## LINEAR PARTIAL q-DIFFERENCE EQUATIONS ON q-LINEAR LATTICES AND THEIR BIVARIATE q-ORTHOGONAL POLYNOMIAL SOLUTIONS

I. AREA, N. ATAKISHIYEV, E. GODOY, AND J. RODAL

ABSTRACT. Orthogonal polynomial solutions of an admissible potentially self-adjoint linear second-order partial q-difference equation of the hypergeometric type in two variables on q-linear lattices are analyzed. A q-Pearson's system for the orthogonality weight function, as well as for the difference derivatives of the solutions are presented, giving rise to a solution of the q-difference equation under study in terms of a Rodriguestype formula. The monic orthogonal polynomial solutions are treated in detail, giving explicit formulae for the matrices in the corresponding recurrence relations they satisfy. Lewanowicz and Woźny S. Lewanowicz, P. Woźny, J. Comput. Appl. Math. 233 (2010) 1554-1561 have recently introduced a (non-monic) bivariate extension of big q-Jacobi polynomials together with a partial q-difference equation of the hypergeometric type that governs them. This equation is analyzed in the last section: we provide two more orthogonal polynomial solutions, namely, a second non-monic solution from the Rodrigues' representation, and the monic solution both from the recurrence relation that govern them and also explicitly given in terms of generalized bivariate basic hypergeometric series. Limit relations as  $q \uparrow 1$  for the partial q-difference equation and for the all three q-orthogonal polynomial solutions are also presented.

## 1. INTRODUCTION

As recalled in [1], Hahn [20] considered the operator

(1.1) 
$$L_{q,\omega}f(x) = (D_{q,\omega}f)(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x+\omega}, \quad x \in \mathbf{R} \setminus \left\{\frac{\omega}{1-q}\right\},$$

for all  $q \in \mathbf{R} \setminus \{-1, 0\}$ ,  $\omega \in \mathbf{R}$  and  $(q, \omega) \neq (1, 0)$ , which for q = 1 becomes the finite difference operator  $\Delta_{\omega}$ , and if  $q \neq 1$  and  $\omega = 0$  then  $L_{q,0}$  is the q-derivative operator [20, Eq. (2.3)]

$$L_{q,0}f(x) = (D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad q \neq 1,$$

and  $(D_q f)(0) := f'(0)$  by continuity, provided f'(0) exists. Note that  $\lim_{q \uparrow 1} (D_q f)(x) = f'(x)$  if f is differentiable.

Hahn seems to have been the first to realize that the characterizations of classical orthogonal polynomial sequences (OPS) based on derivatives and differential equations

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are too restrictive [8]. He posed and solved the following problems: find all OPS such that one of the following holds

- (1)  $\{L_{q,0}P_n(x)\}$  is also OPS;
- (2)  $\{P_n(x)\}$  satisfy the functional equation

$$\sigma(x)L_{q,0}^2 P_n(x) + \tau(x)L_{q,0}P_n(x) + \lambda_n P_n(x) = 0;$$

(3)  $P_n(x)$  has the representation

$$P_n(x) = \frac{1}{\varrho(x)} L_{q,0}^n \left\{ f_0(x) f_1(x) \cdots f_{n-1}(x) \varrho(x) \right\}$$

where  $f_k(x) = f_{k+1}(qx)$ ;

- (4) If  $P_n(x) = \sum a_{nk} x^k$  then  $a_{nk}/a_{n,k-1}$  is a rational function of  $q^n$  and  $q^k$ ;
- (5) The moments associated with  $\{P_n(x)\}$  satisfy

(1.2) 
$$M_n = \frac{a + bq^n}{c + dq^n} M_{n-1}, \quad a d - b c \neq 0$$

where  $M_n$  are either the power moments (moments against  $x^k$ ) or the generalized moments (moments against the q-shifted factorial  $(x; q)_k = \prod_{j=0}^{k-1} (1 - xq^j)$ ).

Hahn's investigation led him to the most general set of polynomials belonging to this class [8], now called q-Hahn polynomials:

(1.3) 
$$Q_n(x; a, b, N; q) = {}_{3}\phi_2 \left( \begin{array}{c} q^{-n}, abq^{n+1}, x \\ aq, q^{-N} \end{array} \middle| q; q \right), \quad n = 0, 1, \dots, N.$$

Following the works of Nikiforov and Uvarov [39, 40, 41], a review of the hypergeometrictype difference equation for a function y(x(s)) on a non-uniform lattice x(s) has been given in [8]. Note that the difference-derivatives of y(x(s)) satisfy similar (to the initial ones for y(x(s))) equations if and only if the lattice x(s) has the form

(1.4) 
$$x(s) = c_1 q^s + c_2 q^{-s} + c_3, \qquad \text{or}$$

(1.5) 
$$x(s) = c_4 s^2 + c_5 s + c_6,$$

where q,  $c_1$ ,  $c_2$  and  $c_3$  are constants. Depending on the particular choice of constants, the lattices are commonly referred to as

- (1) Linear lattices if we choose in (1.5)  $c_4 = 0$  and  $c_2 \neq 0$ .
- (2) Quadratic lattices if we choose in (1.5)  $c_4 \neq 0$ .
- (3) q-linear lattices (or q-exponential lattices) if we choose in (1.4)  $c_2 = 0$  and  $c_1 \neq 0$ .
- (4) q-quadratic lattices if we choose in (1.4)  $c_1c_2 \neq 0$ .

The Askey tableau of hypergeometric orthogonal polynomials contains the classical orthogonal polynomials, which can be written in terms of a hypergeometric function, starting at the top with Wilson and Racah polynomials on quadratic lattices and ending at the bottom with Hermite polynomials [33]. Hahn [20] actually studied the q-analogue of this scheme. So, there are q-analogues of all the families in the Askey tableau, often several q-analogues for one classical family. The master class of all these q-analogues is formed by the Askey-Wilson polynomials [7] on the q-quadratic lattice  $x(s) = (q^s + q^{-s})/2$ , which contain all other families as special or limit cases [3]. In [28] Koornwinder gave a q-Hahn tableau: a q-analogue of the part of the Askey tableau dominated by

q-Hahn polynomials, in the q-linear lattice  $x(s) = q^s$ . Basic hypergeometric functions and q-orthogonal polynomials for arbitrary (including complex) values of q are connected with quantum algebras and groups [54].

Recently, Koekoek, Lesky and Swarttouw [26] presented a classification of all families of classical orthogonal polynomials and their q-analogues, the classical q-orthogonal polynomials, as orthogonal polynomial solutions of the eigenvalue problem

(1.6) 
$$\phi(x)(D_{q,w})^2 y_n(x) + \psi(x)(D_{q,w})y_n(x) = \lambda_n y_n(qx+w),$$

where  $D_{q,w}$  is the Hahn's operator (1.1),  $\phi(x)$  is a polynomial of at most degree 2,  $\psi(x)$  is a polynomial of exact degree 1, and  $\lambda_n$  is the spectral parameter.

Besides well-known three-term recurrences, that q-orthogonal polynomial solutions of the latter equation do satisfy [11, 41, 49], these solutions can be characterized in a number of ways, e.g. k-th q-derivatives of each family are again orthogonal and belong to the same family [1, 41]. Moreover, the orthogonality weight functions satisfy q-Pearson equations [9, 39], giving rise to Rodrigues' formulae [1, 41] for the corresponding orthogonal polynomials and their derivatives of any order. Also, the orthogonal polynomials posess a number of algebraic and q-difference properties, expressed as q-derivative representations [1, 2, 9] and structure relations [1, 11, 29]. The list of the above references is not exhaustive but only indicative for the kind of references that could be consulted on this topic.

It is quite remarkable that in these classical settings the coefficients, appearing in all of the aforementioned algebraic and differential characterizations, can be explicitly computed in terms of the polynomial coefficients  $\phi(x)$  and  $\psi(x)$  of the hypergeometrictype q-difference equation (1.6) [26, 41], which governs those q-classical families.

Orthogonal polynomials in several variables have been analyzed since a long time ago [22] and we refer to the books of Suetin [48] and Dunkl and Xu [13], as basic references on this topic. Various multivariate extensions have been used in many applications such as image description and pattern recognition [59], or ternary drug mixtures [38], among others.

In 1991 Tratnik introduced some multivariable extensions of univariate orthogonal polynomials (see [51, 52] and references therein). Moreover, q-analogues of these systems have been constructed by Gasper and Rahman [15, 16, 17], yielding systems of multivariable orthogonal Askey-Wilson polynomials and their special and limit cases. Bispectrality of multivariable Racah-Wilson and Askey-Wilson polynomials has been studied in [18] and [19], respectively.

As indicated in [18], a beautiful extension of univariate orthogonal polynomials to the multivariate case is exemplified by symmetric Macdonald-Koornwinder polynomials, see, for instance, [21, 27, 37, 53].

In more recent papers the second-order linear partial differential equations of the hypergeometric type [5] and their discretization on uniform lattices [4, 6, 44, 45], as well as a general way of introducing orthogonal polynomial families in two discrete variables on the simplex [43], have been analyzed. Therefore, it is possible to generalize the univariate classical orthogonal polynomials to the bivariate and multivariate versions by requiring that they obey a second-order partial differential equation of the hypergeometric type

(continuous case) [32, 48], or a second-order partial difference equation of the hypergeometric type (discrete case), as indicated before. Thus, the "continuous" polynomials can be analyzed as limits of the "discrete" ones [6]. Likewise, the corresponding differential operator will appear as a scaling limit of an appropriate difference operator, and the continuous distribution (the weight measure for the bivariate continuous polynomials) is obtained through a scaling limit from the discrete distribution (the weight for the bivariate discrete polynomials).

The main goal of this paper is to extend the latter results on continuous and discrete bivariate cases to an admissible potentially self-adjoint linear second-order partial qdifference equation of the hypergeometric type on particular non-uniform lattices, and to study their orthogonal polynomial solutions.

The paper is organized as follows. In Section 2 the linear second-order partial qdifference equations of the hypergeometric type are introduced, giving explicitly the coefficients of the equation for the partial q-derivatives (of arbitrary order) of any solution in terms of the coefficients of the initial equation. In Section 3 we study the admissibility conditions for partial q-difference equations of the hypergeometric type. Next, in Section 4, the partial q-difference equation is written in the self-adjoint form, which gives a number of useful identities for the orthogonality weight function for the polynomial solutions (q-Pearson's system), as well as of the q-difference derivatives of the polynomial solutions. The key point is to determine the orthogonality weight function from the polynomial coefficients of the initial equation, which is also explicitly worked out. In the remaining part of the paper we deal with admissible potentially self-adjoint linear second-order partial q-difference equations of the hypergeometric type. In Section 5 an analogue of the well-known Rodrigues' formula for classical orthogonal polynomials is presented for orthogonal polynomial solutions of the partial q-difference equation. The monic orthogonal polynomial solutions of the partial q-difference equation are analyzed in detail in Section 6, where we give explicitly the matrices of the corresponding three-term recurrence relations for the most general equation, which belongs to the class under study. Section 7 is related with bivariate big q-Jacobi polynomials and a partial q-difference equation, that govern them [34]. Two novel bivariate q-orthogonal polynomial solutions of this equation are explicitly given. The first (non-monic) one is constructed from the Rodrigues' representation, derived in Section 5. The second novel (monic) solution of the equation is obtained from the general analysis, given in Section 6, i.e. by employing in this particular case the matrices of the three-term recurrence relations for the vector column of polynomials. Besides, this monic solution is also explicitly given in terms of generalized bivariate basic hypergeometric series. Finally, limit relations as  $q \uparrow 1$  for the partial q-difference equation, as well as for the three above-mentioned solutions, are analyzed in detail.

# 2. A linear second-order partial q-difference equation of the hypergeometric type

In what follows we shall assume that 0 < q < 1 and we shall consider disconnected or independent non-uniform lattices [39] as

(2.1) 
$$x = x(s) = q^s, \quad y = y(t) = q^t.$$

Related with these q-linear lattices (2.1), let us introduce the following partial q-difference operators

(2.2) 
$$D_q^1 f(x,y) = \frac{f(qx,y) - f(x,y)}{(q-1)x}, \quad D_q^2 f(x,y) = \frac{f(x,qy) - f(x,y)}{(q-1)y},$$

$$(2.3) \quad D_{q^{-1}}^1 f(x,y) = \frac{q \left(f(x,y) - f\left(x/q,y\right)\right)}{(q-1)x}, \quad D_{q^{-1}}^2 f(x,y) = \frac{q \left(f(x,y) - f\left(x,y/q\right)\right)}{(q-1)y}$$

The rules for the partial q-derivatives of a product of two functions f(x, y) and g(x, y) are given by

(2.4) 
$$D_q^1(fg)(x,y) = f(x,y)D_q^1g(x,y) + g(qx,y)D_q^1f(x,y),$$

(2.5) 
$$D_q^2(fg)(x,y) = f(x,y)D_q^2g(x,y) + g(x,qy)D_q^2g(x,y).$$

The following readily verified relations will also be used

$$(2.6) \qquad \begin{cases} D_q^1 D_{q^{-1}}^2 f(x,y) = D_{q^{-1}}^2 D_q^1 f(x,y), & D_q^2 D_{q^{-1}}^1 f(x,y) = D_{q^{-1}}^2 D_q^2 f(x,y), \\ D_q^1 D_q^2 f(x,y) = D_q^2 D_q^1 f(x,y), & D_{q^{-1}}^1 D_{q^{-1}}^2 f(x,y) = D_{q^{-1}}^2 D_{q^{-1}}^1 f(x,y), \\ D_{q^{-1}}^1 f(x,y) = D_q^1 f(x,y) + (1-q) x D_q^1 D_{q^{-1}}^1 f(x,y), \\ D_{q^{-1}}^2 f(x,y) = D_q^2 f(x,y) + (1-q) y D_q^2 D_{q^{-1}}^2 f(x,y). \end{cases}$$

The following linear second-order partial differential equation has been considered in [5, 36]

$$(2.7) \quad \tilde{a}_{11}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 x} + \tilde{a}_{12}(x,y)\frac{\partial^2 u(x,y)}{\partial x \partial y} + \tilde{a}_{22}(x,y)\frac{\partial^2 u(x,y)}{\partial^2 y} + \tilde{b}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \tilde{b}_2(x,y)\frac{\partial u(x,y)}{\partial y} + \lambda u(x,y) = 0.$$

Among many methods of approximating (2.7), we shall discuss a linear partial q-difference equation, obtained from (2.7) via the simplest q-difference schemes of the second-order precision [42, 46]:

$$(2.8) \quad a_{11}(x,y)\sqrt{q}D_q^1D_{q^{-1}}^1u(x,y) + a_{22}(x,y)\sqrt{q}D_q^2D_{q^{-1}}^2u(x,y) + a_{12a}(x,y)D_q^1D_q^2u(x,y) + a_{12d}(x,y)D_{q^{-1}}^1D_{q^{-1}}^2u(x,y) + b_1(x,y)D_q^1u(x,y) + b_2(x,y)D_q^2u(x,y) + \lambda u(x,y) = 0.$$

It is important to note here that from the cross second partial derivative we have obtained two second-order q-difference operators. As is shown below through an example (see Section 7), the associated polynomial coefficients  $a_{12a}(x, y)$  and  $a_{12d}(x, y)$  can be distinct.

**Definition 2.1.** We shall refer to

$$u^{(k,\ell)}(x,y) := [D_q^1]^{(k)} [D_q^2]^{(\ell)} u(x,y) = D_q^1 \stackrel{k}{\cdots} D_q^1 D_q^2 \stackrel{\ell}{\cdots} D_q^2 u(x,y)$$

as generalized difference of order  $(k, \ell)$  for the function u(x, y).

**Definition 2.2.** We shall say that equation (2.8) is a partial q-difference equation of the hypergeometric type if all the generalized differences  $u^{(k,\ell)}(x,y)$  for any solution u = u(x,y) of (2.8) are also solutions of equations of the same type.

In a similar way as Lyskova [36] introduced the so-called basic class in the continuous case, we have:

**Lemma 2.3.** Equation (2.8) is a partial q-difference equation of the hypergeometric type if and only if it has the form

$$(2.9) \quad q \left(a_{1}x^{2} + b_{1}x + c_{1}\right) D_{q}^{1} D_{q^{-1}}^{1} u(x, y) + q \left(a_{2}y^{2} + b_{2}y + c_{2}\right) D_{q}^{2} D_{q^{-1}}^{2} u(x, y) + \left(a_{3a}xy + b_{3a}x + c_{3a}y + d_{3a}\right) D_{q}^{1} D_{q}^{2} u(x, y) + \left(a_{3d}xy + b_{3d}x + c_{3d}y + d_{3d}\right) D_{q^{-1}}^{1} D_{q^{-1}}^{2} u(x, y) + \left(f_{1}x + g_{1}\right) D_{q}^{1} u(x, y) + \left(f_{2}y + g_{2}\right) D_{q}^{2} u(x, y) + \lambda u(x, y) = 0,$$

that is,

 $\begin{aligned} a_{11}(x,y) &= a_{11}(x) = \sqrt{q} \left( a_1 x^2 + b_1 x + c_1 \right), \quad a_{22}(x,y) = a_2(y) = \sqrt{q} \left( a_2 y^2 + b_2 y + c_2 \right), \\ a_{12a}(x,y) &= a_{3a} xy + b_{3a} x + c_{3a} y + d_{3a}, \quad a_{12d}(x,y) = a_{3d} xy + b_{3d} x + c_{3d} y + d_{3d}, \\ b_1(x,y) &= b_1(x) = f_1 x + g_1, \quad b_2(x,y) = b_2(y) = f_2 y + g_2. \end{aligned}$ 

*Proof.* Apply the operator  $D_q^1$  to (2.8) in order to reveal that the lemma is simply a consequence of the following seven partial results, based on the relations (2.4)–(2.6):

(1) 
$$D_q^1[a_{11}(x,y)q^{1/2}D_q^1D_{q^{-1}}^1u(x,y)]$$
  
=  $a_{11}(x,y)q^{-1/2}D_q^1D_{q^{-1}}^1u^{(1,0)}(x,y) + D_q^1a_{11}(x,y)q^{-1/2}D_q^1u^{(1,0)}(x,y);$ 

(2)  $D_q^1[a_{22}(x,y)q^{1/2}D_q^2D_{q^{-1}}^2u(x,y)] = a_{22}(x,y)q^{1/2}D_q^2D_{q^{-1}}^2u^{(1,0)}(x,y),$ 

provided that  $a_{22}(x, y)$  does not depend on x (in order to preserve the same structure of the equation);

(3) 
$$D_q^1[a_{12a}(x,y)D_q^1D_q^2u(x,y)]$$
  
=  $D_q^1a_{12a}(x,y)D_q^2u^{(1,0)}(x,y) + a_{12a}(qx,y)D_q^1D_q^2u^{(1,0)}(x,y);$ 

(4) 
$$D_q^1[a_{12d}(x,y)D_{q^{-1}}^1D_{q^{-1}}^2u(x,y)]$$
  
=  $D_q^1a_{12d}(x,y)D_{q^{-1}}^2u^{(1,0)}(x,y) + a_{12d}(x,y)q^{-1/2}Q_{q^{-1}}^1D_{q^{-1}}^2u^{(1,0)}(x,y);$ 

(5) 
$$D_q^1[b_1(x,y)D_q^1u(x,y)] = b_1(qx,y)q^{1/2}q^{-1/2}D_q^1u^{(1,0)}(x,y) + D_q^1b_1(x,y)u^{(1,0)}(x,y);$$

(6)  $D_q^1[b_2(x,y)D_q^2u(x,y)] = b_2(x,y)D_q^2u^{(1,0)}(x,y),$ 

provided that  $b_2(x, y)$  does not depend on x;

and finally,

(7) 
$$D_q^1[\lambda u(x,y)] = \lambda u^{(1,0)}(x,y).$$

Repeating this process k times in x and  $\ell$  times in y, one obtains the following partial q-difference equation for the generalized difference of order  $(k, \ell)$  of the function u(x, y):

$$(2.10) \quad a_{11}^{(k,\ell)}(x)\sqrt{q}D_q^1D_{q^{-1}}^1u^{(k,\ell)}(x,y) + a_{22}^{(k,\ell)}(y)\sqrt{q}D_q^2D_{q^{-1}}^2u^{(k,\ell)}(x,y) + a_{12a}^{(k,\ell)}(x,y)D_q^1D_q^2u^{(k,\ell)}(x,y) + a_{12d}^{(k,\ell)}(x,y)D_{q^{-1}}^1D_{q^{-1}}^2u^{(k,\ell)}(x,y) + b_1^{(k,\ell)}(x)D_q^1u^{(k,\ell)}(x,y) + b_2^{(k,\ell)}(y)D_q^2u^{(k,\ell)}(x,y) + \mu^{(k,\ell)}u^{(k,\ell)}(x,y) = 0,$$

where

$$\begin{cases} a_{11}^{(k+1,\ell)}(x) = q^{-1}a_{11}^{(k,\ell)}(x) , & a_{22}^{(k+1,\ell)}(y) = a_{22}^{(k,\ell)}(y) + q^{-1/2}(1-q)yD_q^1a_{12d}^{(k,\ell)}(x,y) , \\ a_{12a}^{(k+1,\ell)}(x,y) = a_{12a}^{(k,\ell)}(qx,y) , & a_{12d}^{(k+1,\ell)}(x,y) = q^{-1}a_{12d}^{(k,\ell)}(x,y) , \\ b_1^{(k+1,\ell)}(x) = b_1^{(k,\ell)}(qx) + q^{-1/2}D_q^1a_{11}^{(k,\ell)}(x) , \\ b_2^{(k+1,\ell)}(y) = b_2^{(k,\ell)}(y) + D_q^1a_{12a}^{(k,\ell)}(x,y) + D_q^1a_{12d}^{(k,\ell)}(x,y) , \\ \mu^{(k+1,\ell)} = \mu^{(k,\ell)} + D_q^1b_1^{(k,\ell)}(x) , \end{cases}$$

and

$$\begin{cases} a_{11}^{(k,\ell+1)}(x) = a_{11}^{(k,\ell)}(x) + q^{-1/2}(1-q)xD_q^2 a_{12d}^{(k,\ell)}(x,y) \,, & a_{22}^{(k,\ell+1)}(y) = q^{-1}a_{22}^{(k,\ell)}(y) \,, \\ a_{12a}^{(k,\ell+1)}(x,y) = a_{12a}^{(k,\ell)}(x,qy) \,, & a_{12d}^{(k,\ell+1)}(x,y) = q^{-1}a_{12d}^{(k,\ell)}(x,y) \,, \\ b_1^{(k,\ell+1)}(x) = b_1^{(k,\ell)}(x) + D_q^2 a_{12a}^{(k,\ell)}(x,y) + D_q^2 a_{12d}^{(k,\ell)}(x,y) \,, \\ b_2^{(k,\ell+1)}(y) = b_2^{(k,\ell)}(qy) + q^{-1/2}D_q^2 a_{22}^{(k,\ell)}(y) \,, \\ \mu^{(k,\ell+1)} = \mu^{(k,\ell)} + D_q^2 b_2^{(k,\ell)}(y) \,. \end{cases}$$

If one computes the action of  $D_q^1 D_q^2 = D_q^2 D_q^1$  on the equation (2.8), then one obtains that

$$D_q^1 D_q^2 a_{12i}(x, y) = 0,$$

or equivalently, the polynomials  $a_{12a}(x, y)$  and  $a_{12d}(x, y)$  should not contain the terms  $x^2$  and  $y^2$ .

It is not hard to prove by induction that

$$(2.11) \quad a_{11}^{(k,\ell)}(x,y) = \frac{a_{11}(x,y)}{q^k} + \frac{(1-q^\ell)x(c_{3d}+a_{3d}x)}{q^{k+\ell-1/2}}, \quad a_{12a}^{(k,\ell)}(x,y) = a_{12a}(q^kx,q^\ell y),$$

$$(2.12) \quad a_{22}^{(k,\ell)}(x,y) = \frac{a_{22}(x,y)}{q^{\ell}} + \frac{(1-q^k)y(b_{3d}+a_{3d}y)}{q^{k+\ell-1/2}}, \quad a_{12d}^{(k,\ell)}(x,y) = q^{-(k+\ell)}a_{12d}(x,y),$$

$$(2.13) \qquad b_{1}^{(k,\ell)}(x,y) = b_{1}(q^{k}x,q^{\ell}y) + \frac{(1-q)[k]_{q}[\ell]_{q}(c_{3d} + (a_{3d} - a_{3a}q^{k+\ell-1})x)}{q^{k+\ell-1}}, \\ + \frac{[k]_{q}(b_{1} + a_{1}(q^{k} + 1)x)}{q^{k-1}} + \frac{[\ell]_{q}(c_{3d} + a_{3d}x + q^{\ell-1}(c_{3a} + a_{3a}x))}{q^{\ell-1}} \\ (2.14) \qquad b_{2}^{(k,\ell)}(x,y) = b_{2}(q^{k}x,q^{\ell}y) + \frac{(1-q)[k]_{q}[\ell]_{q}(b_{3d} + (a_{3d} - a_{3a}q^{k+\ell-1})y)}{q^{k+\ell-1}} \\ + \frac{[\ell]_{q}(b_{2} + a_{2}(q^{\ell} + 1)y)}{q^{\ell-1}} + \frac{[k]_{q}(b_{3d} + a_{3d}y + q^{k-1}(b_{3a} + a_{3a}y))}{q^{k-1}}, \end{cases}$$

(2.15) 
$$\mu^{(k,\ell)} = \lambda + \frac{[k]_q (f_1 q^{k-2} + a_1 [k-1]_q)}{q^{k-2}} + \frac{[\ell]_q (f_2 q^{\ell-2} + a_2 [\ell-1]_q)}{q^{\ell-2}} + \frac{[k]_q [\ell]_q (a_{3d} + a_{31} q^{k+\ell-2})}{q^{k+\ell-2}},$$

where the q-number is

(2.16) 
$$[z]_q = \frac{q^z - 1}{q - 1}, \qquad z \in \mathbb{C}.$$

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## 3. Admissible equations

**Definition 3.1.** The partial q-difference equation of the hypergeometric type (2.8) will be called admissible if there exists an infinite sequence  $\{\lambda_n\}$  (n = 0, 1, ...) such that for each  $\lambda = \lambda_n$ , there are precisely n + 1 linearly independent polynomial solutions of total degree n and no non-trivial solutions in the form of polynomials, whose total degree is less than n.

This concept was introduced by Krall and Sheffer [32] in the case of second-order partial differential equations and also by Y. Xu in [58, Section 2] for the case of second-order partial difference equations (without assuming that equations are of the hypergeometric type), and analyzed later on in [5] and [4, 44, 46], for the continuous and discrete cases, respectively.

In the case n = 0, the equation (2.8) also implies that a non-trivial solution can only exist when  $\lambda_0 = 0$ .

Observe that this definition of admissibility of equation (2.8) implies that all numbers

$$\lambda_0 = 0, \lambda_1, \lambda_2, \dots, \lambda_n, \dots,$$

are distinct,  $\lambda_m \neq \lambda_n, \ m \neq n$ .

From Lemma 2.3 one can deduce

**Theorem 3.2.** The partial q-difference equation of the hypergeometric type (2.9) is admissible if and only if

(3.1) 
$$f_2 = f_1, \quad a_2 = a_1, \quad a_{3a} = a_1q + f_1(q-1), \quad a_{3d} = a_1,$$

and

(3.2) 
$$\lambda_n = -[n]_q \left( f_1 - a_1 q [1 - n]_q \right),$$

and the numbers  $a_1$  and  $f_1$  are such that for any non-negative integer m the following condition holds

$$f_1 - a_1 q [1 - m]_q \neq 0.$$

*Proof.* A proof can be given in a similar way as in [48, pp. 93–97] for the multivariate continuous situation.  $\Box$ 

It is therefore plain that with the notations of Lemma 2.3 the equation (2.9) can be written as

$$(3.3) \quad q \left(a_{1}x^{2} + b_{1}x + c_{1}\right) D_{q}^{1} D_{q^{-1}}^{1} u(x, y) + q \left(a_{1}y^{2} + b_{2}y + c_{2}\right) D_{q}^{2} D_{q^{-1}}^{2} u(x, y) + \left((a_{1}q + f_{1}(q - 1))xy + b_{3a}x + c_{3a}y + d_{3a}\right) D_{q}^{1} D_{q}^{2} u(x, y) + \left(a_{1}xy + b_{3d}x + c_{3d}y + d_{3d}\right) D_{q^{-1}}^{1} D_{q^{-1}}^{2} u(x, y) + \left(f_{1}x + g_{1}\right) D_{q}^{1} u(x, y) + \left(f_{1}y + g_{2}\right) D_{q}^{2} u(x, y) + \lambda_{n} u(x, y) = 0$$

i.e.

(3.4) 
$$\begin{cases} a_{11}(x) = \sqrt{q} (a_1 x^2 + b_1 x + c_1), & b_1(x) = f_1 x + g_1, & b_2(y) = f_1 y + g_2, \\ a_{12a}(x, y) = (a_1 q + f_1(q - 1))xy + b_{3a} x + c_{3a} y + d_{3a}, \\ a_{22}(y) = \sqrt{q} (a_1 y^2 + b_2 y + c_2), & a_{12d}(x, y) = a_1 xy + b_{3d} x + c_{3d} y + d_{3d}. \end{cases}$$

## 4. POTENTIALLY SELF-ADJOINT OPERATOR

From the admissible linear second-order partial q-difference equation of the hypergeometric type (3.3) we introduce the following second-order partial q-difference operator:

$$(4.1) \quad \mathcal{D}_q[f(x,y)] = a_{11}(x)\sqrt{q}D_q^1 D_q^{1-1}f(x,y) + a_{22}(y)\sqrt{q}D_q^2 D_{q^{-1}}^2 f(x,y) + a_{12a}(x,y)D_q^1 D_q^2 f(x,y) + a_{12d}(x,y)D_{q^{-1}}^1 D_{q^{-1}}^2 f(x,y) + b_1(x)D_q^1 f(x,y) + b_2(y)D_q^2 f(x,y).$$

This enables us to write (3.3) as

(4.2) 
$$\mathcal{D}_q f(x, y) + \lambda_n f(x, y) = 0.$$

**Lemma 4.1.** The adjoint operator  $\mathcal{D}_q^{\dagger}$  of  $\mathcal{D}_q$ , defined by (4.1), is given by

$$\begin{aligned} (4.3) \quad \mathcal{D}_{q}^{\dagger}[f(x,y)] &= \sqrt{q} D_{q}^{1} D_{q^{-1}}^{1}(a_{11}(x) f(x,y)) + \sqrt{q} D_{q}^{2} D_{q^{-1}}^{2}(a_{22}(y) f(x,y)) \\ q^{2} D_{q}^{1} D_{q}^{2}(a_{12d}(x,y) f(x,y)) + \frac{1}{q^{2}} D_{q^{-1}}^{1} D_{q^{-1}}^{2}(a_{12a}(x,y) f(x,y)) \\ &- \frac{1}{q} D_{q^{-1}}^{1}(b_{1}(x) f(x,y)) - \frac{1}{q} D_{q^{-1}}^{2}(b_{2}(y) f(x,y)). \end{aligned}$$

*Proof.* The result is a direct consequence of

$$\left[D_{q}^{i}\right]^{\dagger} = -\frac{1}{q}D_{q^{-1}}^{i}, \quad \left[D_{q^{-1}}^{i}\right]^{\dagger} = -qD_{q}^{i}, \quad i = 1, 2.$$

**Definition 4.2.** The operator  $\mathcal{D}_q$  is potentially self-adjoint in a domain G, if there exists a positive real function  $\varrho(\mathbf{x}) = \varrho(x, y)$  in this domain, such that the operator  $\varrho(\mathbf{x})\mathcal{D}_q$  is self-adjoint in the domain G, i.e.,  $(\varrho(\mathbf{x})\mathcal{D}_q)^{\dagger} = \varrho(\mathbf{x})\mathcal{D}_q$  (see [48, Chapter V]).

In order that  $\mathcal{D}_q$  be potentially self-adjoint, we multiply (3.3) through by a positive function  $\varrho(\mathbf{x}) = \varrho(x, y)$  in some domain G, to be chosen later, to arrive at

$$(4.4) \quad a_{11}(x)\varrho(x,y)\sqrt{q}D_q^1D_{q^{-1}}^1f(x,y) + a_{22}(y)\varrho(x,y)\sqrt{q}D_q^2D_{q^{-1}}^2f(x,y) + a_{12a}(x,y)\varrho(x,y)D_q^1D_q^2f(x,y) + a_{12d}(x,y)\varrho(x,y)D_{q^{-1}}^1D_{q^{-1}}^2f(x,y) + b_1(x)\varrho(x,y)D_q^1f(x,y) + b_2(y)\varrho(x,y)D_q^2f(x,y) + \lambda\varrho(x,y)f(x,y) = 0,$$

which can be written in the self-adjoint form if the following q-Pearson's system of equations is satisfied:

(4.5) 
$$\begin{cases} \varrho(x,y)a_{12a}(x,y) = q^2 \varrho(qx,qy)a_{12d}(qx,qy),\\ \varrho(x,y)\phi_1(x,y) = \varrho(qx,y)\omega_1(qx,y),\\ \varrho(x,y)\phi_2(x,y) = \varrho(x,qy)\omega_2(x,qy), \end{cases}$$

where

(4.6) 
$$\begin{cases} \omega_1(qx,y) = \sqrt{q}ya_{11}(qx) - xq^2a_{12d}(qx,y), \\ \omega_2(x,qy) = \sqrt{q}xa_{22}(qy) - yq^2a_{12d}(x,qy), \\ \phi_1(x,y) = \sqrt{q}ya_{11}(x) - xa_{12a}(x,y) + (q-1)xyb_1(x), \\ \phi_2(x,y) = \sqrt{q}xa_{22}(y) - ya_{12a}(x,y) + (q-1)xyb_2(y). \end{cases}$$

The q-Pearson's system (4.5) can be also written as

(4.7) 
$$\begin{cases} \sqrt{q}D_q^1(\varrho(x,y)a_{11}(x)) + q^{-1}D_{q^{-1}}^2(\varrho(x,y)a_{12a}(x,y)) = \varrho(x,y)b_1(x,y), \\ \sqrt{q}D_q^2(\varrho(x,y)a_{22}(y)) + q^{-1}D_{q^{-1}}^1(\varrho(x,y)a_{12a}(x,y)) = \varrho(x,y)b_2(x,y), \\ q^4D_q^1D_q^2(\varrho(x,y)a_{12d}(x,y)) = D_{q^{-1}}^1D_{q^{-1}}^2(\varrho(x,y)a_{12a}(x,y)), \end{cases}$$

or equivalently,

(4.8) 
$$\begin{cases} D_q^1(\omega_1(x,y)\varrho(x,y)) = \frac{1}{q}D_{q^{-1}}^1(\phi_1(x,y)\varrho(x,y)), \\ D_q^2(\omega_2(x,y)\varrho(x,y)) = \frac{1}{q}D_{q^{-1}}^2(\phi_2(x,y)\varrho(x,y)), \\ q^4D_q^1D_q^2(\varrho(x,y)a_{12d}(x,y)) = D_{q^{-1}}^1D_{q^{-1}}^2(\varrho(x,y)a_{12a}(x,y)). \end{cases}$$

4.1. Computation of the weight function. By introducing the functions

(4.9) 
$$\mathcal{G}_1(x,y) = \frac{\phi_1(x,y)}{\omega_1(qx,y)}, \qquad \mathcal{G}_2(x,y) = \frac{\phi_2(x,y)}{\omega_2(x,qy)},$$

where  $\phi_j(x, y)$  and  $\omega_j(x, y)$  are defined in (4.6), j = 1, 2, and using the *q*-Pearson's system (4.5), one obtains that

(4.10) 
$$\varrho(qx,y) = \mathcal{G}_1(x,y)\varrho(x,y), \quad \varrho(x,qy) = \mathcal{G}_2(x,y)\varrho(x,y),$$

$$(4.11) \quad q^2 \mathcal{G}_1(x,y) \mathcal{G}_2(qx,y) a_{12d}(qx,qy) = a_{12a}(x,y) = q^2 \mathcal{G}_1(x,qy) \mathcal{G}_2(x,y) a_{12d}(qx,qy),$$

(4.12) 
$$y\mathcal{G}_2(x,y)D_q^2(\mathcal{G}_1(x,y)) = x\mathcal{G}_1(x,y)D_q^1(\mathcal{G}_2(x,y)).$$

From (4.10) it follows then that

ar

$$\frac{\ln\left[\varrho(qx,y)\right] - \ln\left[\varrho(x,y)\right]}{(q-1)x} = \frac{\ln\left[\mathcal{G}_1(x,y)\right]}{(q-1)x},$$

or, equivalently,

$$D_q^1 [\ln(\varrho(x, y))] = \ln \left[ (\mathcal{G}_1(x, y))^{1/((q-1)x)} \right],$$

and therefore

(4.13) 
$$D_q^1 \left[ \ln(\varrho(x,y)) \right] - D_q^1 \left[ \ln \varrho(x,y_0) \right] = \frac{1}{(q-1)x} \ln \left[ \frac{\mathcal{G}_1(x,y)}{\mathcal{G}_1(x,y_0)} \right].$$

Upon using the q-integral due to J. Thomae [50] and F.H. Jackson [24] (see also [14, 23, 26]), this yields

$$\int_{x_0}^x D_q^1 \left[ \ln(\varrho(s, y)) \right] d_q s = \ln \left[ \varrho(x, y) \right] - \ln \left[ \varrho(x_0, y) \right],$$

and

$$(4.14) \quad \ln\left[\varrho(x,y)\right] - \ln\left[\varrho(x_0,y)\right] = \int_{x_0}^x \ln\left[\left(\mathcal{G}_1(s,y)\right)^{1/((q-1)s)}\right] d_q s$$
$$= (1-q)x \sum_{j=0}^\infty q^j \ln\left[\left(\mathcal{G}_1(q^j x,y)\right)^{1/((q-1)q^j x)}\right] - (1-q)x_0 \sum_{j=0}^\infty q^j \ln\left[\left(\mathcal{G}_1(q^j x_0,y)\right)^{1/((q-1)q^j x_0)}\right]$$
$$= \sum_{j=0}^\infty \ln\left[\frac{\mathcal{G}_1(q^j x_0,y)}{\mathcal{G}_1(q^j x,y)}\right] + c_1(y).$$

In a similar way, one has

(4.15) 
$$\ln \left[ \varrho(x,y) \right] - \ln \left[ \varrho(x,y_0) \right] = \sum_{j=0}^{\infty} \ln \left[ \frac{\mathcal{G}_2(x,q^j y_0)}{\mathcal{G}_2(x,q^j y)} \right] + c_2(x).$$

From (4.12) we deduce that

$$\frac{(q-1)xD_q^1(\mathcal{G}_2(x,yq^j))}{\mathcal{G}_2(x,yq^j)} = \frac{(q-1)yq^jD_q^2(\mathcal{G}_1(x,yq^j))}{\mathcal{G}_1(x,yq^j)}.$$

Upon using

$$(q-1)xD_q^1(\ln|f|) = \ln\left[\frac{(q-1)xD_q^1f(x,y)}{f(x,y)} + 1\right],$$

and then applying the operator  $D_q^1$  to (4.15), from (4.13) one obtains that

$$\begin{split} \ln\left[\frac{\mathcal{G}_{1}(x,y)}{\mathcal{G}_{1}(x,y_{0})}\right] \\ &= \sum_{j=0}^{\infty} (q-1)x D_{q}^{1}[\ln(\mathcal{G}_{2}(x,q^{j}y_{0}))] - (q-1)x D_{q}^{1}[\ln(\mathcal{G}_{2}(x,q^{j}y))] + (q-1)x D_{q}^{1}(c_{2}(x)) \\ &= \sum_{j=0}^{\infty} \ln\left[\frac{(q-1)x D_{q}^{1} \mathcal{G}_{2}(x,q^{j}y_{0})}{\mathcal{G}_{2}(x,q^{j}y_{0})} + 1\right] - \ln\left[\frac{(q-1)x D_{q}^{1} \mathcal{G}_{2}(x,q^{j}y)}{\mathcal{G}_{2}(x,q^{j}y)} + 1\right] + (q-1)x D_{q}^{1}(c_{2}(x)) \\ &= \sum_{j=0}^{\infty} \ln\left[\frac{(q-1)y_{0}q^{j} D_{q}^{2} \mathcal{G}_{1}(x,q^{j}y_{0})}{\mathcal{G}_{1}(x,q^{j}y_{0})} + 1\right] - \ln\left[\frac{(q-1)yq^{j} D_{q}^{2} \mathcal{G}_{1}(x,q^{j}y)}{\mathcal{G}_{1}(x,q^{j}y)} + 1\right] + (q-1)x D_{q}^{1}(c_{2}(x)) \\ &= \sum_{j=0}^{\infty} (q-1)y_{0}q^{j} D_{q}^{2}(\ln(\mathcal{G}_{1}(x,q^{j}y_{0}))) - (q-1)yq^{j} D_{q}^{2}(\ln(\mathcal{G}_{1}(x,q^{j}y))) + (q-1)x D_{q}^{1}(c_{2}(x)) \\ &= \int_{y_{0}}^{y} D_{q}^{2}(\ln(\mathcal{G}_{1}(x,t))) d_{q}t + (q-1)x D_{q}^{1}(c_{2}(x)) \\ &= \ln(\mathcal{G}_{1}(x,y)) - \ln(\mathcal{G}_{1}(x,y_{0})) + (q-1)x D_{q}^{1}(c_{2}(x)), \end{split}$$

and therefore

$$(q-1)xD_q^1(c_2(x)) = 0,$$

which implies that  $c_2(x) = c_3$  is a constant. In a similar way one verifies that  $c_1(y) = c_4$  is also constant.

Substituting in (4.14) and (4.15),

$$\ln\left[\varrho(x,y)\right] - \ln\left[\varrho(x_0,y)\right] = \sum_{j=0}^{\infty} \ln\left[\frac{\mathcal{G}_1(q^j x_0,y)}{\mathcal{G}_1(q^j x,y)}\right] + c_4,$$
$$\ln\left[\varrho(x,y)\right] - \ln\left[\varrho(x,y_0)\right] = \sum_{j=0}^{\infty} \ln\left[\frac{\mathcal{G}_2(x,q^j y_0)}{\mathcal{G}_2(x,q^j y)}\right] + c_3,$$

we finally obtain (up to a multiplicative constant) the explicit expression for the weight function solution of the q-Pearson's system of equations (4.5),

(4.16) 
$$\varrho(x,y) = \prod_{j=0}^{\infty} \frac{\mathcal{G}_1(q^j x_0, y) \, \mathcal{G}_2(x_0, q^j y_0)}{\mathcal{G}_1(q^j x, y) \, \mathcal{G}_2(x_0, q^j y)},$$

where  $\mathcal{G}_1(x, y)$  and  $\mathcal{G}_2(x, y)$  are defined in (4.9).

In a similar way one can obtain the following representation for the orthogonality weight function, associated with the q-derivatives of any order:

(4.17) 
$$\varrho^{(k,\ell)}(x,y) = \prod_{j=0}^{\infty} \frac{\mathcal{G}_1^{(k,\ell)}(q^j x_0, y) \, \mathcal{G}_2^{(k,\ell)}(x_0, q^j y_0)}{\mathcal{G}_1^{(k,\ell)}(q^j x, y) \, \mathcal{G}_2^{(k,\ell)}(x_0, q^j y)}$$

Here  $\mathcal{G}_{1}^{(k,\ell)}(x,y)$  and  $\mathcal{G}_{2}^{(k,\ell)}(x,y)$  are defined by inserting into (4.9) the polynomial coefficients  $a_{11}^{(k,\ell)}(x,y)$ ,  $a_{22}^{(k,\ell)}(x,y)$ ,  $a_{12a}^{(k,\ell)}(x,y)$ ,  $a_{12d}^{(k,\ell)}(x,y)$ ,  $b_{1}^{(k,\ell)}(x,y)$ , and  $b_{2}^{(k,\ell)}(x,y)$ , introduced in Section 2 and given explicitly in terms of the coefficients of the initial equation (3.3) in (2.11)–(2.14). It is important to note here that, for example,  $\varrho^{(1,1)}(x,y)$  can be computed in two ways: as the  $D_q^1$  derivative of the  $D_q^2$  derivative or vice versa. The following relation ensures that one arrives at the same result:

(4.18) 
$$\omega_1^{(k,\ell+1)}(qx,y)\omega_2^{(k,\ell)}(qx,qy) = \omega_2^{(k+1,\ell)}(x,qy)\omega_1^{(k,\ell)}(qx,qy), \qquad k,\ell \ge 0.$$

We shall refer to the latter equation as the coupling hypergeometric condition, analogous to [44, Eq. (50)].

## 5. Rodrigues' formula

Rodrigues' formula for classical orthogonal polynomials in one variable is an important tool for analyzing the fundamental properties of these polynomials [10, 12, 39]. The great advantage of the Rodrigues' formula is its form as *n*th derivative of the orthogonality weight function. In [48], an analogue of the Rodrigues' formula for orthogonal polynomials over a domain in two variables, which are solutions of admissible and potentially self-adjoint equations, is presented. Kwon et al. [25] succeeded in deriving a (functional) Rodrigues-type formula for multivariable orthogonal polynomial solutions of a secondorder partial differential equation. In recent papers appropriate Rodrigues' formulae for polynomials solutions of second-order admissible, hypergeometric and potentially selfadjoint partial differential and difference equations have been presented [5, 44].

By using the results of the previous sections in a similar vein as was elaborated by Suetin [48, Theorem 3, p. 151] for the continuous case, it is not hard to arrive at an explicit expression for a polynomial solution of an admissible potentially self-adjoint second-order partial q-difference equation of the hypergeometric type (3.3). The expression

(5.1) 
$$\tilde{P}_{n,m}(x,y) = \frac{\Lambda_{n,m}}{\varrho(x,y)} [D_{q^{-1}}^{1}]^{(n)} [D_{q^{-1}}^{2}]^{(m)} \left[\varrho^{(n,m)}(x,y)\right]$$
$$= \frac{q^{n(1-n)/2+m(1-m)/2}\Lambda_{n,m}}{\varrho(x,y)} [D_{q}^{1}]^{(n)} [D_{q}^{2}]^{(m)} \left[\varrho^{(n,m)}(q^{-n}x,q^{-m}y)\right]$$
$$= \frac{q^{n(1-n)/2+m(1-m)/2}\Lambda_{n,m}}{\varrho(x,y)} [D_{q}^{1}]^{(n)} [D_{q}^{2}]^{(m)} \left[\varrho(x,y)\prod_{k=0}^{n-1}\omega_{1}(q^{-k}x,y)\prod_{s=0}^{m-1}\omega_{2}(x,q^{-s}y)\right],$$

defines an algebraic polynomial of total degree n+m in the variables x and y, called Rodrigues' formula for the bivariate q-orthogonal polynomials  $\tilde{P}_{n,m}(x, y)$ , that are solutions of (3.3). In (5.1) the  $\Lambda_{n,m}$  are normalizing constants,  $\varrho(x, y)$  and  $\varrho^{(n,m)}(x, y)$  are defined by (4.16) and (4.17), respectively, and  $\omega_1(x, y)$  and  $\omega_2(x, y)$  are defined in (4.6). In the limit as q tends to 1, the Rodrigues' formula (5.1) reduces to the one derived in [48] for the continuous case. Moreover, in the bivariate discrete case a Rodrigues' formula has been also given in [44] upon employing the same approach as in [48].

## 6. Monic orthogonal polynomial solutions

One essential difference between polynomials in one variable and in several variables is the lack of an obvious basis in the latter [13]. One possibility to avoid this problem is to consider graded lexicographical order and use the matrix vector representation, first introduced by Kowalski [30, 31] and later on studied by Xu [55, 56].

Let  $\mathbf{x} = (x, y) \in \mathbb{R}^2$ , and let  $\mathbf{x}^n$   $(n \in \mathbb{N}_0)$  denote the column vector of the monomials  $x^{n-k}y^k$ , whose elements are arranged in graded lexicographical order (see [13, p. 32]):

(6.1) 
$$\mathbf{x}^n = (x^{n-k}y^k), \quad 0 \le k \le n, \quad n \in \mathbb{N}_0.$$

Let  $\{P_{n-k,k}^n(x,y)\}$  be a sequence of polynomials in the space  $\Pi_n^2$  of all polynomials of total degree at most n in two variables,  $\mathbf{x} = (x, y)$ , with real coefficients. Such polynomials are finite sums of terms of the form  $ax^{n-k}y^k$ , where  $a \in \mathbb{R}$ .

From now on  $\mathbb{P}_n$  will denote the (column) polynomial vector

(6.2) 
$$\mathbb{P}_n = (P_{n,0}^n(x,y), P_{n-1,1}^n(x,y), \dots, P_{1,n-1}^n(x,y), P_{0,n}^n(x,y))^{\mathrm{T}}.$$

Then, each polynomial vector  $\mathbb{P}_n$  can be written in terms of the basis (6.1) as:

(6.3) 
$$\mathbb{P}_n = G_{n,n} \mathbf{x}^n + G_{n,n-1} \mathbf{x}^{n-1} + \dots + G_{n,0} \mathbf{x}^0,$$

where  $G_{n,j}$  are  $(n+1) \times (j+1)$ -matrices and  $G_{n,n}$  is a nonsingular square matrix of the size  $(n+1) \times (n+1)$ .

A polynomial vector  $\widehat{\mathbb{P}}_n$  is said to be monic if its leading matrix coefficient  $\widehat{G}_{n,n}$  is the identity matrix (of the size  $(n+1) \times (n+1)$ ), that is,

(6.4) 
$$\widehat{\mathbb{P}}_n = \mathbf{x}^n + \widehat{G}_{n,n-1}\mathbf{x}^{n-1} + \dots + \widehat{G}_{n,0}\mathbf{x}^0.$$

Then each of its polynomial entries  $\widehat{P}_{n-k,k}^n(x,y)$  are of the form:

(6.5) 
$$\widehat{P}_{n-k,k}^{n}(x,y) = x^{n-k}y^{k} + \text{terms of lower total degree}.$$

In what follows the "hat" notation  $\widehat{\mathbb{P}}_n$  will represent monic polynomials.

The existence of a recurrence relation for any vector of bivariate discrete orthogonal polynomial family can be established in more general settings than those considered here [57]. The following existence theorem, proved in [13], can be applied for infinite or finite (n = 0, 1, ..., N) sequences of polynomials (see examples 4.1 and 4.2 in [57]), because we are using graded lexicographical order (6.1).

**Theorem 6.1.** Let  $\mathcal{L}$  be a positive definite moment linear functional acting on the space  $\Pi_n^2$  of all polynomials of total degree at most n in two variables, and  $\{\mathbb{P}_n\}_{n\geq 0}$  be an orthogonal family with respect to  $\mathcal{L}$ . Then, for  $n \geq 0$ , there exist unique matrices  $A_{n,j}$  of the size  $(n+1) \times (n+2)$ ,  $B_{n,j}$  of the size  $(n+1) \times (n+1)$ , and  $C_{n,j}$  of the size  $(n+1) \times n$ , such that

(6.6) 
$$x_{j}\mathbb{P}_{n} = A_{n,j}\mathbb{P}_{n+1} + B_{n,j}\mathbb{P}_{n} + C_{n,j}\mathbb{P}_{n-1}, \quad j = 1, 2,$$

with the initial conditions  $\mathbb{P}_{-1} = 0$  and  $\mathbb{P}_0 = 1$ . Here the notation  $x_1 = x$  and  $x_2 = y$  is used.

In this section we give explicit expressions for the matrices  $A_{n,j}$ ,  $B_{n,j}$  and  $C_{n,j}$ , which appear in the three-term recurrence relations (6.6), in terms of the coefficients of  $a_{ii}$ ,  $a_{12j}$ and  $b_i$  in (3.3). These matrices enable one to compute the monic orthogonal polynomial solutions of an admissible potentially self-adjoint second-order partial q-difference equation of the hypergeometric type.

The weight function (4.16) determines the moment linear functional  $\mathcal{L}$ , defined in the space  $\Pi_n^2$  of all polynomials of total degree at most n in two variables, in terms of a double q-integral

$$\mathcal{L}(P) = \iint_G P(x, y)\varrho(x, y)d_q x d_q y,$$

in an appropriate domain  $G \subset \mathbb{R}^2$ , which can be applied to polynomial vectors. Thus, in what follows  $\{\widehat{\mathbb{P}}_n\}_{n\in\mathbb{N}_0}$  denotes a monic vector polynomial family solution of (3.3), that is orthogonal with respect to  $\varrho(x, y)$ ,

(6.7) 
$$\mathcal{L}(\mathbf{x}^m \ \widehat{\mathbb{P}}_n^T) = \iint_G \mathbf{x}^m \ \widehat{\mathbb{P}}_n^T \varrho(x, y) d_q x d_q y = \begin{cases} 0 \in \mathcal{M}^{(m+1, n+1)}, \text{ if } n > m, \\ H_n \in \mathcal{M}^{(n+1, n+1)}, \text{ if } m = n, \end{cases}$$

where  $H_n$  (of the size  $(n+1) \times (n+1)$ ) is nonsingular.

Let us first introduce the matrices  $L_{n,j}$  of the size  $(n+1) \times (n+2)$ 

(6.8) 
$$L_{n,1} = \begin{pmatrix} 1 & \bigcirc & 0 \\ & \ddots & & \vdots \\ & \bigcirc & 1 & 0 \end{pmatrix}$$
 and  $L_{n,2} = \begin{pmatrix} 0 & 1 & \bigcirc \\ \vdots & & \ddots & \\ & 0 & \bigcirc & 1 \end{pmatrix}$ 

so that

(6.9) 
$$x \mathbf{x}^n = L_{n,1} \mathbf{x}^{n+1}, \quad y \mathbf{x}^n = L_{n,2} \mathbf{x}^{n+1}.$$

Observe that

(6.10) 
$$x^{2} \mathbf{x}^{n} = L_{n,1} L_{n+1,1} \mathbf{x}^{n+2}, \quad y^{2} \mathbf{x}^{n} = L_{n,2} L_{n+1,2} \mathbf{x}^{n+2},$$
$$L_{n,2} L_{n+1,1} = L_{n,1} L_{n+1,2},$$

and for j = 1, 2,

(6.11) 
$$L_{n,j} L_{n,j}^{\mathrm{T}} = I_{n+1},$$

where  $I_{n+1}$  denotes the identity matrix of the size n+1.

From the definition of the partial q-difference operators in (2.2) and (2.3), one obtains that

$$D_q^j \mathbf{x}^n = \mathbb{E}_{n,j} \mathbf{x}^{n-1}, \quad D_{q^{-1}}^j \mathbf{x}^n = \mathbb{K}_{n,j} \mathbf{x}^{n-1}, \qquad j = 1, 2,$$

where the matrices  $\mathbb{E}_{n,j}$  of the size  $(n+1) \times n$  are given by

the matrices  $\mathbb{K}_{n,j}$  of the size  $(n+1) \times n$  are given by (6.13)

and the q-number is defined in (2.16).

Substitute the expansion (6.4) into (3.3) and then equate the coefficients of  $\mathbf{x}^{n-1}$  and  $\mathbf{x}^{n-2}$  to arrive at the following explicit expressions for the matrices  $\widehat{G}_{n,n-1}$  and  $\widehat{G}_{n,n-2}$ :

(6.14) 
$$\widehat{G}_{n,n-1} = \mathbb{S}_n \mathbb{F}_{n-1}^{-1}(\lambda_n),$$

(6.15) 
$$\widehat{G}_{n,n-2} = \left(\mathbb{T}_n + \widehat{G}_{n,n-1}\mathbb{S}_{n-1}\right)\mathbb{F}_{n-2}^{-1}(\lambda_n),$$

where the nonsingular matrix  $\mathbb{F}_n(\lambda_\ell)$  is given by

(6.16) 
$$\mathbb{F}_n(\lambda_\ell) = (\lambda_n - \lambda_\ell) \mathbb{I}_{n+1}$$

 $\lambda_n$  is given in (3.2),  $\mathbb{I}_{n+1}$  denotes the identity matrix of the size  $(n+1) \times (n+1)$ , and the matrix  $\mathbb{S}_n$  of the size  $(n+1) \times n$  is given in terms of the coefficients of the polynomials  $a_{ii}$ ,  $a_{12i}$  and  $b_i$  from the partial q-difference equation (3.3), explicitly written down in (3.4), as

(6.17) 
$$\mathbb{S}_{n} = \begin{pmatrix} s_{1,1} & & & & \\ s_{2,1} & s_{2,2} & & & \\ & \ddots & \ddots & & \\ & & s_{n-1,n-2} & s_{n-1,n-1} & \\ & & & s_{n,n-1} & s_{n,n} \\ & & & & 0 & s_{n+1,n} \end{pmatrix} \quad (n \ge 1) \,.$$

Here, for  $1 \leq i \leq n$ ,

$$s_{i,i} = [n-i+1]_q \left( g_1 + q^{1+i-n} b_1 [n-i]_q + c_{3a} [i-1]_q + q^{2-n} c_{3d} [i-1]_q \right) ,$$
  
$$s_{i+1,i} = [i-1]_q \left( b_2 q^{3-i} [i-2]_q + [n+1-i]_q \left( b_{3a} + b_{3d} q^{2-n} \right) + g_2 \right) .$$

Besides, the matrix  $\mathbb{T}_n$  of the size  $(n+1) \times (n-1)$ , which appears in (6.15), is given in terms of the coefficients of the partial q-difference equation (3.3) as

(6.18) 
$$\mathbb{T}_{n} = d_{3a}\mathbb{E}_{n,2}\mathbb{E}_{n-1,1} + c_{1}q\mathbb{K}_{n,1}\mathbb{E}_{n-1,1} + c_{2}q\mathbb{K}_{n,2}\mathbb{E}_{n-1,2} + d_{3d}\mathbb{K}_{n,2}\mathbb{K}_{n-1,1} \quad (n \ge 2),$$

where the matrices  $\mathbb{E}_{n,i}$  and  $\mathbb{K}_{n,i}$ , i = 1, 2, are defined by (6.12) and (6.13), respectively. Now, in this monic situation, it is possible to generalize the well-known explicit expressions for the coefficients in the three-term recurrence relation for the one variable case [39, p. 14] to the *q*-bivariate case. This is achieved with the aid of the auxiliary matrices  $L_{n,j}$ , defined in (6.8) and (6.9), and the following result, proved in [5] in the continuous bivariate situation and therefore valid also in this *q*-bivariate situation, because it is a consequence of the three-term recurrence relations (6.6).

**Theorem 6.2.** In the monic case, the explicit expressions of the matrices  $A_{n,j}$ ,  $B_{n,j}$  and  $C_{n,j}$  (j = 1, 2), that appear in (6.6) in terms of the values of the leading coefficients  $\widehat{G}_{n,n-1}$  and  $\widehat{G}_{n,n-2}$  (see (6.14) and (6.15), respectively), are given by

(6.19) 
$$\begin{cases} A_{n,j} = L_{n,j}, \quad n \ge 0, \\ B_{0,j} = -L_{0,j}\widehat{G}_{1,0}, \quad B_{n,j} = \widehat{G}_{n,n-1}L_{n-1,j} - L_{n,j}\widehat{G}_{n+1,n}, \quad n \ge 1, \\ C_{1,j} = -(L_{1,j}\widehat{G}_{2,0} + B_{1,j}\widehat{G}_{1,0}), \\ C_{n,j} = \widehat{G}_{n,n-2}L_{n-2,j} - L_{n,j}\widehat{G}_{n+1,n-1} - B_{n,j}\widehat{G}_{n,n-1}, \quad n \ge 2, \end{cases}$$

where the matrices  $L_{n,j}$  have been introduced in (6.8).

It is of interest to remark here that, as it is described in [13], since

(6.20) 
$$\operatorname{rank}(L_{n,j}) = n + 1 = \operatorname{rank}(C_{n+1,j}), \quad j = 1, 2, \quad n \ge 0,$$

the columns of the joint matrices

$$L_n = (L_{n,1}^T, L_{n,2}^T)^T$$
 and  $C_n = (C_{n,1}^T, C_{n,2}^T)^T$ ,

of the size  $(2n+2) \times (n+2)$  and  $(2n+2) \times n$ , respectively, are linearly independent, that is,

(6.21) 
$$\operatorname{rank}(L_n) = n + 2, \quad \operatorname{rank}(C_n) = n$$

Therefore the matrix  $L_n$  has full rank, so that there exists a unique matrix  $D_n^{\dagger}$  of the size  $(n+2) \times (2n+2)$ , called the generalized inverse of  $L_n$ ,

(6.22) 
$$D_n^{\dagger} = (D_{n,1}|D_{n,2}) = (L_n^T L_n)^{-1} L_n^T$$

such that

$$D_n^{\dagger}L_n = I_{n+2}.$$

Moreover, using the left inverse  $D_n^{\dagger}$  of the joint matrix  $L_n$ ,

$$D_n^{\dagger} = \begin{pmatrix} 1 & & 0 & & \\ & 1/2 & \bigcirc & 1/2 & \bigcirc & \\ & \ddots & & \ddots & & \\ & \bigcirc & & 1/2 & \bigcirc & & 1/2 & \\ & & & & 0 & & & 1 \end{pmatrix},$$

one can write a recursive formula for the monic orthogonal polynomials

(6.23) 
$$\widehat{\mathbb{P}}_{n+1} = D_n^{\dagger} \left[ \begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n \right] \widehat{\mathbb{P}}_n - D_n^{\dagger} C_n \widehat{\mathbb{P}}_{n-1}, \quad n \ge 0,$$

with the initial conditions  $\widehat{\mathbb{P}}_{-1} = 0$ ,  $\widehat{\mathbb{P}}_0 = 1$ . In (6.23) the symbol  $\otimes$  denotes the Kronecker product and

(6.24) 
$$B_n = (B_{n,1}^T, B_{n,2}^T)^T, \quad C_n = (C_{n,1}^T, C_{n,2}^T)^T,$$

are matrices of the size  $(2n + 2) \times (n + 1)$  and  $(2n + 2) \times n$ , respectively, which can be obtained by using (6.19) in terms of the coefficients of the partial q-difference equation (3.3), explicitly given in (3.4). This means that the recurrence (6.23) gives another realisation of [13, (3.2.10)], already appeared in the bivariate discrete case in [45].

Therefore, from (6.23) it is possible to compute a monic orthogonal polynomial solution of an admissible potentially self-adjoint linear second-order partial q-difference equation of the hypergeometric type (3.3).

## 7. Illustrative example

In this section we discuss in detail an example, which is related to the admissible potentially self-adjoint linear second-order partial q-difference equation of the hypergeometric type, satisfied by a (non-monic) bivariate extension of big q-Jacobi polynomials, recently introduced in [34, 35]. The monic orthogonal polynomial solutions are expressed by means of the three-term recurrence relations, that govern them, and also explicitly in terms of generalized bivariate basic hypergeometric series. Moreover, a third (non-monic) solution of the same partial q-difference equation is provided by using the Rodrigues' representation (5.1). In the limit when  $q \uparrow 1$  the partial q-difference equation reduces to a second-order partial differential equation of the hypergeometric type, with monic Appell polynomials as solutions. Limit relations between the three orthogonal polynomial solutions of the partial q-difference equation are explicitly given. Besides, the corresponding orthogonality weight functions are also linked by appropriate limit relations as  $q \uparrow 1$ .

**Example.** Lewanowicz and Woźny [34] have recently introduced the following bivariate extension of the big q-Jacobi polynomials

(7.1) 
$$P_{n,k}(x,y;a,b,c,d;q) := P_{n-k}(y;a,bcq^{2k+1},dq^k;q)y^k(dq/y;q)_kP_k(x/y;c,b,d/y;q),$$
$$n \in \mathbb{N}, \quad k = 0, 1, \dots, n, \quad q \in (0,1), \quad 0 < aq, bq, cq < 1, \quad d < 0,$$

where the univariate big q-Jacobi polynomials [3] (see also [26, Eq. (19.5.1)]) are defined by means of the  $_{3}\phi_{2}$  basic hypergeometric series as

$$P_m(t; A, B, C; q) := {}_{3}\phi_2 \begin{pmatrix} q^{-m}, ABq^{m+1}, t \\ Aq, Cq \end{pmatrix}, \ 0 < A < 1/q, \ 0 < B < 1/q, \ C < 0.$$

They have demonstrated that the polynomials (7.1) satisfy a linear second-order partial q-difference equation of the form (3.3), with coefficients given by

$$(7.2) \begin{cases} a_{11}(x) = \sqrt{q} \ (d q - x) \ (a c q^{2} - x), \quad a_{22}(y) = \sqrt{q} \ (a q - y) \ (d q - y), \\ a_{12a}(x, y) = a c q^{4} \ (d - b x) \ (1 - y), \quad a_{12d}(x, y) = (d q - x) \ (a q - y), \\ b_{1}(x) = \frac{q \ (q \ (d - a c d q^{2} + a c q \ (1 + b q \ (-1 + x))) - x))}{-1 + q}, \\ b_{2}(y) = \frac{q \ (d q + a q \ (1 - d q + b c q^{2} \ (-1 + y)) - y)}{-1 + q}, \\ \lambda_{n} = q^{2-n} [n]_{q} \frac{(1 - abcq^{n+2})}{q - 1}. \end{cases}$$

Thus this q-difference equation is admissible, potentially self-adjoint and of the hypergeometric type.

In [34] it has been proved that the polynomials (7.1) satisfy the following orthogonality relation

(7.3) 
$$\int_{dq}^{aq} \int_{dq}^{cqy} W(x,y;a,b,c,d;q) P_{n,k}(x,y;q) P_{m,l}(x,y;q) d_q x d_q y = H_{n,k}(a,b,c,d;q) \delta_{n,m} \delta_{k,l},$$

where 0 < aq, bq, cq < 1, d < 0, and the weight function is defined by

(7.4) 
$$W(x,y;a,b,c,d;q) := \frac{(dq/y,c^{-1}x/y,x/d,y/a,y/d;q)_{\infty}}{y(c^{-1}d/y,cqy/d,x/y,bx/d,y;q)_{\infty}}$$

Here the q-shifted factorial  $(a; q)_k$  is equal to

$$(a; q)_0 = 1,$$
  $(a; q)_k = \prod_{j=0}^{k-1} (1 - aq^j),$   $(k = 1, 2, \dots, \text{ or } \infty).$ 

and we have employed the conventional notation

$$(a_1,\ldots,a_r; q)_k = (a_1; q)_k \cdots (a_r; q)_k$$

for products of q-shifted factorials.

It is easy to check that the W(x, y; a, b, c, d; q) is a solution of the q-Pearson's system (4.5), which has been presented in a different form in [35, Lemma 3.1] for this particular example. From the explicit expression for the weight function (4.16), we have (up to a normalization constant), upon taking into account that  $x_0 = dq$ ,

(7.5) 
$$\varrho(x,y) = \frac{(y/a, x/d, dq/y, x/(cy), y/d; q)_{\infty}}{(y, x/y, bx/d, dq/(cy), cy/d; q)_{\infty}},$$

which coincides with the weight function, given in (7.4), up to the positive multiplicative constant -d/c. Observe that equations (4.11), (4.12), and (4.18) also hold.

The partial q-difference (3.3) with polynomial coefficients (7.2) has another (nonmonic) orthogonal polynomial solution which can be computed from the Rodrigues' formula (5.1) as

(7.6) 
$$\tilde{P}_{n,m}(x,y;a,b,c,d;q) = \frac{\Lambda_{n,m}}{\varrho(x,y)} \times [D_q^1]^{(n)} [D_q^2]^{(m)} \left[ \varrho(x,y) x^{2n} y^{2m} (dq/x;q)_n (aq/y;q)_m (x/y;q)_m (cqy/x;q)_n \right],$$

where  $\rho(x, y)$  is given in (7.5) and  $\Lambda_{n,m}$  is a normalizing constant.

The partial q-difference equation (3.3), with coefficients given in (7.2), has a third (monic) orthogonal polynomial solution, which can be computed recursively from Theorem 6.2. From (6.19) it follows that the matrix  $B_{n,1}$  has entries (7.7)

$$\begin{cases} \frac{dq^{-i+n+2} \left(acq^{2i-1} \left(q^{-i+n+1} \left(b \left(acq^{i+n+1} + q^{-i+n+2} - q - 1\right) - q - 1\right) + 1\right) + 1\right)}{\left(abcq^{2n+1} - 1\right) \left(abcq^{2n+3} - 1\right)} \\ + \frac{acq^{n+2} \left(-b(q+1)q^{n-i} \left(acq^{2i} + q\right) + ab(b+1)cq^{2n+2} + b + 1\right)}{\left(abcq^{2n+1} - 1\right) \left(abcq^{2n+3} - 1\right)}, \quad (i = j), \\ - \frac{\left(q^{i-1} - 1\right) \left(aq^{i-1} - 1\right) q^{-i+n+2} \left(abcdq^{2n+2} - abc(q+1)q^{n+1} + d\right)}{\left(abcq^{2n+1} - 1\right) \left(abcq^{2n+3} - 1\right)}, \quad (i = j + 1), \\ 0, \quad \text{otherwise}, \end{cases}$$

the matrix 
$$B_{n,2}$$
 has entries  
(7.8)  

$$\begin{cases}
q^{-i} \left( \frac{q[i]_q (aq^i - 1) (dq^{n+1} - 1)}{abcq^{2n+3} - 1} + \frac{q[i - 1]_q (q - aq^i) (dq^n - 1)}{abcq^{2n+1} - 1} + q \right), & (i = j), \\
- \frac{acq^{i+1} (q^{-i+n+1} - 1) (bq^{-i+n+1} - 1) (abcq^{2(n+1)} - d(q + 1)q^n + 1)}{(abcq^{2n+1} - 1) (abcq^{2n+3} - 1)}, & (i = j - 1), \\
0, & \text{otherwise,} 
\end{cases}$$

the matrix  $C_{n,1}$  has entries (7.9)

$$\begin{cases} -\frac{acq^{n+2} (dq^n - 1) (q^{-i+n+1} - 1) (bq^{-i+n+1} - 1)}{(abcq^{2n} - 1) (abcq^{2n+1} - 1)^2 (abcq^{2n+2} - 1)} \\ \times (acq^{i+n} - 1) (abcq^{n+1} - d) (abcq^{i+n} - 1), \quad (i = j), \\ -\frac{ac (q^{i-1} - 1) (aq^{i-1} - 1) (dq^n - 1) q^{-i+2n+3} (abcq^{n+1} - d)}{(abcq^{2n} - 1) (abcq^{2n+1} - 1)^2 (abcq^{2n+2} - 1)} \\ \times (-b(q + 1)q^{-i+n-1} (acq^{2i} + q^2) + ab(b + 1)cq^{2n+1} + b + 1), \quad (i = j + 1), \\ -\frac{abc (aq^i - q) (aq^i - q^2) (dq^n - 1) q^{-2i+3n+2} (q; q)_{i-1} (abcq^{n+1} - d)}{(q; q)_{i-3} (abcq^{2n} - 1) (abcq^{2n+1} - 1)^2 (abcq^{2n+2} - 1)}, \quad (i = j + 2), \\ 0, \quad \text{otherwise}, \end{cases}$$

and the matrix  $C_{n,2}$  has entries (7.10)

$$\begin{cases} -\frac{ac(q-1)(dq^{n}-1)q^{n-i}[-i+n+1]_{q}(bq^{n+1}-q^{i})(abcq^{n+1}-d)}{(abcq^{2n}-1)(abcq^{2n+1}-1)^{2}(abcq^{2n+2}-1)}, \\ \times \left((a+1)q^{i+1}(abcq^{2n+1}+1)-abc(q+1)q^{2n+2}-a(q+1)q^{2i}\right), \quad (i=j), \\ -\frac{a(q^{i-1}-1)q^{n+1}(aq^{i-1}-1)(dq^{n}-1)(bcq^{-i+2n+2}-1)}{(abcq^{2n}-1)(abcq^{2n+1}-1)^{2}(abcq^{2n+2}-1)} \\ \times \left(abcq^{n+1}-d\right)(abcq^{-i+2n+2}-1), \quad (i=j+1), \\ -\frac{a^{2}c^{2}q^{n+2}(dq^{n}-1)(q^{i}-bq^{n})(q^{i}-bq^{n+1})(q;q)_{n-i+1}(abcq^{n+1}-d)}{(q;q)_{n-i-1}(abcq^{2n}-1)(abcq^{2n+1}-1)^{2}(abcq^{2n+2}-1)}, \quad (i=j-1), \\ 0, \quad \text{otherwise.} \end{cases}$$

From (6.23) each column polynomial vector (6.2) can be obtained by using the above matrices  $B_{n,i}$  and  $C_{n,i}$ , i = 1, 2.

Remark 1. In [34] the authors obtained the matrices of the three-term recurrence relations (6.6), satisfied by the non-monic solution (7.1), by using the recurrence relation that governs the univariate q-Jacobi polynomials. Notice that we have derived the matrices in the monic case from our approach given in Section 6, with different shapes as in the non-monic case.

Remark 2. Observe that the monic polynomial solution  $\widehat{P}_{n,m}(x, y; a, b, c, d; q)$  of the partial q-difference equation (3.3), with coefficients, given in (7.2) and obtained from Theorem 6.2 (with the matrix coefficients that are given above), can be also written in terms of generalized bivariate basic hypergeometric series as

$$(7.11) \quad \widehat{P}_{n,m}(x,y;a,b,c,d;q) = \frac{\left(\frac{d}{b}\right)^{n} (aq;q)_{m}(bq;q)_{n}(dq^{n+1};q)_{m}(abcq^{m+2}/d;q)_{n}}{(abcq^{m+n+2};q)_{n+m}} \\ \times \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(-1)^{-i-j} {n \brack j}_{q} {m \brack j}_{q} q^{\frac{1}{2}(i(i-2n+1)+j(j-2m+1))} (abcq^{m+n+2};q)_{i+j}}{(aq;q)_{j}(bq;q)_{i}(dq^{n+1};q)_{j}(abcq^{m+2}/d;q)_{i}} (bx/d;q)_{i}(y;q)_{j}} \\ = \frac{\left(\frac{d}{b}\right)^{n} (aq;q)_{m}(bq;q)_{n}(dq^{n+1};q)_{m}(abcq^{m+2}/d;q)_{n}}{(abcq^{m+n+2};q)_{n+m}}}{(abcq^{m+n+2};q)_{n+m}} \\ \times \Phi_{0:2;2}^{1:2;2} \left[ \begin{array}{c} abcq^{n+m+2}:q^{-n},bx/d;q^{-m},y \\ -:bq,abcq^{m+2}/d;aq,dq^{n+1} \end{array} \middle| \begin{array}{c} q:q,q \\ 0,0,0 \end{array} \right],$$

where the generalized bivariate basic hypergeometric series is defined by [47]

$$\Phi_{\mu:u;v}^{\lambda:r;s} \begin{bmatrix} \alpha_1, \dots, \alpha_\lambda : a_1, \dots, a_r; c_1, \dots, c_s & q : x, y \\ \beta_1, \dots, \beta_\mu : b_1, \dots, b_u; d_1, \dots, d_v & i, j, k \end{bmatrix}$$
$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1, \dots, \alpha_\lambda; q)_{m+n}(a_1, \dots, a_r; q)_m(c_1, \dots, c_s; q)_n}{(\beta_1, \dots, \beta_\mu; q)_{m+n}(b_1, \dots, b_u; q)_m(d_1, \dots, d_v; q)_n} \frac{x^m y^n q^{i\binom{m}{2} + j\binom{n}{2} + kmn}}{(q; q)_m (q; q)_n}$$

**Limit relations.** Let us consider the case when  $a = q^{\alpha}$ ,  $b = q^{\beta}$ ,  $c = q^{\gamma}$  and  $d = -q^{\delta}$ . As  $q \uparrow 1$  the second-order partial q-difference equation goes formally to the following

$$(7.12) \quad (x^{2}-1) \frac{\partial^{2}}{\partial x^{2}} f(x,y) + (y^{2}-1) \frac{\partial^{2}}{\partial y^{2}} f(x,y) + 2((x+1)(y-1)) \frac{\partial^{2}}{\partial x \partial y} f(x,y) + (x(\alpha+\beta+\gamma+3)+\alpha-\beta+\gamma+1) \frac{\partial}{\partial x} f(x,y) + (y(\alpha+\beta+\gamma+3)+\alpha-\beta-\gamma-1) \frac{\partial}{\partial y} f(x,y) - n(\alpha+\beta+\gamma+n+2) f(x,y) = 0.$$

An orthogonality weight function for the polynomial solutions of the above equation can be computed in the same way as in [5], giving rise to

(7.13) 
$$\varrho^{(\alpha,\beta,\gamma)}(x,y) = (1-y)^{\alpha}(x+1)^{\beta}(y-x)^{\gamma},$$

second-order partial differential equation of the hypergeometric type

in the triangular domain

(7.14) 
$$\mathcal{R} = \{ (x, y) \in \mathbf{R}^2 \mid x \le y \le 1, \ -1 \le x \le 1 \}.$$

It is important to note that

$$\lim_{q\uparrow 1} W(x,y;q^{\alpha},q^{\beta},q^{\gamma},-q^{\delta};q) = (1-y)^{\alpha}(x+1)^{\beta}(y-x)^{\gamma} = \varrho^{(\alpha,\beta,\gamma)}(x,y).$$

The monic polynomial solutions of (7.12) satisfy a three-term recurrence relation of the form (6.6), where the matrix coefficients can be easily computed by considering the limit as  $q \uparrow 1$  in (7.7)—(7.10) for  $a = q^{\alpha}$ ,  $b = q^{\beta}$ ,  $c = q^{\gamma}$  and  $d = -q^{\delta}$ , or, eventually, from [4]. The monic orthogonal polynomial solutions of (7.12) can be written in terms of generalized Kampé de Fériet hypergeometric series as

(7.15) 
$$\widehat{A}_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = (-1)^n 2^{n+m} \frac{(\alpha+1)_m (\beta+1)_n}{(\alpha+\beta+\gamma+m+n+2)_{n+m}} \times F_{0:1;1}^{1:1;1} \left( \begin{array}{c} \alpha+\beta+\gamma+m+n+2:-n;-m\\ -:\beta+1;\alpha+1 \end{array} \middle| \frac{x+1}{2}, \frac{1-y}{2} \end{array} \right).$$

*Remark* 3. We would like to mention the following limit relation between the monic bivariate big q-Jacobi polynomials in (7.11) and the monic bivariate Jacobi polynomials, defined in (7.15),

(7.16) 
$$\lim_{q \to 1} \widehat{P}_{n,m}(x,y;q^{\alpha},q^{\beta},q^{\gamma},-q^{\delta};q) = \widehat{A}_{n,m}^{(\alpha,\beta,\gamma)}(x,y).$$

Remark 4. Note the following limit relation for the non-monic bivariate big q-Jacobi polynomials, defined in (7.1),

$$\lim_{q \to 1} P_{n,m}(x,y;q^{\alpha},q^{\beta},q^{\gamma},-q^{\delta};q) = (y+1)^m {}_2F_1\left(\begin{array}{c} -m,\beta+\gamma+m+1 \\ \gamma+1 \end{array} \middle| \frac{y-x}{y+1} \right) \\ \times {}_2F_1\left(\begin{array}{c} m-n,\alpha+\beta+\gamma+m+n+2 \\ \alpha+1 \end{array} \middle| \frac{1-y}{2} \right) =: J_{n,m}(x,y;\alpha,\beta,\gamma), \quad 0 \le m \le n$$

The polynomials  $J_{n,m}(x, y; \alpha, \beta, \gamma)$  are a non-monic polynomial solution of (7.12) and they are orthogonal on the same domain (7.14) with respect to the weight (7.13). This non-monic polynomial solution can be written as

$$J_{n,m}(x,y;\alpha,\beta,\gamma) = \frac{m!(y+1)^m(n-m)!}{(\gamma+1)_m(\alpha+1)_{n-m}} P_m^{(\gamma,\beta)} \left(\frac{2x-y+1}{y+1}\right) P_{n-m}^{(\alpha,\beta+\gamma+2m+1)}(y),$$

where

$$P_n^{(a,b)}(x) = \frac{(a+1)_n}{n!} {}_2F_1\left(\begin{array}{c} -n, n+a+b+1\\ a+1 \end{array} \middle| \frac{1-x}{2} \right), \qquad a > -1, \quad b > -1,$$

are the Jacobi polynomials [26, Eq. (9.8.1)].

There exists at least a third family of orthogonal polynomial solutions of the partial differential equation (7.12) (on the same domain  $\mathcal{R}$ , defined in (7.14), and with respect to the weight function  $\rho^{(\alpha,\beta,\gamma)}$ , given in (7.13)). The non-monic polynomials which can be computed from the Rodrigues' formula [4, Eq. (36)],

(7.17) 
$$\tilde{A}_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\varrho^{(\alpha,\beta,\gamma)}(x,y)} \frac{\partial^{n+m}}{\partial x^n \, \partial y^m} \left[ (x+1)^{\beta+n} \, (1-y)^{\alpha+m} \, (y-x)^{\gamma+n+m} \right] \, .$$

*Remark* 5. Observe the following limit relation between the non-monic bivariate big q-Jacobi polynomials, derived from the Rodrigues' formula (5.1), and the non-monic bivariate Jacobi polynomials, defined by the Rodrigues' formula (7.17),

(7.18) 
$$\lim_{q \to 1} \tilde{P}_{n,m}(x,y;q^{\alpha},q^{\beta},q^{\gamma},-q^{\delta};q) = \tilde{A}_{n,m}^{(\alpha,\beta,\gamma)}(x,y).$$

## 8. CONCLUSIONS AND AN OUTLOOK OF FUTURE RESEARCH

In the present work we have initiated a general approach to the study of solutions of bivariate linear second-order partial q-difference equations on non-uniform lattices and concentrated our efforts on the particular case of q-linear lattices of the form  $x(s) = q^s$ and  $y(t) = q^t$ . We have dealt with those bivariate polynomials, written in vector representation (and graded lexicographical order), that are solutions of admissible potentially self-adjoint linear second-order partial q-difference equation of the hypergeometric type. In this context, we have proved that (similar to the one variable hypergeometric-type case) the coefficients of the three-term recurrence relations, obeyed by the vector polynomials, can be written explicitly in terms of the coefficients of the partial q-difference equation, they satisfy. It has been shown that the orthogonality weight function is completely determined by the coefficients of the partial q-difference equation.

The results obtained suggest that this approach can be extended to the quadratic lattices; this direction has been already taken up in [46] and will be a subject of future research.

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(Area) DEPARTAMENTO DE MATEMÁTICA APLICADA II, E.E. TELECOMUNICACIÓN, UNIVERSIDADE DE VIGO, 36310-VIGO, SPAIN.

*E-mail address*, Area: area@uvigo.es

(Atakishiyev) Instituto de Matemáticas, Unidad Cuernavaca, Universidad Nacional Autó-Noma de México, C.P. 62251 Cuernavaca, Morelos, México

*E-mail address*, Atakishiyev: natig@matcuer.unam.mx

(Godoy) DEPARTAMENTO DE MATEMÁTICA APLICADA II, E.E. INDUSTRIAL, UNIVERSIDADE DE VIGO, 36310-VIGO, SPAIN.

*E-mail address*, Godoy: egodoy@dma.uvigo.es

(Rodal) DEPARTAMENTO DE MATEMÁTICA APLICADA II, E.E. TELECOMUNICACIÓN, UNIVERSI-DADE DE VIGO, 36310-VIGO, SPAIN.

*E-mail address*, Rodal: jrodal@edu.xunta.es