STRONG CONVERGENCE FOR THE MODIFIED MANN'S ITERATION OF λ -STRICT PSEUDOCONTRACTION

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Abstract. In this paper, for an λ -strict pseudocontraction T, we prove strong convergence of the modified Mann's iteration defined by

$$x_{n+1} = \beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n) [\alpha_n T x_n + (1 - \alpha_n) x_n]$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in (0, 1) satisfy:

(i)
$$0 \le \alpha_n \le \frac{\lambda}{K^2}$$
 with $\liminf_{n \to \infty} \alpha_n (\lambda - K^2 \alpha_n) > 0;$
(ii) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty;$

(iii) $\limsup \gamma_n < 1.$

Our results unify and improve some existing results. **Key Words and Phrases:** λ -strict pseudocontraction, modified Mann's iteration, 2-uniformly smooth Banach space. **2000 AMS Subject Classification:** 49J40, 47H05, 47H04, 65J15, 47H10.

1. Introduction

Throughout this paper, let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^* and $\langle y, x^* \rangle$ denote the value of $x^* \in E^*$ at $y \in E$. The normalized duality mapping J from E into 2^{E^*} is defined by the following equation:

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x\| = \|x^*\|\}.$$

Let $F(T) = \{x \in E : Tx = x\}$, the set of all fixed point of a mapping T.

Recall that a mapping T with domain D(T) and range R(T) in Banach space E is called *Lipschitzian* if there exists L > 0 such that

$$||Tx - Ty|| \le L||x - y|| \text{ for all } x, y \in D(T).$$

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T is said to be *nonexpansive* if L = 1 in the above inequality. T is called λ -strictly pseudocontractive if there exists $\lambda \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \le ||x - y||^2 - \lambda ||x - y - (Tx - Ty)||^2 \text{ for all } x, y \in D(T).$$
 (1.1)

T is called *pseudocontractive* if $\lambda \equiv 0$ in (1.1). Obviously, each λ -strictly pseudocontractive mapping is a Lipschitzian and pseudocontractive mapping with $L = \frac{\lambda+1}{\lambda}$. In particular, a nonexpansive mapping is λ -strictly pseudocontractive mapping in a Hilbert space, but the conversion may be false.

For finding a fixed point of λ -strictly pseudocontractive mapping T, a strong convergence theorem was obtained by Zhou [22] in a 2-uniformly smooth Banach space.

Theorem Z. (Zhou [22, Theorem 2.3]) Let C be a closed convex subset of a real 2-uniformly smooth Banach space E and let $T: C \to C$ be a λ -strict pseudocontraction with $F(T) \neq \emptyset$. Given $u, x_0 \in C$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = \beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n) [\alpha_n T x_n + (1 - \alpha_n) x_n], \quad (1.2)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in (0, 1) satisfy:

(i) $\alpha_n \in [a, \mu], \mu = \min\{1, \frac{\lambda}{K^2}\}$ for some constant $a \in (0, \mu);$ (ii) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty;$ (iii) $\lim_{n \to \infty} |\alpha_{n+1} - \alpha_n| = 0;$ (iv) $0 < \liminf_{n \to \infty} \gamma_n \le \limsup_{n \to \infty} \gamma_n < 1.$ Then the sequence $\{x_n\}$ converges strongly to a fixed point z of T.

Recently, Zhang and Su [23] extended Zhou's results to q-uniformly smooth Banach space. We also note that the above results excluded $\gamma_n \equiv 0$ and $\gamma_n = \frac{1}{n+1}$. Very recently, Chai and Song [1] studied the strong convergence of the modified Mann's iteration (1.2) with $\gamma_n \equiv 0$.

Theorem CS. (Chai and Song [1, Theorem 3.1]) Let C be a closed convex subset of a real 2-uniformly smooth Banach space E and let $T: C \to C$ be a λ -strict pseudo-contraction with $F(T) \neq \emptyset$. Given $u, x_0 \in C$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = \beta_n u + (1 - \beta_n) [\alpha_n T x_n + (1 - \alpha_n) x_n], \qquad (1.3)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in (0,1) satisfy the following control conditions:

- (i) $\alpha_n \in [a, \mu], \mu = \min\{1, \frac{\lambda}{K^2}\}$ for some constant $a \in (0, \mu)$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} \alpha_n| < \infty;$

(iii) $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point z of T.

In this paper, we will deal with strong convergence of the modified Mann's iteration (1.2) under more relaxed conditions on the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in (0,1),

(i) $\alpha_n \in [0, \mu], \ \mu = \min\{1, \frac{\lambda}{K^2}\}$ with $\liminf_{n \to \infty} \alpha_n(\lambda - K^2 \alpha_n) > 0;$ (ii) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;

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(iii) $\limsup_{n \to \infty} \gamma_n < 1.$

Our results obviously develop and complement the corresponding ones of Zhou [22], Song and Chai [19], Chai and Song [1], Zhang and Su [23] and others. Moreover, our conditions are simpler, which contain $\gamma_n \equiv 0$ and $\gamma_n = \frac{1}{n+1}$ as special cases. Our conclusions may be regarded as a unification of the some existing results.

2. Preliminaries and basic results

For achieving our purposes, the following facts and results are needed. Let $\rho_E: [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E), \|y\| \le t\right\}.$$

Let q > 1. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant c > 0 such that $\rho_E(t) \leq ct^q$ and uniformly smooth if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$. Clearly, a q-uniformly smooth space must be uniformly smooth. Typical example of uniformly smooth Banach spaces is L_p (p > 1). More precisely, L_p is min $\{p, 2\}$ uniformly smooth for every p > 1.

Lemma 2.1. (Zhou [22, Lemma 1.2]) Let C be a nonempty subset of a real 2uniformly smooth Banach space E with the best smooth constant K, and let T : $C \to C$ be a λ -strict pseudocontraction. For any $\alpha \in (0,1)$, we define $T_{\alpha} = (1-\alpha)x + \alpha Tx$. Then,

$$\|T_{\alpha}x - T_{\alpha}y\|^{2} \leq \|x - y\|^{2} - 2\alpha(\lambda - K^{2}\alpha)\|Tx - Ty - (x - y)\|^{2} \text{ for all } x, y \in C.$$
(2.1)
In particular, as $\alpha \in (0, \frac{\lambda}{K^{2}}], T_{\alpha}: C \to C$ is nonexpansive such that $F(T_{\alpha}) = F(T).$

Lemma 2.2 was shown and used by several authors. For detail proofs, see Liu [12] and Xu [20, 21]. Furthermore, a variant of Lemma 2.1 has already been used by Reich in [16, Theorem 1].

Lemma 2.2. Let $\{a_n\}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1-t_n)a_n + t_n c_n, \quad \forall \ n \ge 0.$$

Assume that $\{t_n\} \subset [0,1]$ and $\{c_n\} \subset (0,+\infty)$ satisfy the restrictions:

$$\sum_{n=0}^{\infty} t_n = \infty \text{ and } \limsup_{n \to \infty} c_n \le 0.$$

Then as $n \to \infty$, $\{a_n\}$ converges to zero.

Morales and Jung [13], in 2000, proved the following behavior for pseudocontractive mappings. Also see Song and Chen [17, 18] for more details. The same result of nonexpansive mapping was shown by Reich [15] in 1980.

Lemma 2.3. ([13, 17, 18]) Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space E, and let $T : C \to C$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Suppose that for $t \in (0, 1)$ and $u \in C$, x_t defined by

$$x_t = tu + (1-t)Tx_t.$$
 (2.2)

Then, as $t \to 0$, x_t converges strongly to a fixed point of T.

This following results play a key role in proving our main results, which was proved by Song and Chen [17].

Lemma 2.4. (Song and Chen [17, Theorm 2.3]) Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space E, and let $T: C \to C$ be a continuous pseudocontractive mapping with a fixed point. Assume that there exists a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and $z = \lim_{t\to 0} z_t$ exists, where $\{z_t\}$ is defined by (2.2). Then

$$\limsup_{n \to \infty} \langle u - z, J(x_n - z) \rangle \le 0.$$

We also need the following results that showed by Mainge in 2008.

Lemma 2.5. (Mainge [14, Lemma 3.1]) Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that

$$\Gamma_{n_k} < \Gamma_{n_k+1}$$
 for all $k \ge 0$

Also consider the sequence of integers $\{\tau(n)\}_{n>n_0}$ defined by

 $\tau(n) = \max\{k \le n; \Gamma_k \le \Gamma_{k+1}\}.$

Then $\tau(n)$ is a nondecreasing sequence verifying

$$\lim_{n \to \infty} \tau(n) = +\infty,$$

and, for all $n \ge n_0$, the following two estimates hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$$
 and $\Gamma_n \leq \Gamma_{\tau(n)+1}$.

3. Main results

In this section, we will present our main results of this paper.

Theorem 3.1. Let C be a closed convex subset of a real 2-uniformly smooth Banach space E and let $T: C \to C$ be a λ -strict pseudo-contraction with $F(T) \neq \emptyset$. Given $u, x_0 \in C$, a sequence $\{x_n\}$ is generated by the modified Mann's iteration (1.2), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in (0,1) satisfy:

(i) $\alpha_n \in [0, \mu], \ \mu = \min\{1, \frac{\lambda}{K^2}\}$ with $\liminf_{n \to \infty} \alpha_n(\lambda - K^2 \alpha_n) > 0;$

- (ii) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$; (*iii*) $\limsup \gamma_n < 1$.
- $n \rightarrow \infty$

Then the sequence $\{x_n\}$ converges strongly to a fixed point z of T.

Proof. Let $y_n = T_{\alpha_n} x_n = \alpha_n T x_n + (1 - \alpha_n) x_n$. Then for each n, T_{α_n} is nonexpansive and $F(T) = F(T_{\alpha_n})$ by Lemma 2.1. So, the sequence $\{x_n\}$ is bounded since for given $p \in F(T) = F(T_{\alpha_n}),$

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n(u-p) + \gamma_n(x_n - p) + (1 - \beta_n - \gamma_n)(T_{\alpha_n}x_n - p)\| \\ &\leq \beta_n \|u - p\| + \gamma_n \|x_n - p\| + (1 - \beta_n - \gamma_n)\|T_{\alpha_n}x_n - T_{\alpha_n}p\| \\ &\leq \beta_n \|u - p\| + \gamma_n \|x_n - p\| + (1 - \beta_n - \gamma_n)\|x_n - p\| \\ &\leq \beta_n \|u - p\| + (1 - \beta_n)\|x_n - p\| \\ &\leq \max\{\|x_n - p\|, \|u - p\|\} \\ &\vdots \\ &\leq \max\{\|x_0 - p\|, \|u - p\|\}. \end{aligned}$$

Now we show $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. It follows from Lemma 2.1 that

$$||y_n - p|| = ||T_{\alpha_n} x_n - p||^2 \le ||x_n - p||^2 - 2\alpha_n (\lambda - K^2 \alpha_n) ||x_n - Tx_n||^2.$$
(3.1)

Furthermore, we also have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\beta_n(u-p) + \gamma_n(x_n-p) + (1-\beta_n - \gamma_n)(y_n-p)\|^2 \\ &\leq \beta_n \|u-p\|^2 + \gamma_n \|x_n-p\|^2 \\ &+ (1-\beta_n - \gamma_n)(\|x_n-p\|^2 - 2\alpha_n(\lambda - K^2\alpha_n)\|x_n - Tx_n\|^2) \\ &\leq \beta_n \|u-p\|^2 + (1-\beta_n)\|x_n-p\|^2 \\ &- 2\alpha_n(1-\beta_n - \gamma_n)(\lambda - K^2\alpha_n)\|x_n - Tx_n\|^2 \\ &\leq \|x_n-p\|^2 - (2\alpha_n(1-\beta_n - \gamma_n)(\lambda - K^2\alpha_n)\|x_n - Tx_n\|^2 - \beta_n\|u-p\|^2). \end{aligned}$$

Then we obtain

$$2\alpha_n(1-\beta_n-\gamma_n)(\lambda-K^2\alpha_n)\|x_n-Tx_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \beta_n\|u-p\|^2$$

It follows from Lemma 2.3 that there exist $z \in F(T)$ and $x_t = tu + (1-t)Tx_t$ such that $\lim_{t\to 0} x_t = z$. Then we also have

$$2\alpha_n(1-\beta_n-\gamma_n)(\lambda-K^2\alpha_n)\|x_n-Tx_n\|^2 \le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\beta_n\|u-z\|^2.$$
(3.2)

Following the proof technique in Mainge [14, Lemma 3.2, Theorem 3.1], the proof may be divided two cases.

Case 1. If there exists N_0 such that the sequence $\{||x_n - z||^2\}$ is nonincreasing for $n \ge N_0$, then the limit $\lim_{n \to \infty} ||x_n - z||^2$ exists, and hence $\lim_{n \to \infty} (||x_n - z||^2 - ||x_{n+1} - z||^2) = 0$. So by the condition (ii) and the inquality (3.2), it is obvious that

$$\limsup_{n \to \infty} \alpha_n (1 - \beta_n - \gamma_n) (\lambda - K^2 \alpha_n) \| x_n - T x_n \|^2 = 0.$$

It follows from the conditions (i), (ii) and (iii) that

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$
 (3.3)

Then by Lemma 2.4, we obtain

$$\limsup_{n \to \infty} \langle u - z, J(x_{n+1} - z) \rangle \le 0.$$
(3.4)

Finally, we show that $x_n \to z$. Indeed, since

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \langle (\beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n) T_{\alpha_n} x_n) - z, J(x_{n+1} - z) \rangle \\ &\leq \beta_n \langle u - z, J(x_{n+1} - z) \rangle + \gamma_n \|x_n - z\| \|J(x_{n+1} - z)\| \\ &+ (1 - \beta_n - \gamma_n) \|T_{\alpha_n} x_n - z\| \|J(x_{n+1} - z)\| \\ &\leq \beta_n \langle u - z, J(x_{n+1} - z) \rangle + (1 - \beta_n) \|x_n - z\| \|x_{n+1} - z\| \\ &\leq \beta_n \langle u - z, J(x_{n+1} - z) \rangle + (1 - \beta_n) \frac{\|x_n - z\|^2 + \|x_{n+1} - z\|^2}{2}, \end{aligned}$$

then, we have

$$||x_{n+1} - z||^2 \le (1 - \beta_n) ||x_n - z||^2 + 2\beta_n \langle u - z, J(x_{n+1} - z) \rangle.$$
(3.5)

So, an application of Lemma 2.2 onto (3.5) yields that $\lim_{n\to\infty} ||x_n - z|| = 0$. **Case 2.** Assume that there exists a subsequence $\{||x_{n_k} - z||^2\}$ of $\{||x_n - z||^2\}$ such that $||x_{n_k} - z||^2 < ||x_{n_k+1} - z||^2$ for for all $k \ge 0$. Let

$$\Gamma_n = \|x_n - z\|^2 \text{ and } \tau(n) = \max\{k \le n; \Gamma_k < \Gamma_{k+1}\}.$$

It follows from Lemma 2.5 that $\tau(n)$ is a nondecreasing sequence verifying

$$\lim_{n \to \infty} \tau(n) = +\infty$$

and for n large enough,

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \ \Gamma_n = ||x_n - z||^2 \le \Gamma_{\tau(n)+1}.$$
 (3.6)

In light of Eq. (3.2), we have

$$2\alpha_{\tau(n)}(1-\beta_{\tau(n)}-\gamma_{\tau(n)})(\lambda-K^{2}\alpha_{\tau(n)})\|x_{\tau(n)}-Tx_{\tau(n)}\|^{2} \leq \beta_{\tau(n)}\|u-z\|^{2},$$

and so by the condition (i),(ii) and (iii), we have

$$\lim_{n \to \infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$$

Then as $n \to \infty$,

$$\begin{aligned} \|x_{\tau(n)+1} - Tx_{\tau(n)}\| &\leq \beta_{\tau(n)} \|u - Tx_{\tau(n)}\| + \gamma_{\tau(n)} \|x_{\tau(n)} - Tx_{\tau(n)}\| \\ &+ (1 - \beta_{\tau(n)} - \gamma_{\tau(n)})(1 - \alpha_{\tau(n)}) \|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0. \end{aligned}$$

Since

$$\begin{aligned} \|x_{\tau(n)+1} - Tx_{\tau(n)+1}\| &\leq \|x_{\tau(n)+1} - Tx_{\tau(n)}\| + \|Tx_{\tau(n)} - Tx_{\tau(n)+1}\| \\ &\leq \|x_{\tau(n)+1} - Tx_{\tau(n)}\| + \|x_{\tau(n)} - x_{\tau(n)+1}\| \\ &\leq 2\|x_{\tau(n)+1} - Tx_{\tau(n)}\| + \|x_{\tau(n)} - Tx_{\tau(n)}\|, \end{aligned}$$

we have

$$\lim_{t \to \infty} \|x_{\tau(n)+1} - Tx_{\tau(n)+1}\| = 0.$$
(3.7)

Then by Lemma 2.4, we obtain

$$\limsup_{n \to \infty} \langle u - z, J(x_{\tau(n)+1} - z) \rangle \le 0.$$
(3.8)

Using the similar proof techniques of Case 1, the only modification is that n is replaced by $\tau(n)$, we have

$$\|x_{\tau(n)+1} - z\|^2 \le (1 - \beta_{\tau(n)}) \|x_{\tau(n)} - z\|^2 + 2\beta_{\tau(n)} \langle u - z, J(x_{\tau(n)+1} - z) \rangle.$$
(3.9)
Together with (3.6), we have

Together with (3.6), we have

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \le (1 - \beta_{\tau(n)})\Gamma_{\tau(n)} + 2\beta_{\tau(n)} \langle u - z, J(x_{\tau(n)+1} - z) \rangle,$$

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and so,

$$\Gamma_{\tau(n)} = \|x_{\tau(n)} - z\|^2 \le 2\langle u - z, J(x_{\tau(n)+1} - z)\rangle.$$

Along with (3.8), we have

$$\lim_{n \to \infty} \Gamma_{\tau(n)} = \lim_{n \to \infty} \|x_{\tau(n)} - z\| = 0.$$

It follows from (3.9), (3.8) and the condition (ii) that

$$\lim_{n \to \infty} \Gamma_{\tau(n)+1} = \lim_{n \to \infty} \|x_{\tau(n)+1} - z\| = 0.$$

Now it follows from (3.6) that

$$\lim_{n \to \infty} \Gamma_n = \lim_{n \to \infty} \|x_n - z\| = 0.$$

The proof is completed.

Clearly, Theorem 3.1 contains $\gamma_n \equiv 0$ and $\gamma_n = \frac{1}{n+1}$ as special cases. So the following result is obtained easily.

Corollary 3.2. Let C be a closed convex subset of a real 2-uniformly smooth Banach space E and let $T: C \to C$ be a λ -strict pseudo-contraction with $F(T) \neq \emptyset$. Given $u, x_0 \in C$, a sequence $\{x_n\}$ is generated by the modified Mann's iteration (1.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ in (0,1) satisfy:

(i)
$$\alpha_n \in [0, \mu], \ \mu = \min\{1, \frac{\lambda}{K^2}\}\ \text{with } \liminf \alpha_n(\lambda - K^2\alpha_n) > 0;$$

(ii)
$$\lim_{n \to \infty} \beta_n = 0$$
 and $\sum_{n=1}^{\infty} \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point z of T.

Using the same proof techniques as Theorem 3.1, we easily obtain the following result. Since the only difference is that $\alpha_n(\lambda - K^2\alpha_n)$ is replaced by $\alpha_n(q\lambda - C_q\alpha_n^{q-1})$ in its proof (Lemma 2.1 is replaced by Lemma 2.2 of Zhang and Su [23]), so we omit its proof.

Theorem 3.3. Let C be a closed convex subset of a real q-uniformly smooth Banach space E (q > 1) and let $T: C \to C$ be a λ -strict pseudo-contraction with $F(T) \neq \emptyset$. Given $u, x_0 \in C$, a sequence $\{x_n\}$ is generated by the modified Mann's iteration (1.2) or (1.3), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in (0,1) satisfy: (i) $\alpha_n \in [0,\mu], \mu = \min\{1, \{\frac{q\lambda}{C_q}\}^{\frac{1}{q-1}}\}$ with $\liminf_{n \to \infty} \alpha_n(q\lambda - C_q\alpha_n^{q-1}) > 0;$

(*ii*) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$; (*iii*) $\limsup_{n \to \infty} \gamma_n < 1$. $n \rightarrow \infty$

Then the sequence $\{x_n\}$ converges strongly to a fixed point z of T.

Remark 1. If E is q-uniformly smooth, then $1 < q \leq 2$ and E is uniformly smooth[11], and hence Theorem 3.1 may be regarded as a special case of Theorem 3.3.

Remark 2. In Theorem 3.1 and Corollary 3.3, the sequence $\{\gamma_n\}$ only need satisfy $\limsup \gamma_n < 1$. Then Theorem 3.1 may properly contain Theorem 2.3 of Zhou [22] and Theorem 3.1 of Chai and Song [1] as a special case, and Theorem 4.1 of Zhang and Su [23] may be obtained from Corollary 3.3. So our conclusions may be regarded as a unification of the some existing results.

Remark 3. Our main results are obtained in the frame of q-uniformly smooth Banach space, then in the future work, we may consider the results of this paper in n-Banach space. For more details on n-Banach space, see Dutta [2, 3, 4, 5, 6, 7, 8, 9, 10].

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