This is a preprint of: "On the periodic orbit bifurcating from a Hopf bifurcation in systems with two slow and one fast variables", Isaac A. García, Jaume Llibre, Susanna Maza, *Appl. Math. Comput.*, vol. 232, 84–90, 2014.
DOI: [10.1016/j.amc.2013.12.184]

ON THE PERIODIC ORBIT BIFURCATING FROM A HOPF BIFURCATION IN SYSTEMS WITH TWO SLOW AND ONE FAST VARIABLES

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Abstract. The Hopf bifurcation in slow-fast systems with two slow variables and one fast variable has been studied recently, mainly from a numerical point of view. Our goal is to provide an analytic proof of the existence of the zero Hopf bifurcation exhibited for such systems, and to characterize the stability or instability of the periodic orbit which borns in such zero Hopf bifurcation. Our proofs use the averaging theory.

1. Introduction and statement of the results

Hopf bifurcations have been studied intensively in two dimensional differential systems with one slow and one fast variable, see for instance [4, 6, 7, 11, 8].

Canard explosions are associated with these singular Hopf bifurcations, manifested by a very rapid growth in the amplitude of periodic orbits, see for instance [2, 3, 5, 12].

There has been less analysis of the Hopf bifurcations in slow-fast systems with two slow variables and one fast variable, see for instance [9, 13, 16].

In this paper we shall study this last kind of Hopf bifurcation using the averaging theory and in particular we shall characterize the stability or instability of the periodic orbit which bifurcates in the Hopf bifurcation. We shall follow the work of Guckenheimer [9] who reduces the study of the mentioned Hopf bifurcation to study the zero Hopf bifurcation of the differential system

$$X' = Y - X^2, \qquad Y' = Z - X, \qquad Z' = -\mu - AX - BY - CZ, \tag{1}$$

where the prime denotes derivative with respect to τ. A summary of this reduction process is done in appendix I, for more details see [9].

We note that when

$$B \neq 0 \quad \text{and} \quad (A+C)^2 - 4B\mu \geq 0, \tag{2}$$

the differential system (1) possesses exactly one equilibrium point with coordinates (X^*, X^{*2}, X^*) where

$$X^* = -\frac{A + C - \sqrt{(A+C)^2 - 4B\mu}}{2B}.$$

The *zero Hopf bifurcation* (also called *saddle-node Hopf bifurcation* or *fold Hopf*) occurs at the equilibrium point (X^*, X^{*2}, X^*) when it has one zero eigenvalue and

2010 *Mathematics Subject Classification.* 37G15, 37G10, 34C07.
Key words and phrases. zero Hopf bifurcation, averaging theory, singular perturbation.

a pair of pure imaginary eigenvalues $\pm iw$ with $w \neq 0$. The parameter values for which system (1) has such an equilibrium are

$$A = C(B-1), \qquad \mu = BC^2/4; \tag{3}$$

the pure imaginary eigenvalues are $\pm iw = \pm i\sqrt{1+B-C^2}$ where $w^2 = 1+B-C^2 > 0$ since B and C are small as explained in Appendix I, and the equilibrium point is $(X^*, {X^*}^2, X^*)$ with $X^* = -C/2$.

In the next theorem we study the periodic orbit which bifurcates in the mentioned zero Hopf bifurcation of the differential system (1). Additionally we also study the stability or instability of this periodic orbit.

Theorem 1. *Consider the differential system* (1) *written in the new parameters* (R, w, ε, μ) *defined in* (4) *that substitute to the initial parameters* (A, B, C, μ)*. Assume that the following conditions hold:*

$$w > 0, \quad C^2 + w^2 - 1 \neq 0 \quad \textit{and} \quad 2(1-C^2)R - 1 > 0.$$

(a) *For* $\varepsilon \neq 0$ *sufficiently small there exists one periodic solution of* (1) *that shrinks to the equilibrium* $(X^*, {X^*}^2, X^*)$ *with* $X^* = -C/2$ *when* $\varepsilon \to 0$.
(b) *Such a periodic orbit is stable if either* $\Delta \geq 0$, $\Omega_1 > 0$ *and* $\Omega_2 < 0$*; or* $\Delta < 0$ *and* $\Omega_4 < 0$*. Otherwise, it is unstable. The constants* Ω_k *and* Δ *are defined in* (12).

Theorem 1 is proved in section 2.

The main reason for introducing the parameters (R, w, ε, μ) is that then the eigenvalues of the equilibrium point of the differential system (1) are εR and $\pm wi + O(\varepsilon)$, and consequently for $\varepsilon = 0$ we have a zero Hopf point, which will be able to exhibit a zero Hopf bifurcation.

2. Proof of Theorem 1

Now, we do the following reparametrization

$$(\mu, A, B) \mapsto (R, \varepsilon, w), \tag{4}$$

defined by

$$\begin{aligned}
\mu &= \frac{\displaystyle\sum_{i=0}^{7} P_i(C,R,w)\varepsilon^i}{4[w^2-1+(C+R\varepsilon)^2]^2} = \frac{1}{4}C^2(C^2+w^2-1) + O(\varepsilon), \\
A &= \varepsilon(1-R) + C\left[-2+w^2+(C+R\varepsilon)^2\right] = C(-2+C^2+w^2) + O(\varepsilon), \\
B &= -1 + C^2 + w^2 + CR\varepsilon,
\end{aligned}$$

where

$$\begin{aligned}
P_0(C,R,w) &= C^2(C^2+w^2-1)^3, \\
P_1(C,R,w) &= C(C^2+w^2-1)^2(2-2R+7C^2R),
\end{aligned}$$

$$
\begin{array}{rcl}
P_2(C,R,w) & = & (C^2+w^2-1)\Big(1+R\big(21RC^4+2C^2(5+R(2w^2-7))\\
& & -Rw^4+R-2\big)\Big),\\
P_3(C,R,w) & = & CR\Big(R\big(35RC^4+20C^2(1+R(w^2-2))+9R\\
& & +w^2(8-3R(w^2+2))-12\big)+3\Big),\\
P_4(C,R,w) & = & R^2\Big(2+R\big(35RC^4+5(R(w^2-5)+4)C^2\\
& & +2(-Rw^4+w^2+R-2)\big)\Big),\\
P_5(C,R,w) & = & CR^4(10-8R+21C^2R-2w^2R),\\
P_6(C,R,w) & = & R^5(2-R+7C^2R-w^2R),\\
P_7(C,R,w) & = & CR^7.
\end{array}
$$

We observe that when $\varepsilon \to 0$ the conditions (3) for a zero Hopf bifurcation at the equilibrium of (1) are satisfied because

$$A-C(B-1)=O(\varepsilon), \qquad \mu-BC^2/4=O(\varepsilon).$$

Moreover, since $(A+C)^2-4B\mu=w^4R^2\varepsilon^2+O(\varepsilon^3)$ and $B=-1+C^2+w^2+O(\varepsilon)$, it is clear that restrictions (2) for the existence of the equilibrium of (1) are also satisfied when $|\varepsilon|$ is sufficiently small if we assume

$$-1+C^2+w^2\neq 0. \tag{5}$$

With the reparametrization (4) we get that the equilibrium of (1) becomes

$$(X^*,X^{*2},X^*) \quad \text{where} \quad X^*=-\frac{C}{2}+O(\varepsilon), \tag{6}$$

and moreover when $\varepsilon \to 0$ this equilibrium tends to have the eigenvalues 0 and $\pm iw$. We shall assume in addition that

$$w>0. \tag{7}$$

Now first we translate the equilibrium at the origin by means of the change $(X,Y,Z)\mapsto(X-X^*,Y-X^{*2},Z-X^*)$, and after we do the linear change of variables

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} \dfrac{C^2-1}{w} & \dfrac{C}{w} & \dfrac{1}{w}\\ C & 1 & 0\\ -1+C^2+w^2 & C & 1\end{pmatrix}\begin{pmatrix} X\\ Y\\ Z\end{pmatrix},$$

which will write the linear part at the origin in its real Jordan normal form

$$\begin{pmatrix} 0 & -w & 0\\ w & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

when $\varepsilon \to 0$. Finally we do the rescaling $(x,y,z)=(\varepsilon U,\varepsilon V,\varepsilon W)$. In the new coordinates system (1) becomes

$$
\begin{array}{rcl}
U' & = & -wV+\varepsilon f_1(U,V,W)+O(\varepsilon^2),\\
V' & = & wU+\varepsilon f_2(U,V,W)+O(\varepsilon^2),\\
W' & = & \varepsilon f_3(U,V,W)+O(\varepsilon^2),
\end{array} \tag{8}
$$

with

$$
\begin{aligned}
f_1(U,V,W) &= \frac{-1}{w^5(C^2+w^2-1)}\Big(w^4\big(CRw^2V - wU + C(C^2-1)RV + W\big) \\
&\quad + (C^2+w^2-1)(C^2-1)\big(w^2U^2 - 2wUW + W^2\big)\Big), \\
f_2(U,V,W) &= \frac{-C}{w^4(C^2+w^2-1)}\Big((C^2+w^2-1)(-w^2U^2 + 2wUW - W^2) \\
&\quad - w^2\big(R(C^2+w^2-1)+1\big)(wU - W)\Big), \\
f_3(U,V,W) &= -\frac{(C^2+w^2-1)(-wU+W)^2}{w^4} + R(-wU - CV + W).
\end{aligned}
$$

Now performing the change to cylindrical coordinates $(U,V,W) \mapsto (\theta, r, Z)$ with $U = r\cos\theta$, $V = r\sin\theta$ and $W = Z$ system (8) is transformed into

$$
\begin{aligned}
r' &= \varepsilon g_1(\theta,r,Z) + O(\varepsilon^2), \\
\theta' &= w + \varepsilon g_2(\theta,r,Z) + O(\varepsilon^2), \\
Z' &= \varepsilon g_3(\theta,r,Z) + O(\varepsilon^2),
\end{aligned} \tag{9}
$$

where we emphasize that this system is only well defined for $r > 0$ due to the expression of $g_2(\theta,r,Z)$. Moreover, in this region, since for sufficiently small ε we have $\theta' > 0$ in a big ball centered at the origin, we can rewrite the differential equations of the orbits of (9) in this big ball in the form

$$
\frac{dr}{d\theta} = \varepsilon\frac{g_1(\theta,r,Z)}{w} + O(\varepsilon^2), \qquad \frac{dZ}{d\theta} = \varepsilon\frac{g_3(\theta,r,Z)}{w} + O(\varepsilon^2), \tag{10}
$$

where

$$
\begin{aligned}
g_1(\theta,r,Z) &= -\frac{1}{w^5(1-C^2-w^2)}\sum_{i,j} R_{ij}(r,Z)\cos^i\theta\,\sin^j\theta, \\
g_3(\theta,r,Z) &= \frac{1}{w^4}[(1-C^2-w^2)(Z - wr\cos\theta)^2 + \\
&\quad R(Z - r(w\cos\theta + C\sin\theta))],
\end{aligned}
$$

and the only non identically zero coefficients $R_{ij}(r,Z)$ are

$$
\begin{aligned}
&R_{01}(r,Z) = Cw^3(1 - R(1-C^2-w^2))Z + Cw(1-C^2-w^2)Z^2, \\
&R_{10}(r,Z) = -w^4Z - (C^2-1)(-1+C^2+w^2)Z^2 \\
&R_{11}(r,Z) = -Cw^4(1 - 2R(-1+C^2+w^2)r + 2Cw^2(-1+C^2+w^2)rZ, \\
&R_{20}(r,Z) = w^5r + 2w(C^2-1)(-1+C^2+w^2)rZ, \\
&R_{21}(r,Z) = -Cw^3(-1+C^2+w^2)r^2, \\
&R_{30}(r,Z) = -w^2(C^2-1)(-1+C^2+w^2)r^2.
\end{aligned}
$$

System (10) is 2π–periodic is the variable θ and is in the standard form for applying the averaging theory, see Theorem 2 of the appendix II. The averaged functions are

$$
\begin{aligned}
\mathcal{R}(r,Z) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{g_1(\theta,r,Z)}{w}d\theta = \frac{r}{2w(C^2+w^2-1)} + \frac{(C^2-1)rZ}{w^5}, \\
\mathcal{Z}(r,Z) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{g_3(\theta,r,Z)}{w}d\theta = -\frac{(C^2+w^2-1)(w^2r^2+2Z^2)}{2w^5} + \frac{RZ}{w}.
\end{aligned}
$$

The only real zero (r_*, Z_*) with $r_* > 0$ of the map $(r, Z) \mapsto \mathcal{F}(r, Z)$ defined as $\mathcal{F}(r, Z) = (\mathcal{R}(r, Z), \mathcal{Z}(r, Z))$ is

$$(r_*, Z_*) = \left(\frac{w^3 \sqrt{2(1-C^2)R - 1}}{\sqrt{2}\,|(C^2-1)(C^2+w^2-1)|}, -\frac{w^4}{2(C^2-1)(C^2+w^2-1)} \right).$$

Recalling that C is small (see Appendix I) and therefore $C^2 - 1 \neq 0$, we notice that such a solution only exists if

$$2(1-C^2)R - 1 > 0 \quad \text{and} \quad C^2 + w^2 - 1 \neq 0. \tag{11}$$

In addition the Jacobian of $\mathcal{F}$ at the point (r_*, Z_*) is

$$\det(D\mathcal{F}(r_*, Z_*)) = \frac{2(1-C^2)R - 1}{2w^2(C^2-1)(C^2+w^2-1)} \neq 0,$$

from (11). Therefore (r_*, Z_*) is a simple zero of $\mathcal{F}$. Hence, the averaging theory of Theorem 2 predicts for $\varepsilon \neq 0$ sufficiently small the existence of a 2π–periodic orbit $\gamma_\varepsilon = \{(r(\theta, \varepsilon), Z(\theta, \varepsilon)) : 0 \leq \theta \leq 2\pi\}$ of system (10) such that $(r(0, \varepsilon), Z(0, \varepsilon)) \to (r_*, Z_*)$ as $\varepsilon \to 0$.

Now going back through the changes of variables and time rescaling which keep the stability of the periodic orbit γ_ε. Thus we have that system (8) for $\varepsilon \neq 0$ sufficiently small has one periodic orbit γ_ε such that $\gamma_\varepsilon \to \{W = Z_*\} \cap \{U^2 + V^2 = {r_*}^2\}$ as $\varepsilon \to 0$, the intersection of the plane $\mathcal{P} = \{W = Z_*\}$ and the cylinder $\mathcal{C} = \{U^2 + V^2 = {r_*}^2\}$. Undoing the remaining changes of variable to reach the coordinates of system (1) we get that the former plane and cylinder are transformed into $\mathcal{P}_\varepsilon = \{(-1+C^2+w^2)(X - X^*) + C(Y - {X^*}^2) + Z - X^* = \varepsilon Z_*\}$ and $\mathcal{C}_\varepsilon = \{w^2[C(X - X^*) + (Y - {X^*}^2)]^2 + [(C^2-1)(X - X^*) + C(Y - {X^*}^2) + (Z - X^*)]^2 = \varepsilon^2 w^2 {r_*}^2\}$. We note that the radius of the cylinder $\mathcal{C}_\varepsilon$ goes to zero as $\varepsilon \to 0$ and therefore $\mathcal{C}_0$ is a straight line. Finally, system (1) has one periodic orbit γ_ε such that $\gamma_\varepsilon \to \mathcal{P}_0 \cap \mathcal{C}_0$ as $\varepsilon \to 0$ where $\mathcal{P}_0 \cap \mathcal{C}_0 = (X^*, {X^*}^2, X^*)$ is just the equilibrium of (1). This proves statement (a).

Define now the following functions

$$\begin{aligned}
\Omega_1(C, R, w) &= \frac{2(1-C^2)R - 1}{2w^2(C^2-1)(C^2+w^2-1)}, \\
\Omega_2(C, R, w) &= \frac{1 + R(C^2-1)}{w(C^2-1)}, \\
\Delta(C, R, w) &= (C^2+w^2-1)\Omega_3(C, R, w), \\
\Omega_3(C, R, w) &= -3 + 3C^2 + w^2 + 2(C^2-1)(-3+3C^2+w^2)R \\
&\quad +(C^2-1)^2(C^2+w^2-1)R^2, \\
\Omega_4(C, R, w) &= \frac{1 + R(C^2-1)}{2w(C^2-1)}.
\end{aligned} \tag{12}$$

Denoting by λ_1 and λ_2 the eigenvalues of $D\mathcal{F}(r_*, Z_*)$, we obtain that

$$\lambda_1\lambda_2 = \det(D\mathcal{F}(r_*, z_*)) = \Omega_1(C, R, w) \neq 0, \qquad \lambda_1 + \lambda_2 = \Omega_2(C, R, w).$$

More detailed we have

$$\lambda_{1,2} = \frac{1 + R(C^2-1)}{2w(C^2-1)} \pm \sqrt{\Delta}.$$

Therefore two cases arise depending on the nature of the eigenvalues:

(i) Assume that the eigenvalues are real, that is, $\Delta \geq 0$. Then both eigenvalues are negative if and only if $\lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 0$.

(ii) Assume that the eigenvalues are not real, that is, $\Delta < 0$. Then its real part is

$$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \Omega_4(C, R, w).$$

In summary, from Theorem 2 the periodic orbit γ_ε is stable if either $\Delta \geq 0$, $\lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 0$; or $\Delta < 0$ and $\mathrm{Re}(\lambda_i) < 0$. Otherwise, it is unstable. Hence statement (b) is proved.

Appendix I: Hopf bifurcations in slow-fast systems

Consider a *slow–fast* vector field of the form

$$\varepsilon\dot{x} = f(x, y, z, \varepsilon), \qquad \dot{y} = g(x, y, z, \varepsilon), \qquad \dot{z} = h(x, y, z, \varepsilon), \tag{13}$$

with one fast variable $x \in \mathbb{R}$ and two slow variables $(y, z) \in \mathbb{R}^2$. Here ε is a small parameter that represents the ratio of time scales and the overdot denotes as usual derivative with respect to the time t. The set of points satisfying $f = 0$ is the *critical manifold* of system (13) and the slow motion of the trajectories of (13) can occur only near the critical manifold. Points on the critical manifold where $\partial f/\partial x$ vanishes are called *fold points.*

In [9] the author explores the dynamics of a singular Hopf bifurcation of systems (13) via analysis of normal forms. In order to derive a normal form for a singular Hopf bifurcation in a generic system (13), in [9] it is assumed that an equilibrium point crosses a simple fold transversally. In [1] it is shown that the fast equation near a simple fold can be reduced to $\varepsilon\dot{x} = y - x^2$, perhaps using a rescaling of time. Next the system is approximated truncating nonlinear terms in the Taylor series of g and h. The truncated component $\dot{y} = \alpha + \beta x + \gamma y + \delta z$ is further reduced via the affine change $z \mapsto \alpha + \gamma y + \delta z$ obtaining $\dot{y} = \beta x + z$ while the other component $\dot{z}$ is still an affine function. Hopf bifurcation occurs when $\beta < 0$. The rescaling $(x, y, z, t) \mapsto (|\beta|^{1/2}x, |\beta|y, |\beta|^{3/2}z, |\beta|^{-1/2}t)$ reduces the study to the case $\beta = -1$. In short, after all these transformations the truncated normal form becomes the differential system

$$\varepsilon\dot{x} = y - x^2, \qquad \dot{y} = z - x, \qquad \dot{z} = -\mu - ax - by - cz. \tag{14}$$

The final rescaling $(x, y, z, t) = (\varepsilon^{1/2}X, \varepsilon Y, \varepsilon^{1/2}Z, \varepsilon^{1/2}\tau)$ and $(A, B, C) = (\varepsilon^{1/2}a, \varepsilon b, \varepsilon^{1/2}c)$ eliminates ε from system (14) and yields the differential system (1). Note that $(A, B, C) \to (0, 0, 0)$ as ε goes to zero and that B tends to zero faster that A and C. Moreover, as μ varies near zero, the equilibrium point of system (14) crosses the fold curve of the critical manifold at the origin. The origin is always a folded singularity (recall that folded singularities are regular points of (13) when $\varepsilon > 0$) that is a saddle when $\mu < 0$, a node when $0 < \mu < 1/8$ and a focus when $1/8 < \mu$. Hopf bifurcations of systems (14) and (1) typically occur at nonzero values of μ. See [9] for more details on the deduction of the differential system (1).

The equilibria of (1) occur at points (X_μ, X_μ^2, X_μ) where X_μ is a solution of $\mu = -AX_\mu - BX_\mu^2 - CX_\mu$. The linearization of (1) at this equilibrium is

$$\begin{pmatrix} -2X_\mu & 1 & 0 \\ -1 & 0 & 1 \\ -A & -B & -C \end{pmatrix},$$

whose characteristic polynomial is

$$P(\lambda) = \lambda^3 + (C + 2X_\mu)\lambda^2 + (B + 2X_\mu C + 1)\lambda + (A + 2X_\mu B + C).$$

Hence a Hopf bifurcation occurs when $B + 2X_\mu C + 1 > 0$ (note that this condition is always satisfied when B and C are small) and

$$(C + 2X_\mu)(B + 2X_\mu C + 1) = A + 2X_\mu B + C.$$

Under these assumptions the real eigenvalue is $-(C+2X_\mu)$ and the pure imaginary eigenvalues are $\pm i\sqrt{B + 2X_\mu C + 1}$.

We emphasize that zero eigenvalues occur near the origin only if $a + c$ is small in system (14) because B is $O(\varepsilon)$ while A and C are $O(\varepsilon^{1/2})$. A subtle aspect of the normal form is that terms of higher order contribute to the first Lyapunov coefficient in an essential way.

Appendix II: Averaging Theory of First Order

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [15].

Consider the differential equation in the standard form

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{15}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{16}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t, \mathbf{y})\, dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (15).

Theorem 2. *Consider the two initial value problems (15) and (16). Suppose:*

(i) F_1, *its Jacobian* $\partial F_1/\partial \mathbf{x}$, *its Hessian* $\partial^2 F_1/\partial \mathbf{x}^2$, F_2 *and its Jacobian* $\partial F_2/\partial \mathbf{x}$ *are defined, continuous and bounded by a constant independent of* ε *in* $[0, \infty) \times D$ *and* $\varepsilon \in (0, \varepsilon_0]$.

(ii) F_1 *and* F_2 *are* $T-$*periodic in* t *(*T *independent of* ε*).*

Then the following statements hold.

(a) *If p is an equilibrium point of the averaged equation* (16) *and*

$$\det \left(\frac{\partial f_1}{\partial \mathbf{y}} \right) \bigg|_{\mathbf{y}=p} \neq 0,$$

then there exists a $T-$periodic solution $\varphi(t,\varepsilon)$ of equation (15) *such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *If the eigenvalues of the equilibrium point p all have negative real part, the corresponding periodic orbit $\varphi(t,\varepsilon)$ is asymptotically stable for ε sufficiently small. If one of the eigenvalues has positive real part, then $\varphi(t,\varepsilon)$ is unstable.*

Acknowledgments

The first and third authors are partially supported by a MICINN grant number MTM2011-22877 and by an AGAUR grant number 2009SGR 381. The second author is partially supported by a MICINN/FEDER grant number MTM2008-03437, by an AGAUR grant number 2009SGR 410 and by ICREA Academia.

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