbb{R} \times \mathbb{R}^{d}, f^{2}(t, y, z, \cdot)$ is increasing, i.e., $f^{2}\left(t, y, z, \theta_{r}\right) \geq$ $f^{2}\left(t, y, z, \theta_{r}^{\prime}\right)$, if $\theta \geq \theta^{\prime}, \theta, \theta^{\prime} \in L_{\mathcal{G}}^{2}(t, T+K ; \mathbb{R}), r \in[t, T+K]$.

Remark 4.9 The coefficients $f^{1}$ and $f^{2}$ will satisfy 4.10), if

$$
f^{1}\left(t, y, z, \theta_{r}\right) \geq \tilde{f}\left(t, y, z, \theta_{r}\right) \geq f^{2}\left(t, y, z, \theta_{r}\right)
$$

for any $(t, y, z) \in[0, T] \times \mathbb{R} \times \mathbb{R}^{d}, \theta \in L_{\mathcal{G}}^{2}(t, T+K ; \mathbb{R}), r \in[t, T+K]$. Here the function $\tilde{f}(t, y, z, \cdot)$ is increasing, for any $(t, y, z) \in[0, T] \times \mathbb{R} \times \mathbb{R}^{d}$, i.e., $\tilde{f}\left(t, y, z, \theta_{r}\right) \geq \tilde{f}\left(t, y, z, \theta_{r}^{\prime}\right)$, if $\theta_{r} \geq \theta_{r}^{\prime}, \theta, \theta^{\prime} \in L_{\mathcal{G}}^{2}(t, T+K ; \mathbb{R}), r \in[t, T+K]$.
Example 4.10 The following three functions satisfy the conditions in Remark 4.9: $f^{1}\left(t, y, z, \theta_{r}\right)=E^{\mathcal{F}_{t}}\left[\theta_{r}-\sin \left(2 \theta_{r}\right)+2\right], \tilde{f}\left(t, y, z, \theta_{r}\right)=E^{\mathcal{F}_{t}}\left[\theta_{r}+\cos \theta_{r}\right], f^{2}\left(t, y, z, \theta_{r}\right)=$ $E^{\mathcal{F}_{t}}\left[\theta_{r}+2 \cos \theta_{r}-1\right]$.

## 5 A duality result between delayed DSDEs and anticipated BDSDEs

In this part we will establish a duality between the following anticipated BDSDE

$$
\left\{\begin{array}{rlrl}
-d Y_{t}= & \left(\left[\mu_{t}+\kappa_{t}^{2}\right] Y_{t}+\bar{\mu}_{t} E^{\mathcal{F}_{t, T}^{B}}\left[Y_{t+\delta}\right]+\sigma_{t} Z_{t}+\bar{\sigma}_{t} E^{\mathcal{F}_{t, T}^{B}}\left[Z_{t+\delta}\right]+\rho_{t}\right) d t &  \tag{5.1}\\
& +\kappa_{t} Y_{t} d \overleftarrow{B}_{t}-Z_{t} d W_{t}, & & t \in\left[t_{0}, T\right] ; \\
Y_{t}= & \xi_{t}, & t \in[T, T+\delta] ; \\
Z_{t}= & \eta_{t}, & t \in[T, T+\delta]
\end{array}\right.
$$

and the delayed DSDE

$$
\left\{\begin{array}{rlrl}
d X_{s} & =\left(\mu_{s} X_{s}+\bar{\mu}_{s-\delta} X_{s-\delta}\right) d s+\kappa_{s} X_{s} d \overleftarrow{B}_{s}+\left(\sigma_{s} X_{s}+\bar{\sigma}_{s-\delta} X_{s-\delta}\right) d W_{s}, & & s \in[t, T]  \tag{5.2}\\
X_{t} & =1 \\
X_{s} & =0 & & s \in[t-\delta, t)
\end{array}\right.
$$

where we suppose that $t_{0} \geq \delta>0$ are fixed constants, $(\xi, \eta) \in S_{\mathcal{G}}^{2}(T, T+\delta ; \mathbb{R}) \times$ $L_{\mathcal{G}}^{2}\left(T, T+\delta ; \mathbb{R}^{d}\right)$ with $\xi_{T} \in L^{2}\left(\mathcal{F}_{T, T}^{B} ; \mathbb{R}\right), \mu_{t}, \bar{\mu}_{t} \in L_{\mathcal{F}_{t, T}^{B}}^{2}\left(t_{0}-\delta, T+\delta ; \mathbb{R}\right), \sigma_{t}, \bar{\sigma}_{t} \in$ $L_{\mathcal{F}_{t, T}^{B}}^{2}\left(t_{0}-\delta, T+\delta ; \mathbb{R}^{d}\right), \kappa_{t} \in L_{\mathcal{F}_{t, T}^{B}}^{2}\left(t_{0}, T ; \mathbb{R}^{l}\right), \rho_{t} \in L_{\mathcal{F}_{t, T}^{B}}^{2}\left(t_{0}, T ; \mathbb{R}\right)$, and $\mu, \bar{\mu}, \sigma, \bar{\sigma}, \kappa$ are uniformly bounded. Then by Theorem [3.2, (5.1) has a unique solution.

Proposition 5.1 Let $(Y, Z) \in S_{\mathcal{G}}^{2}\left(t_{0}, T+\delta ; \mathbb{R}\right) \times L_{\mathcal{G}}^{2}\left(t_{0}, T+\delta ; \mathbb{R}^{d}\right)$ be the unique solution of $A B D S D E$ (5.1). Then for $t \in\left[t_{0}, T\right], Z_{t} \equiv 0$, and $Y_{t}$ is $\mathcal{F}_{t, T}^{B}$-progressively measurable.

Proof. First we show that $Y_{t}$ is $\mathcal{F}_{t, T^{-}}^{B}$-progressively measurable. For this we introduce the following auxiliary equation:

$$
\left\{\begin{array}{rlrl}
-d Y_{t}^{\prime}= & E^{\mathcal{F}_{t, T}^{B}}\left[\left(\mu_{t}+\kappa_{t}^{2}\right) Y_{t}^{\prime}+\bar{\mu}_{t} Y_{t+\delta}^{\prime}+\sigma_{t} Z_{t}^{\prime}+\bar{\sigma}_{t} Z_{t+\delta}^{\prime}+\rho_{t}\right] d t &  \tag{5.3}\\
& +\kappa_{t} E^{\mathcal{F}_{t, T}^{B}}\left[Y_{t}^{\prime}\right] d \overleftarrow{B}_{t}-Z_{t}^{\prime} d W_{t}, & & t \in\left[t_{0}, T\right] ; \\
Y_{t}^{\prime}= & \xi_{t}, & & t \in[T, T+\delta] ; \\
Z_{t}^{\prime}= & \eta_{t}, & & t \in[T, T+\delta]
\end{array}\right.
$$

which has a unique solution according to Theorem 3.2.
In fact, it is obvious that

$$
Y_{t}^{\prime}=E^{\mathcal{G}_{t}}\left[\Theta^{\prime}\right],
$$

where
$\Theta^{\prime}:=\xi_{T}+\int_{t}^{T} E^{\mathcal{F}_{s, T}^{B}}\left[\left(\mu_{s}+\kappa_{s}^{2}\right) Y_{s}^{\prime}+\bar{\mu}_{s} Y_{s+\delta}^{\prime}+\sigma_{s} Z_{s}^{\prime}+\bar{\sigma}_{s} Z_{s+\delta}^{\prime}+\rho_{s}\right] d s+\int_{t}^{T} \kappa_{s} E^{\mathcal{F}_{s, T}^{B}}\left[Y_{s}^{\prime}\right] d \overleftarrow{B}_{s}$
is $\mathcal{F}_{t, T}^{B}$ measurable thanks to the fact that $\xi_{T} \in L^{2}\left(\mathcal{F}_{T T}^{B} ; \mathbb{R}\right)$ and $\mu_{t}, \bar{\mu}_{t}, \sigma_{t}, \bar{\sigma}_{t}, \kappa_{t}$, $\rho_{t}$ are all $\mathcal{F}_{t, T}^{B}$-progressively measurable. Note that $\mathcal{F}_{0, t}^{W} \vee \mathcal{F}_{T, T+K}^{B}$ is independent of $\mathcal{F}_{t, T}^{B} \vee \sigma\left(\Theta^{\prime}\right)$, hence we know

$$
Y_{t}^{\prime}=E^{\mathcal{F}_{t, T}^{B}}\left[\Theta^{\prime}\right] .
$$

Thus obviously $Z_{t}^{\prime} \equiv 0$, and moreover, $E^{\mathcal{F}_{t, T}^{B}}\left[Y_{t}^{\prime}\right]=Y_{t}^{\prime}, E^{\mathcal{F}_{t, T}^{B}}\left[Z_{t}^{\prime}\right]=Z_{t}^{\prime}$. Then by Comparing the anticipated BDSDE (5.1) with (5.3), together with the uniqueness of their solutions, we immediately get the desired conclusion.

The next is our main result.

Theorem 5.2 For any $(\xi, \eta) \in S_{\mathcal{G}}^{2}(T, T+\delta ; \mathbb{R}) \times L_{\mathcal{G}}^{2}\left(T, T+\delta ; \mathbb{R}^{d}\right)$ with $\xi_{T} \in$ $L^{2}\left(\mathcal{F}_{T, T}^{B} ; \mathbb{R}\right)$, the solution $Y$. of the anticipated BDSDE (5.1) can be given by
$Y_{t}=E^{\mathcal{F}_{t, T}^{B}}\left[X_{T} \xi_{T}+\int_{t}^{T} \rho_{s} X_{s} d s\right]+E^{\mathcal{F}_{t, T}^{B}}\left[\int_{T}^{T+\delta}\left(\bar{\mu}_{s-\delta} X_{s-\delta} E^{\mathcal{F}_{s-\delta, T}^{B}}\left[\xi_{s}\right]+\bar{\sigma}_{s-\delta} X_{s-\delta} E^{\mathcal{F}_{s-\delta, T}^{B}}\left[\eta_{s}\right]\right) d s\right]$,
where $X$. is the unique solution of delayed $D S D E$ (5.2).
Proof. We first show that DSDE (5.2) has a unique solution. In fact, when $s \in[t, t+\delta]$,

$$
\left\{\begin{align*}
d X_{s} & =\mu_{s} X_{s} d s+\kappa_{s} X_{s} d \overleftarrow{B}_{s}+\sigma_{s} X_{s} d W_{s}, \quad s \in[t, t+\delta]  \tag{5.4}\\
X_{t} & =1
\end{align*}\right.
$$

Then we can easily obtain a unique solution $\varsigma{ }^{1}$ for (5.4). When $s \in[t+\delta, t+2 \delta]$,

$$
\left\{\begin{align*}
d X_{s} & =\left(\mu_{s} X_{s}+\bar{\mu}_{s-\delta} \varsigma_{s-\delta}^{1}\right) d s+\kappa_{s} X_{s} d \overleftarrow{B}_{s}+\left(\sigma_{s} X_{s}+\bar{\sigma}_{s-\delta} \varsigma_{s-\delta}^{1}\right) d W_{s}, \quad s \in[t+\delta, t+2 \delta]  \tag{5.5}\\
X_{t+\delta} & =\varsigma_{t+\delta}^{1}
\end{align*}\right.
$$

Then we can easily obtain a unique solution $\varsigma^{2}$ for (5.5). Similarly , we can consider all the other cases when $t \in[t+2 \delta, t+3 \delta],[t+3 \delta, t+4 \delta], \cdots,\left[t+\left[\frac{T-t}{\delta}\right] \delta, T\right]$. Thus $\operatorname{DSDE}$ (5.2) has a unique solution $X \in \mathcal{S}_{\tilde{\mathcal{G}}}^{2}(t-\delta, T ; \mathbb{R})$ where $\tilde{\mathcal{G}}_{t}:=\mathcal{F}_{t}^{W} \vee \mathcal{F}_{t}^{B}$.

Applying Itô's formula to $X_{s} Y_{s}$, according to Proposition 5.1, we have

$$
\begin{align*}
& X_{T} Y_{T}-X_{t} Y_{t}-\int_{t}^{T}\left(X_{s} Z_{s}+\sigma_{s} X_{s} Y_{s}+\bar{\sigma}_{s-\delta} X_{s-\delta} Y_{s}\right) d W_{s} \\
= & \int_{t}^{T}\left(\bar{\mu}_{s-\delta} X_{s-\delta} Y_{s}-\bar{\mu}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[Y_{s+\delta}\right]+\bar{\sigma}_{s-\delta} X_{s-\delta} Z_{s}-\bar{\sigma}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[Z_{s+\delta}\right]-\rho_{s} X_{s}\right) d s \\
= & \int_{t}^{T-\delta}\left(\bar{\mu}_{s-\delta} X_{s-\delta} Y_{s}-\bar{\mu}_{s} X_{s} Y_{s+\delta}\right) d s \\
& +\int_{T-\delta}^{T}\left(\bar{\mu}_{s-\delta} X_{s-\delta} Y_{s}-\bar{\mu}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[\xi_{s+\delta}\right]-\bar{\sigma}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[\eta_{s+\delta}\right]\right) d s-\int_{t}^{T} \rho_{s} X_{s} d s . \tag{5.6}
\end{align*}
$$

Write $\Delta=\int_{t}^{T-\delta}\left(\bar{\mu}_{s-\delta} X_{s-\delta} Y_{s}-\bar{\mu}_{s} X_{s} Y_{s+\delta}\right) d s$, then

$$
\begin{align*}
\Delta & =\int_{t}^{T-\delta}\left(\bar{\mu}_{s-\delta} X_{s-\delta} Y_{s}-\bar{\mu}_{s} X_{s} Y_{s+\delta}\right) d s=\int_{t}^{T-\delta} \bar{\mu}_{s-\delta} X_{s-\delta} Y_{s} d s-\int_{t+\delta}^{T} \bar{\mu}_{s-\delta} X_{s-\delta} Y_{s} d s \\
& =\int_{t}^{t+\delta} \bar{\mu}_{s-\delta} X_{s-\delta} Y_{s} d s-\int_{T-\delta}^{T} \bar{\mu}_{s-\delta} X_{s-\delta} Y_{s} d s=-\int_{T-\delta}^{T} \bar{\mu}_{s-\delta} X_{s-\delta} Y_{s} d s, \tag{5.7}
\end{align*}
$$

and the last equality is due to the fact that $X_{s}=0, s \in[t-\delta, t)$.
Combining (5.6) and (5.7), we have

$$
\begin{aligned}
& X_{T} Y_{T}-X_{t} Y_{t}-\int_{t}^{T}\left(X_{s} Z_{s}+\sigma_{s} X_{s} Y_{s}+\bar{\sigma}_{s-\delta} X_{s-\delta} Y_{s}\right) d W_{s} \\
= & -\int_{T-\delta}^{T}\left(\bar{\mu}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[\xi_{s+\delta}\right]+\bar{\sigma}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[\eta_{s+\delta}\right]\right) d s-\int_{t}^{T} \rho_{s} X_{s} d s .
\end{aligned}
$$

Take conditional expectation with respect to $\tilde{\tilde{\mathcal{G}}}_{t}:=\mathcal{F}_{t}^{W} \vee \mathcal{F}_{T}^{B}$ on both sides, then $X_{t} Y_{t}=E^{\tilde{\mathcal{G}_{t}}}\left[X_{T} Y_{T}+\int_{t}^{T} \rho_{s} X_{s} d s\right]+E^{\tilde{\tilde{\mathcal{G}}}_{t}}\left[\int_{T-\delta}^{T}\left(\bar{\mu}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[\xi_{s+\delta}\right]+\bar{\sigma}_{s} X_{s} E^{\mathcal{F}_{s, T}^{B}}\left[\eta_{s+\delta}\right]\right) d s\right]$, which implies the desired result when noting that $X_{t}=1$ and $Y_{t} \in \mathcal{F}_{t, T}^{B}$.

## 6 Conclusion and future work

In this paper, we have established the existence/uniqueness theorem and the comparison theorem for the anticipated BDSDEs. Moreover, as an application, we studied a duality between the anticipated BDSDE and delayed DSDE, where the BDSDE is of a special form, thus the duality is somewhat limited. In fact for the general case, it should be admitted that $\mathcal{G}_{t}$, which is not a filtration, not increasing non decreasing, brings the main technical difficulty in working with the duality problem. For the future work, I will go on studying this topic and pay more attention to the applications of such equations.

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