SOME BEST APPROXIMATION FORMULAS AND INEQUALITIES FOR WALLIS RATIO

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Abstract. In the paper, the authors establish some best approximation formulas and inequalities for Wallis ratio. These formulas and inequalities improve an approximation formula and a double inequality for Wallis ratio recently presented in "S. Guo, J.-G. Xu, and F. Qi, *Some exact constants for the approximation of the quantity in the Wallis' formula*, J. Inequal. Appl. 2013, **2013**:67, 7 pages; Available online at `http://dx.doi.org/10.1186/1029-242X-2013-67`".

1. Introduction

Wallis ratio is defined as

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\bigl(n+\frac12\bigr)}{\Gamma(n+1)},$$

where Γ is the classical Euler gamma function which may be defined by

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} \,\mathrm{d}\, u, \quad \Re(z) > 0. \tag{1.1}$$

The study and applications of W_n have a long history, a large amount of literature, and a lot of new results. For detailed information, please refer to the papers [1, 4, 18, 21], related texts in the survey articles [17, 19, 20] and references cited therein. Recently, Guo, Xu, and Qi proved in [5] that the double inequality

$$\sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} < W_n \le \frac{4}{3}\left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} \tag{1.2}$$

for $n \ge 2$ is valid and sharp in the sense that the constants $\sqrt{\frac{e}{\pi}}$ and $\frac43$ in (1.2) are best possible. They also proposed in [5] the approximation formula

$$W_n \sim \chi_n := \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, \quad n \to \infty. \tag{1.3}$$

The sharpness of the double inequality (1.2) was proved in [5] basing on the variation of a function which decreases on $[2,\infty)$ from $\frac43$ to $\sqrt{\frac{e}{\pi}}$. As a consequence, the right-hand side of (1.2) becomes weak for large values of n. Moreover, if we are interested to estimating W_n when n approaches infinity, then the constant $\sqrt{\frac{e}{\pi}}$ should be chosen and inequalities using $\sqrt{\frac{e}{\pi}}$ are welcome.

The aim of this paper is to improve the double inequality (1.2) and the approximation formula (1.3).

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2. A lemma

For improving the double inequality (1.2) and the approximation formula (1.3), we need the following lemma.

Lemma 2.1 ([12, Lemma 1.1]). *If the sequence $\{\omega_n : n \in \mathbb{N}\}$ converges to 0 and*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = \ell \in \mathbb{R} \tag{2.1}$$

for $k > 1$, then

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{\ell}{k-1}. \tag{2.2}$$

Remark 2.1. Lemma 2.1 was first established in [15] and has been effectively applied in many papers such as [2, 3, 6, 7, 8, 9, 10, 11, 13, 14, 16].

3. A best approximation formula

With the help of Lemma 2.1, we first provide a best approximation formula of Wallis ratio W_n.

Theorem 3.1. *The approximation formula*

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{1}{\sqrt{n}}, \quad n\to\infty \tag{3.1}$$

is the best approximation of the form

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n+a}}{n}, \quad n\to\infty, \tag{3.2}$$

where a is a real parameter.

Proof. Define $z_n(a)$ by

$$W_n = \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n+a}}{n}\exp z_n(a), \quad n \ge 1.$$

It is not difficult to see that $z_n(a) \to 0$ as $n \to \infty$, A direct computation gives

$$z_n(a) - z_{n+1}(a) = -\frac{a}{2n^2} + \left(\frac{1}{2}a + \frac{1}{2}a^2 + \frac{1}{12}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$\lim_{n\to\infty}\left\{n^2[z_n(a) - z_{n+1}(a)]\right\} = -\frac{a}{2}.$$

Making use of Lemma 2.1, we immediately see that the sequence $\{z_n(a) : n \in \mathbb{N}\}$ converges fastest only when $a = 0$. The proof of Theorem 3.1 is complete. □

Remark 3.1. The approximation formula (3.1) is an improvement of (1.3), since the approximation formula (1.3) is the special case $a = -1$ in (3.2).

4. An asymptotic series associated to (3.1)

In this section, by discovering an asymptotic series and a single-sided inequality for Wallis ratio, we further generalize the approximation formula (3.1) and improve the left-hand side of the double inequality (1.2).

Theorem 4.1. *As $n \to \infty$, we have*

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{1}{\sqrt{n}}\exp\left(\frac{1}{24n^2} + \frac{1}{48n^3} + \frac{1}{160n^4} + \frac{1}{960n^5} + \cdots\right).$$

Proof. Recall from [15] that, to an approximation formula $f(n) \sim g(n)$, the following asymptotic series is associated

$$f(n) \sim g(n) \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{n^k}\right),$$

where a_k for $k \geq 2$ is a solution of the following infinite triangular system

$$a_1 - \binom{k-1}{1} a_2 + \cdots + (-1)^k \binom{k-1}{k-2} a_{k-1} = (-1)^k x_k \tag{4.1}$$

and x_k are coefficients of the expansion

$$\ln \frac{f(n)g(n+1)}{g(n)f(n+1)} = \sum_{k=2}^{\infty} \frac{x_k}{n^k}.$$

Replacing $f(n)$ and $g(n)$ by W_n and $\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}}$ respectively yields

$$\ln \frac{f(n)g(n+1)}{g(n)f(n+1)} = \sum_{k=2}^{\infty} (-1)^k \left[\frac{1+(-1)^k}{(k+1)2^{k+1}} - \frac{1}{k+1} + \frac{1}{2k}\right] \frac{1}{n^k}.$$

Hence, the system (4.1) becomes

$$a_1 - \binom{k-1}{1} a_2 + \cdots + (-1)^k \binom{k-1}{k-2} a_{k-1} = \frac{1+(-1)^k}{(k+1)2^{k+1}} - \frac{1}{k+1} + \frac{1}{2k}$$

which has a solution

$$a_1 = 0, \quad a_2 = \frac{1}{24}, \quad a_3 = \frac{1}{48}, \quad a_4 = \frac{1}{160}, \quad a_5 = \frac{1}{960}, \quad \dots .$$

The proof of Theorem 4.1 is complete. $\square$

Theorem 4.2. *For every integer $n \geq 1$, we have*

$$W_n > \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{1}{24n^2} + \frac{1}{48n^3} + \frac{1}{160n^4} + \frac{1}{960n^5}\right). \tag{4.2}$$

Proof. It suffices to prove

$$\alpha_n = n \ln\left(1 - \frac{1}{2n}\right) - \frac{1}{2} \ln n - \ln \frac{(2n-1)!!}{(2n)!!} + \ln \sqrt{\frac{e}{\pi}} + h(n) < 0,$$

where

$$h(x) = \frac{1}{24x^2} + \frac{1}{48x^3} + \frac{1}{160x^4} + \frac{1}{960x^5}.$$

Because α_n converges to 0, it is sufficient to show that the sequence $\{\alpha_n : n \in \mathbb{N}\}$ is strictly increasing. It is not difficult to obtain $\alpha_{n+1} - \alpha_n = s(n)$, where

$$\begin{aligned} s(x) &= (x+1) \ln\left(1 - \frac{1}{2x+2}\right) - x \ln\left(1 - \frac{1}{2x}\right) \\ &\quad - \frac{1}{2} \ln\left(1 + \frac{1}{x}\right) - \ln \frac{2x+1}{2x+2} + h(x+1) - h(x), \\ s''(x) &= \frac{C(x-1)}{32x^7(x+1)^7(2x+1)^2(2x-1)^2} \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} C(x) &= 4913 + 33387x + 98177x^2 + 164799x^3 + 174543x^4 \\ &\quad + 121173x^5 + 55197x^6 + 15920x^7 + 2640x^8 + 192x^9. \end{aligned}$$

Accordingly, the function $s(x)$ is strictly convex on $[1,\infty)$. Combing this with the fact that $\lim_{x\to\infty} s(x)=0$ reveals that the function $s(x)$ on $[1,\infty)$, and so the sequence $\{s(n): n\in\mathbb{N}\}$, is positive. The proof of Theorem 4.2 is complete. $\square$

5. A new approximation formula and a double inequality

Finally we will find a new approximation formula and a double inequality for Wallis ratio W_n.

Theorem 5.1. *As $n\to\infty$, we have*

$$W_n \sim \mu_n := \sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3}\frac{1}{\sqrt{n}}. \tag{5.1}$$

Proof. Motivated by (3.1), we now ask for the best approximation of the form

$$W_n \sim \sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2(n+b)}\right]^{n+c}\frac{1}{\sqrt{n}}, \quad n\to\infty,$$

where b and c are real parameters. For this, let

$$W_n = \sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2(n+b)}\right]^{n+c}\frac{1}{\sqrt{n}}\exp\beta_n(b,c).$$

Then an easy calculation leads to

$$\begin{aligned}\beta_n(b,c)-\beta_{n+1}(b,c) &= \frac{1}{2}(c-b)\frac{1}{n^2}+\left(b^2-bc-\frac{1}{4}c+\frac{1}{12}\right)\frac{1}{n^3}\\ &\quad+\left(\frac{1}{4}c-\frac{1}{8}b+\frac{3}{4}bc-\frac{3}{8}b^2-\frac{3}{2}b^3+\frac{3}{2}b^2c-\frac{1}{16}\right)\frac{1}{n^4}+O\left(\frac{1}{n^5}\right).\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\left\{n^2[\beta_n(b,c)-\beta_{n+1}(b,c)]\right\}=\frac{c-b}{2}$$

and

$$\lim_{n\to\infty}\left\{n^2[\beta_n(b,b)-\beta_{n+1}(b,b)]\right\}=\frac{3b-1}{12}.$$

By Lemma 2.1, it follows that the sequence $\{\beta_n(b,c): n\in\mathbb{N}\}$ converges fastest only when $b=c=\frac{1}{3}$. The proof of Theorem 5.1 is complete. $\square$

Remark 5.1. We note that the approximation formula (5.1) is the most accurate possible among a class of approximation formulas mentioned above. The numerical computation in Table 1 shows the superiority of (5.1) over (1.3).

TABLE 1. Numerical computation

n	$W_n-\chi_n$	$W_n-\mu_n$
50	8.0124×10^{-4}	4.4198×10^{-9}
100	2.8269×10^{-4}	3.9124×10^{-10}
250	7.1425×10^{-5}	1.5850×10^{-11}
1000	8.9225×10^{-6}	1.2388×10^{-13}

Theorem 5.2. *For every integer $n\ge1$, we have*

$$\begin{aligned}\sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3}\frac{1}{\sqrt{n}}&<W_n\\ &<\sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3}\frac{1}{\sqrt{n}}\exp\left(\frac{1}{144n^3}\right).\end{aligned} \tag{5.2}$$

Proof. It is sufficient to prove

$$b_n = \left(n + \frac{1}{3}\right) \ln\left(1 - \frac{1}{2(n+1/3)}\right) - \frac{1}{2}\ln n - \ln\frac{(2n-1)!!}{(2n)!!} + \ln\sqrt{\frac{e}{\pi}} < 0$$

and

$$c_n = b_n + \frac{1}{144n^3} > 0.$$

Because b_n and c_n converge to 0, it suffices to show that b_n is strictly increasing and c_n is strictly decreasing. For this, we discuss the differences $b_{n+1} - b_n = p(n)$ and $c_{n+1} - c_n = q(n)$, where

$$\begin{aligned} p(x) = {} & \left(x + \frac{4}{3}\right) \ln\left(1 - \frac{1}{2(x+4/3)}\right) - \left(x + \frac{1}{3}\right) \ln\left(1 - \frac{1}{2(x+1/3)}\right) \\ & - \frac{1}{2}\ln\left(1 + \frac{1}{x}\right) - \ln\frac{2x+1}{2x+2} \end{aligned}$$

and

$$q(x) = p(x) + \frac{1}{144(x+1)^3} - \frac{1}{144x^3}.$$

Since

$$p''(x) = \frac{A(x-1)}{2x^2(3x+1)(3x+4)(x+1)^2(2x+1)^2(6x-1)^2(6x+5)^2} > 0$$

and

$$q''(x) = -\frac{B(x-1)}{12x^5(3x+1)(3x+4)(2x+1)^2(6x-1)^2(x+1)^5(6x+5)^2} < 0,$$

where

$$\begin{aligned} A(x) = {} & 351068 + 1516131x + 2684091x^2 + 2495340x^3 \\ & + 1285956x^4 + 348624x^5 + 38880x^6 \end{aligned}$$

and

$$\begin{aligned} B(x) = {} & 6780036 + 50421819x + 166596550x^2 + 322415601x^3 \\ & + 405307306x^4 + 346439295x^5 + 204449525x^6 + 82629900x^7 \\ & + 22094730x^8 + 3618864x^9 + 305208x^{10} + 7776x^{11}, \end{aligned}$$

it follows that $p(x)$ is strictly convex and $q(x)$ is strictly concave on $[1, \infty)$. As a result, considering the fact that $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} q(x) = 0$, we derive that $p(x) > 0$ and $q(x) < 0$ on $[1, \infty)$. Consequently, the sequences $\{p(n) : n \in \mathbb{N}\}$ and $\{q(n) : n \in \mathbb{N}\}$ are positive. The proof of Theorem 5.2 is complete. □

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