# The Clausenian hypergeometric function ${}_{3}F_{2}$ with unit argument and negative integral parameter differences

M. A. Shpot<sup>a,b,1,\*</sup>, H. M. Srivastava<sup>c,2</sup>

<sup>a</sup>Institute for Condensed Matter Physics, 79011 Lviv, Ukraine <sup>b</sup>Fakultät für Physik, Universität Duisburg-Essen, D-47048 Duisburg, Germany <sup>c</sup>Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

#### Abstract

New explicit as well as manifestly symmetric three-term summation formulas are derived for the Clausenian hypergeometric series  ${}_{3}F_{2}(1)$  with negative integral parameter differences. Our results generalize and naturally extend several similar relations published, in recent years, by many authors. An appropriate and useful connection is established with the quite underestimated 1974 paper by P. W. Karlsson.

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## 1. Introduction

The main object of the present paper is to derive two elegant and manifestly symmetric summation formulas (31) and (32) for Clausen's series  ${}_{3}F_{2}$  with unit argument (see, e.g., [1, 2, 3, 4, 5, 6, 7])

$${}_{3}F_{2}\begin{bmatrix} a, b, c; \\ & 1 \\ b+1+m, c+1+n; \end{bmatrix} \qquad (m, n \in \mathbb{N}_{0} := \mathbb{N} \cup \{0\}), \tag{1}$$

and *negative* integral parameter differences given by the arbitrary non-negative integers m and n,  $\mathbb{N}$  being (as usual) the set of *positive* integers. We also aim at supporting interest in generalized hypergeometric functions  ${}_{p}F_{q}(z)$  of p numerator and q denominator parameters (and with argument z) especially of the type exemplified above.

Our summation formulas (31) and (32) match very well the recent trend of finding new relationships for generalized hypergeometric functions. In fact, they are immediate and natural generalizations of more special formulas suggested, a decade ago, by Milgram [8, 9, 10, 11], which were further proved and employed by Miller and Paris [12] and Rathie and Paris [13] quite recently.

A substantial and wide-spread progress has been recently achieved in the classical field of investigating the generalized hypergeometric functions  ${}_{p}F_{q}(z)$  and producing various relationships between them. Very often the studied functions contain, in different ways, integers in their numerator and denominator parameters (see, for example, [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]).

<sup>\*</sup>Corresponding author

 $<sup>^{1}</sup> http://researchgate.net/profile/Mykola\_Shpot$ 

<sup>&</sup>lt;sup>2</sup>http://www.math.uvic.ca/faculty/harimsri/

More studied, however, are functions with *positive* integral parameter differences in pairs of their numerator and denominator parameters [19, 20, 24, 22, 23] just in the spirit of very well-known early papers by Karlsson [25] and Minton [26]. It may be of interest to recall here the following known reduction formula (see, for example, [25], [27], and also [28, p. 1080] and the references to more general results on hypergeometric reduction formulas, which are cited in [28]):

$${}_{p}F_{q}\left[\begin{array}{c}b_{1}+m_{1},\cdots,b_{r}+m_{r},a_{r+1},\cdots,a_{p};\\b_{1},\cdots,b_{r},b_{r+1},\cdots,b_{q};\end{array}\right]$$

$$=\sum_{j_{1}=0}^{m_{1}}\cdots\sum_{j_{r}=0}^{m_{r}}\Lambda(j_{1},\cdots,j_{r})z^{J_{r}}{}_{p-r}F_{q-r}\left[\begin{array}{c}a_{r+1}+J_{r},\cdots,a_{p}+J_{r};\\b_{r+1}+J_{r},\cdots,b_{q}+J_{r};\end{array}\right]$$

$$(r \leq \min\{p,q\}; p,q \in \mathbb{N}_{0}; p < q+1 \quad \text{when} \quad z \in \mathbb{C}; p = q+1 \quad \text{when} \quad |z| < 1),$$

where

$$J_r := j_1 + \dots + j_r$$

and

$$\Lambda(j_1,\cdots,j_r) = \binom{m_1}{j_1}\cdots\binom{m_r}{j_r}\frac{(b_2+m_2)_{J_1}\cdots(b_r+m_r)_{J_{r-1}}}{(b_1)_{J_1}\cdots(b_r)_{J_r}}(b_{r+1})_{J_r}\cdots(b_q)_{J_r}}$$

The general hypergeometric identity (2) was proved by Karlsson [25] and (in two markedly different *simpler* ways) by Srivastava [27]. More interestingly, various generalizations and *basic* (or q-) extensions of the hypergeometric identity (2) can be found in several sequels to the works by Karlsson [25] and Srivastava [27] (see, for example, [29]). Reference [30], on the other hand, contains *further* general results stemming from the hypergeometric identity (2) including *multivariable* generalizations. Furthermore, Karlsson's proof of the Karlsson-Minton summation formula (see, for details, [25]; see also [26], [19] and [28, p. 1080, Equation (20)]) was based upon the hypergeometric reduction formula (2).

There is another obscure and seemingly forgotten paper by Karlsson [31] in which similar results have been obtained for generalized hypergeometric functions  ${}_{p}F_{q}(z)$  with *negative* integral parameter differences. The papers [8, 11] and [12] mentioned at the beginning, as well as the present one, discuss the summation formulas for the functions  ${}_{3}F_{2}(1)$  that belong to the same category.

Our motivation in doing this work stems from calculations [32, 33, 34, 35, 36] in the field theory of Lifshitz points [37], where  ${}_{3}F_{2}$  functions of the type indicated in (1) appear as a part of the expansion coefficients of certain important functions (see, for explicit formulas, [35, Eqs. (5.69), (5.71)]). Such expansions appear as a result of a term-by term integration of special [38] Appell functions [39, 1, 4, 6, 7]. Owing to global universal features of mathematical description of the underlying systems with anisotropic scaling, similar expansions are expected to inevitably appear in a very broad class of statistical physics and (Lorentz violating) high energy theories as discussed in a review section of [36]. On the other side, a review and further references can be found in [40]. Owing to numerous potential applications, both in theoretical physics and mathematics [12, 13], we believe that functions (1) or the related ones, deserve to be studied in a best way.

In the following section, we explicitly write down the previous results of [31], [8, 11] and [12], which will be needed for establishing the necessary contacts and connections with the present work.

# 2. Background results

In 1974, Karlsson [31] derived a quite general reduction formula for generalized hypergeometric functions  ${}_{p}F_{q}(z)$  with generic negative integral parameter differences and for  $p \leq q+1$ . In the case when p = q+1 = 3, his

equation (6) in [31] may be written as follows:<sup>3</sup>

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+m, c+1+n;\\c+j)=0\end{bmatrix} \frac{m!n!}{(b)_{m+1}(c)_{n+1}} = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{(-m)_{i}(-n)_{j}}{i! j!}$$

$$\cdot \left(\frac{1}{(c+j)(b-c+i-j)} {}_{2}F_{1}\begin{bmatrix}a, c+j;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+1;\\c+j+$$

provided that no denominator parameter equals zero or a negative integer and  $|\arg(1-z)| < \pi$ . Here, and throughout this paper,  $m, n \in \mathbb{N}_0$  are arbitrary non-negative integers. The Pochhammer symbol  $(\lambda)_n$  is given by (see, e.g., [6, Ch. 1] and [19])

$$(0)_0 := 1 \qquad \text{and} \qquad (\lambda)_n \equiv \lambda(\lambda+1)\cdots(\lambda+n-1) \qquad (\lambda \in \mathbb{C}; \ n \in \mathbb{N})$$
(4)

and, in general, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad (n \in \mathbb{Z}).$$
(5)

Finally,  ${}_{2}F_{1}(a, b; c; z)$  is a Gauss hypergeometric function and  $\Gamma(z)$  is the usual Euler Gamma function (see, e.g., [1, 4, 2, 6]).

At unit argument z = 1, the Gauss hypergeometric functions on the right-hand side of (3) are summed by applying the celebrated Gauss summation theorem [1, Sec. 1.3]:

We thus find from (3) that

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+m, c+1+n;\end{bmatrix} \frac{1}{(b)_{m+1}(c)_{n+1}} = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{(-m)_{i}(-n)_{j}}{i! \, j!} \left(\frac{B(1-a, c+j)}{b-c+i-j} + \frac{B(1-a, b+i)}{c-b+j-i}\right).$$
(7)

Here

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad (a,b \neq 0,-1,-2,\cdots)$$
(8)

denotes the familiar Beta function.

In 2004, Milgram [8, Eq. (11)] suggested the following summation formula for Clausen's  ${}_{3}F_{2}(1)$  series:

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+n, c+1;\end{bmatrix} = \frac{(b)_{n}\Gamma(c+1)\Gamma(1-a)}{(b-c)_{n}\Gamma(c+1-a)} + c\Gamma(b+n)\Gamma(c-b+1-n)\sum_{\ell=0}^{n-1}\frac{\Gamma(n-\ell-a)(-1)^{\ell}}{\Gamma(b+n-a-\ell)\Gamma(n-\ell)\Gamma(c-b-n+2+\ell)}.$$
(9)

He further reproduced the summation formula (9) in slightly different forms in [11, 9, 10].

<sup>3</sup>The restriction  $\mu \neq 1$  appearing in (7) has to be read as  $\mu \neq i$ .

Quite recently, Miller and Paris [12] re-derived the summation formula (9) in further two equivalent forms. Their equation (1.6) reads as follows:

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+n, c+1;\end{bmatrix} = \frac{c\Gamma(1-a)(b)_{n}}{(b-c)_{n}} \left(\frac{\Gamma(c)}{\Gamma(1+c-a)} - \frac{\Gamma(b)}{\Gamma(1+b-a)} \sum_{k=0}^{n-1} \frac{(1-a)_{k}(b-c)_{k}}{(1+b-a)_{k}k!}\right),$$
(10)

where it is expressed in terms of the partial sum of a Gauss hypergeometric function  $_2F_1$  of unit argument,

$$_{2}F_{1}\begin{bmatrix}a, b;\\ c; \end{bmatrix}_{n} \equiv \sum_{k=0}^{n} \frac{(a)_{k}(b)_{k}}{(c)_{k} k!}.$$

By Lemma 2 of [12], which reads

the equation (10) can be equivalently written in terms of the terminating Clausen's series  ${}_{3}F_{2}(-n, b, c; d, e; 1)$  (see, for details, [12, Eq. (3.1)]).

In order to facilitate the comparison with our further results, we write down two variants of (10) for  $n \mapsto n+1$ . Following the choice  $a \mapsto 1 - a$  and  $b \mapsto b - c$  with  $c \mapsto 1 + b - a$  of [12] in applying (11) to (10), we obtain

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+n, c+1;\end{bmatrix} = c(b)_{n+1}\left(\frac{B(1-a,c)}{(b-c)_{n+1}} - \frac{B(1-a,b)}{n!}{}_{3}F_{2}\begin{bmatrix}-n, b, b-c;\\1+b-a, 1+b-c;\end{bmatrix}\right),$$
(12)

which exactly matches [12, Eq. (3.1)] in view of (8). Alternatively, upon setting  $a \mapsto b - c$  and  $b \mapsto 1 - a$  with the same c as above (that is, with  $c \mapsto 1 + b - a$ ), we arrive at the following result:

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+n, c+1;\end{bmatrix} = \frac{c(b)_{n+1}}{(b-c)_{n+1}}$$

$$\cdot \left(B(1-a,c) - B(1-a,b)\frac{(2-a)_{n}}{n!} {}_{3}F_{2}\begin{bmatrix}-n, 1-a, 1-a+c;\\2-a, 1-a+b;\end{bmatrix}\right).$$
(13)

In closing this section, let us only note that the  ${}_{3}F_{2}$  functions in the last two equations (12) and (13) are related by means of the following two-term Thomae transformation (see, e.g., [41, Entry (7.4.4.1)]):

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\ & 1\\ d, e;\end{bmatrix} = \frac{\Gamma(d)\,\Gamma(p-c)}{\Gamma(p)\,\Gamma(d-c)}\,{}_{3}F_{2}\begin{bmatrix}e-a, e-b, c;\\ & 1\\ p, e;\end{bmatrix} \qquad (p:=d+e-a-b),$$
(14)

which leaves the last pair of the numerator and denominator parameters (that is, c and e) unaltered.

### 3. Summation theorems

In this section, we prove two theorems that generalize the results quoted in the Section 2. Theorem 1 gives a symmetric variant of (12) and (13) in which the second denominator parameter c + 1 is replaced by c + 1 + n. Our Theorem 2 will go a step further by allowing the negative integral differences in two pairs of parameters of  ${}_{3}F_{2}$  series to be independent. **Theorem 1.** For arbitrary non-negative integers  $n \in \mathbb{N}_0$  and complex numbers  $a, b, c \in \mathbb{C}$ ,

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\ b+1+n, c+1+n; \end{bmatrix} \frac{(c-b)_{n+1}n!}{(b)_{n+1}(c)_{n+1}}$$

$$= B(1-a,b)\frac{(1-a)_{n}}{(1+b-a)_{n}} {}_{3}F_{2}\begin{bmatrix}-n, b, 1+n;\\ 1+b-c, a-n; \end{bmatrix}$$

$$+ (-1)^{1+n}B(1-a,c)\frac{(2-a+n)_{n}}{(1-a+c)_{n}} {}_{3}F_{2}\begin{bmatrix}-n, 1-a+b+n, 1+n;\\ 1+b-c, 2-a+n; \end{bmatrix}, \quad (15)$$

provided that  $\Re(2-a+2n) > 0$ .

**Proof.** In transforming the left-hand side of the assertion (15), we use the following Thomae three-term relation for  ${}_{3}F_{2}$  at unit argument quoted by Bailey [1, p. 21, Eq. (1)], which can be also found in [41, Entry (7.4.4.4)]: <sup>4</sup>

$${}_{3}F_{2}\begin{bmatrix}a, b, c; \\ e, f; \end{bmatrix} = \frac{\Gamma(e)\Gamma(e-a-b)}{\Gamma(e-a)\Gamma(e-b)} {}_{3}F_{2}\begin{bmatrix}a, b, f-c; \\ 1\\ a+b-e+1, f; \end{bmatrix} + \frac{\Gamma(e)\Gamma(f)\Gamma(a+b-e)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(f-c)\Gamma(e+f-a-b)} {}_{3}F_{2}\begin{bmatrix}e-a, e-b, e+f-a-b-c; \\ e-a-b+1, e+f-a-b; \end{bmatrix}, \quad (16)$$

where we have  $\Re(e+f-a-b-c) > 0$  for convergence of the  ${}_{3}F_{2}(1)$  series on the left-hand side and  $\Re(1+c-e) > 0$  for convergence of both of the  ${}_{3}F_{2}(1)$  series on the right-hand side. Up to their coefficients, the two resulting  ${}_{3}F_{2}(1)$  functions are given by

$${}_{3}F_{2}\begin{bmatrix}a, b, 1+n;\\ & & 1\\ a-n, 1+c+n;\end{bmatrix} \text{ and } {}_{3}F_{2}\begin{bmatrix}1-a+b+n, 1+n, 2-a+2n;\\ & & 1\\ 2-a+n, 2-a+c+2n;\end{bmatrix},$$
(17)

which converge when  $\Re(c-b) > n$ .

The crucial property of the applied transformation (16) is that the first of the functions in (17) has a pair of the upper and lower parameters (a, a - n) and the second one has (2 - a + 2n, 2 - a + n). In both cases the denominator parameter differs from the numerator parameter by a negative integer -n. Thus, for any nonnegative finite integer  $n < \Re(c-b)$ , each of the new  ${}_{3}F_{2}$  functions is reducible to a finite sum of products of Euler Gamma functions in light of the following relation [41, Entry (7.4.1.2)]:

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\ a-n, d;\end{bmatrix} = \frac{1}{(1-a)_{n}} \sum_{p=0}^{n} (-z)^{p} \binom{n}{p} (1-a)_{n-p} \frac{(b)_{p}(c)_{p}}{(d)_{p}} {}_{2}F_{1}\begin{bmatrix}b+p, c+p;\\ d+p;\end{bmatrix}$$
(18)

with z = 1 as well as the Gauss summation theorem (6).

Expressing the binomial coefficients in (18) via (see, e.g., [6, p. 22, Eq. (16)])

$$\binom{n}{p} = \frac{(-1)^p (-n)_p}{p!}$$

<sup>&</sup>lt;sup>4</sup>In this reference [41], this is the *next* entry to that employed by Miller and Paris [12] in their proof of the summation formula (10).

and after some algebra, we obtain a linear combination of finite sums as follows:

$$\sum_{p=0}^{n} \frac{(-n)_p (1+n)_p}{(a-n)_p p!} \frac{(b)_p}{(1+b-c)_p} \quad \text{and} \quad \sum_{p=0}^{n} \frac{(-n)_p (1+n)_p (1-a+b+n)_p}{(2-a+n)_p p!} \frac{1}{(1+b-c)_p}$$

Identifying each of these finite sums with terminating Clausen's series  ${}_{3}F_{2}(1)$ , we derive

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+n, c+1+n;\\ \hline 1\end{bmatrix}\frac{(c-b)_{n+1}n!}{(b)_{n+1}(c)_{n+1}}$$

$$=\Gamma(b)\frac{\Gamma(1-a+n)}{\Gamma(1-a+b+n)}{}_{3}F_{2}\begin{bmatrix}-n, b, 1+n;\\1+b-c, a-n;\\ \hline 1\\1+b-c, a-n;\\ \hline 1\\1+b-c, 2-a+n;\\ \hline 1\\1+b-c, 2-a+n;\\ \hline 1\end{bmatrix}.$$
(19)

Finally, upon using the reflection formula (34) to transform  $\Gamma(-1+a-n)$  in the second term, if we rearrange the involved Gamma functions, we obtain the summation formula recorded in (15).

**Remark 1.** By changing the numerator parameter  $a \mapsto a + 2n$  in (19), we are led easily to Corollary 1 below.

**Corollary 1.** The following summation formula:

$${}_{3}F_{2}\begin{bmatrix}a+2n, b, c;\\b+1+n, c+1+n;\end{bmatrix} \frac{(c-b)_{n+1}n!}{(b)_{n+1}(c)_{n+1}}$$

$$= \Gamma(b) \frac{\Gamma(1-a-n)}{\Gamma(1-a+b-n)} {}_{3}F_{2}\begin{bmatrix}-n, b, 1+n;\\1+b-c, a+n;\end{bmatrix}$$

$$+\Gamma(c) \frac{\Gamma(2-a)\Gamma(a+n-1)}{\Gamma(a+2n)\Gamma(1-a+c-n)} {}_{3}F_{2}\begin{bmatrix}-n, 1-a+b-n, 1+n;\\1+b-c, 2-a-n;\end{bmatrix}$$
(20)

holds true when  $\Re(2-a) > 0$ .

As already mentioned in Section 1, there are physical applications (see, for example, [35, Eqs. (5.69)-(5.73)] and [36, 44]), where reduction relations of this form are relevant and potentially useful.

The following Theorem 2 extends the result (15) of Theorem 1.

**Theorem 2.** For arbitrary non-negative integers  $m \in \mathbb{N}_0$  and  $n \in \mathbb{N}_0$ , and for complex parameters  $a, b, c \in \mathbb{C}$ ,

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+m, c+1+n;\end{bmatrix} \frac{(c-b)_{n+1}}{(b)_{m+1}(c)_{n+1}} = T_{m,n}^{(1)} + T_{m,n}^{(2)} \qquad \left(\Re(2-a+m+n)>0\right),\tag{21}$$

where

$$T_{m,n}^{(1)} = B(1-a,b) \frac{(1-a)_m}{(1+b-a)_m m!} {}_3F_2 \begin{bmatrix} -m, b, 1+n; \\ 1 \\ 1+b-c, a-m; \end{bmatrix}$$
(22)

and

$$T_{m,n}^{(2)} = (-1)^{1+m} B(1-a,c) \frac{(2-a+m)_n}{(1-a+c)_n n!} (c-b)_{n-m} {}_{3}F_2 \begin{bmatrix} -n, \ 1-a+b+m, \ 1+m; \\ 2-a+m, \ 1+b-c+m-n; \end{bmatrix},$$
(23)

provided that  $\Re(2-a+m+n) > 0$ .

The proof of Theorem 2 proceeds precisely along the same lines as those of Theorem 1.

It is fairly straightforward to see that, in its special case when m = n, the summation formula (21) reduces rather trivially to (15).

## 4. Special cases and consequences

For n = 0, the equation (15) coincides with that of Milgram (9) and Miller and Paris (10) in the special case when n = 1 and yields the following known result [41, Entry (7.4.4.16)] with a and c interchanged:

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1, c+1;\end{bmatrix} = \frac{bc}{c-b}\Gamma(1-a)\left[\frac{\Gamma(b)}{\Gamma(1-a+b)} - \frac{\Gamma(c)}{\Gamma(1-a+c)}\right]$$

Another known result [41, Entry (7.4.4.17)] with  $a \leftrightarrow c$  is the special case of (15) when n = 1. The summation formula (15) gives access to simple generalizations of [41, Entries (7.4.4.16) and (7.4.4.17)] with arbitrary equal integral enhancements of the denominator parameters.

As  $n \ge 1$ , the formula (15) cannot directly match the equations (12) and (13) for symmetry reasons: In this case, the parameter differences in  ${}_{3}F_{2}(1)$  become asymmetric therein. We next show that these formulas, and hence (9) and (10), follow from the result (21) both for m = 0 and n = 0.

Let us first put m = 0 in (21) to (23). In this case, we have to deal with the Clausenian hypergeometric function

$${}_{3}F_{2}\begin{bmatrix}a, b, c; \\ & 1\\b+1, c+1+n;\end{bmatrix}$$

on the left-hand side. In order to compare it with (9) and (10) or with (12) and (13), the parameters b and c have to be interchanged.

At m = 0, the  ${}_{3}F_{2}$  function in  $T_{m,n}^{(1)}$  given by (22) reduces to 1, and its factor, together with the one of (21), trivially combine to the first term on the right-hand side in (12) or (13) with  $b \leftrightarrow c$ . The remaining non-trivial  ${}_{3}F_{2}$  function from  $T_{m,n}^{(2)}$  given by (23) simplifies to the following form:

$${}_{3}F_{2}\begin{bmatrix} -n, \ 1-a+b, \ 1; \\ 1 \\ 1+b-c-n, \ 2-a; \end{bmatrix}.$$
(24)

It does not match directly any of the  ${}_{3}F_{2}$  functions from (12) or (13) because of a subtraction -n in one of the denominator parameters. To get rid of this term, we use the relation [41, Entry (7.4.4.85)]. When read in the reverse direction, it can be written as follows:

$${}_{3}F_{2}\begin{bmatrix}-n, a, b;\\ c-n, d; \end{bmatrix} = \frac{(1+a-c)_{n}}{(1-c)_{n}} {}_{3}F_{2}\begin{bmatrix}-n, a, d-b;\\ 1+a-c, d; \end{bmatrix}.$$
(25)

The evident choice  $a \mapsto 1 - a + b$  and  $b \mapsto 1$  transforms the  ${}_{3}F_{2}$  function from (24) via

$${}_{3}F_{2}\begin{bmatrix} -n, \ 1-a+b, \ 1; \\ 1+b-c-n, \ 2-a; \end{bmatrix} = \frac{(1-a+c)_{n}}{(c-b)_{n}} {}_{3}F_{2}\begin{bmatrix} -n, \ 1-a, \ 1-a+b; \\ 2-a, \ 1-a+c; \end{bmatrix}.$$
(26)

With  $b \leftrightarrow c$ , the last hypergeometric function in (26) matches that in the second term of (13). Collecting its factors from (26), (23), and (21), it is a matter of simple algebra to bring them to the form required by (13).

If we now take n = 0 in the Eqs. (21)–(23), the connection with (12) is established with the same set of parameters a, b, and c. This time we have to care only about a simple exchange of m and n.

Again, at n = 0, the  ${}_{3}F_{2}$  function from  $T_{m,n}^{(1)}$  given by (22) transforms to a "correct" one from (13) on applying (25) with  $a \mapsto 1$ ,  $b \mapsto b$ ,  $c \mapsto a$ , and  $d \mapsto 1 + b - c$ , while the one from  $T_{m,n}^{(2)}$  given by (23) reduces to 1. Matching the factor at the transformed function is again simple. Reducing  $T_{m,0}^{(2)}$  to the first term in (13) requires the use of the following transformation formula (see, e.g., [6, p. 22, Eq. (19)]):

$$(\lambda)_{-m} = \frac{(-1)^m}{(1-\lambda)_m}, \qquad n \in \mathbb{N}, \qquad \lambda \notin \mathbb{Z}$$

for Pochhammer symbols in (5) with  $\lambda \mapsto c - b$ .

**Remark 2.** We note in passing that the relation [41, Entry (7.4.4.86)] rewritten similarly as (25),

$${}_{3}F_{2}\begin{bmatrix}-n, a, b-n;\\ c-n, d-n; \end{bmatrix} = \frac{(1+a-c)_{n}(1-b)_{n}}{(1-c)_{n}(1-d)_{n}} {}_{3}F_{2}\begin{bmatrix}-n, d-b, 1-c;\\ 1\\ 1+a-c, 1-b; \end{bmatrix},$$
(27)

may be also useful in practical calculations.

For instance, the function  $G_{m,k}(t)$  appearing in [23, Eq. (6)] can be reduced with the help of (27) via

$${}_{3}F_{2}\begin{bmatrix} -m+k, t+k, c-a-b-m; \\ c-a-m+k, c-b-m+k; \end{bmatrix} \longrightarrow {}_{3}F_{2}\begin{bmatrix} -m+k, 1+a-c, c-b-t-m; \\ 1\\ 1-b-m, 1-t-m; \end{bmatrix}$$

to a form where the summation index k appears only once in a numerator parameter.

#### 5. Results in the final form

In the preceding sections, we learned that the standard identities (14) and (25) are useful in dealing with the involved functions. With this in mind, we use first (25) to transform the  ${}_{3}F_{2}$  functions appearing in (22) and (23). For the first of them, with the choice  $a \mapsto b$  and  $b \mapsto 1 + n$ , we obtain

$${}_{3}F_{2}\begin{bmatrix} -m, b, 1+n; \\ 1+b-c, a-m; \end{bmatrix} = \frac{(1-a+b)_{m}}{(1-a)_{m}} {}_{3}F_{2}\begin{bmatrix} -m, b, b-c-n; \\ 1+b-a, 1+b-c; \end{bmatrix}.$$
(28)

Similarly, for the second one, we have

$${}_{3}F_{2}\begin{bmatrix} -n, \ 1-a+b+m, \ 1+m; \\ 2-a+m, \ 1+b-c+m-n; \end{bmatrix} = \frac{(1-a+c)_{n}}{(c-b-m)_{n}} {}_{3}F_{2}\begin{bmatrix} -n, \ 1-a, \ 1-a+b+m; \\ 1+c-a, \ 2-a+m; \end{bmatrix} .$$
(29)

The  ${}_{3}F_{2}$  function in (28) looks very good: It is a natural generalization of that in (12). In our hope to obtain a more symmetric expression for the function appearing in (29), we transform it via (14):

$${}_{3}F_{2}\begin{bmatrix}-n, \ 1-a, \ 1-a+b+m;\\ 1+c-a, \ 2-a+m; \end{bmatrix} = \frac{(1+c-b)_{n}}{(2-a+m)_{n}} {}_{3}F_{2}\begin{bmatrix}-n, \ c, \ c-b-m;\\ 1+c-a, \ 1+c-b; \end{bmatrix}.$$
(30)

Now the  ${}_{3}F_{2}$  functions in (29) and (30) are symmetric with respect to the interchange  $b \leftrightarrow c$  and  $m \leftrightarrow n$ , which is quite satisfactory. Using the last three equations in (21)–(23) we obtain the final result:

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+m, c+1+n;\\ \end{bmatrix}\frac{1}{(b)_{m+1}(c)_{n+1}}$$

$$=\frac{B(1-a,b)}{(c-b)_{n+1}m!} {}_{3}F_{2}\begin{bmatrix}-m, b, b-c-n;\\1+b-a, 1+b-c;\\ \end{bmatrix} +\frac{B(1-a,c)}{(b-c)_{m+1}n!} {}_{3}F_{2}\begin{bmatrix}-n, c, c-b-m;\\1+c-a, 1+c-b;\\ \end{bmatrix} .$$
(31)

Upon setting m = n in (31), we are led to the following

**Corollary 2.** The summation formula (32) follows from (31) for  $m \mapsto n$ :

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1+n, c+1+n;\\-\frac{B(1-a,b)}{(c-b)_{n+1}} {}_{3}F_{2}\begin{bmatrix}-n, b, b-c-n;\\1+b-a, 1+b-c;\\1\end{bmatrix} + \frac{B(1-a,c)}{(b-c)_{n+1}} {}_{3}F_{2}\begin{bmatrix}-n, c, c-b-n;\\1+c-a, 1+c-b;\\1\end{bmatrix}.$$
 (32)

The Clausenian series on the left-hand side of (31) and (32) converge when

$$\Re(2-a+m+n) > 0$$
 and  $\Re(2-a+2n) > 0$ ,

respectively. This implies that both of the summation formulas make sense for generic  $m \in \mathbb{N}_0$  and  $n \in \mathbb{N}_0$ , provided that  $\Re(a) < 2$ . Under these conditions, all functions at both sides are well defined, and, by the principle of analytic continuation, the restriction  $n < \Re(c - b)$  imposed in intermediate calculations (see (17)) can be removed.

**Remark 3.** Additional restrictions (for example,  $a \neq 1$ ) imposed by (8) on the Beta functions in the righthand sides of (31) and (32) do not shrink the applicability region of both equations. Singularities, which arise when the arguments of these Beta functions approach negative integers mutually cancel in the whole two-term combinations. Each of such dangerous cases has to be treated separately, in a manner similar to that of [16, 11]: In these references, the special case when m = 0 and  $b \rightarrow c$  has been considered for arbitrary non-negative integer n. It is not complicated to see that, when a = 1, the  ${}_{3}F_{2}$  functions on the right-hand sides of (31) and (32) reduce to the Gauss functions  ${}_{2}F_{1}$ , which can be summed via the Gauss summation theorem (6), and the whole resulting combinations at singular B(1 - a) vanish as they should. Proceeding similarly as in [16, 11] would lead us to a finite limit as  $a \rightarrow 1$  for the relations (31) and (32).

Let us proceed with a proposition, which gives a simple demonstration of Karlsson's result (7), followed by several remarks.

10

**Proposition 1.** The double-sum representation of (7) can be expressed as a three-term summation formula (31) for the Clausenian hypergeometric function  ${}_{3}F_{2}$ .

**Proof.** In both of Karlsson's formulas (3) and (7), the summations over i in the first terms and over j in the second terms are the same up to notations and they are given by

$$\sum_{k=0}^{m} \frac{(-m)_k}{(a+k)\,k!} = \frac{1}{a} \sum_{k=0}^{m} \frac{(-m)_k(a)_k}{(a+1)_k\,k!} = \frac{1}{a} \,_2F_1 \begin{bmatrix} -m, \ a; \\ \\ \\ \\ a+1; \end{bmatrix} = \frac{\Gamma(a)\,m!}{\Gamma(a+1+m)},\tag{33}$$

where we have used the following simple property of the Pochhammer symbol in (4):

$$(a)_{n+1} = a(a+1)_n = (a)_n(a+n)$$

as well as the Gauss summation theorem (6). We take here a = b - c - j and use the familiar reflection formula (see, for example, [3, Ch. 2.17] and [6, Ch. 1])

$$\Gamma(a-j)\Gamma(1-a+j) = \frac{\pi}{\sin\pi(a-j)} = \frac{\pi(-1)^j}{\sin\pi a} = (-1)^j \Gamma(a)\Gamma(1-a) \qquad (a \notin \mathbb{Z}).$$
(34)

It gives an analytic continuation for the Euler Gamma function and implies that

$$\Gamma(a-j) = \frac{(-1)^j \,\Gamma(a)}{(1-a)_j} \qquad (a \notin \mathbb{Z}).$$

Hence we obtain

$$\sum_{k=0}^{m} \frac{(-m)_k}{(b-c-j+k)\,k!} = \frac{m!}{(b-c)_{m+1}} \,\frac{(c-b-m)_j}{(1+c-b)_j} \tag{35}$$

for the inner sums in the first terms of (3) and (7). Moreover, just as we mentioned above, in the second terms of (3) and (7), we have the same thing up to such replacements as  $b \leftrightarrow c$  and  $i \leftrightarrow j$ . Inserting (35) into (7), and after some algebra, we obtain its compact and elegant representation (31) in terms of the  ${}_{3}F_{2}$  functions, which result from summations over the remaining indices.

**Remark 4.** Proceeding in the same fashion in the case of a z-dependent  ${}_{3}F_{2}$  function (3) and using the following linear transformation of the Gauss hypergeometric function (see, e.g., [6, p. 33, Eq. (19)]):

$${}_{2}F_{1}(a,b;c;z) = (1-z)^{-a} {}_{2}F_{1}\left(a,c-b;c;\frac{z}{z-1}\right) \qquad \left(c \neq 0,-1,-2,\cdots; |\arg(1-z)| < \pi\right), \tag{36}$$

we could simplify the second summation there and this procedure could also lead to some interesting and useful results.

**Remark 5.** Since the parameters of  $_2F_1$  functions in (3) are of the form a, b, b+1, they are just hypergeometric representations, via [41, Entry (7.3.1.28)], of the incomplete Beta function  $B_z(a, b)$  given by (see, e.g., [6, Ch. 1])

$$B_z(a,b) = \int_0^z t^{a-1} (1-t)^{b-1} dt$$

Indeed we have

$$B_{z}(a,b) = a^{-1}z^{a}{}_{2}F_{1}(a,1-b,a+1;z) = a^{-1}z^{a}(1-z)^{b-1}{}_{2}F_{1}\left(1-b,1,a+1;\frac{z}{z-1}\right),$$

where the first equality is given by [42, p. 263, Entry (6.6.8)] and the second one is a result of the linear transformation (36).

**Remark 6.** Our summation formula (31) gives an equivalent alternative representation of (7) by Karlsson [31]. Both (31) and its special case (32) are direct generalizations of several results by other authors quoted in Section 2. Their equations (9), (10), (12) and (13) are *immediate* consequences of (31).

In concluding this section, we note that, in a private communication, Christian Krattenthaler suggested an alternative derivation of the summation formula (31), which employs the equation (3.3.3) in its limit case when  $q \rightarrow 1$  along with the equation (3.1.1) of the book [43].

## 6. Concluding remarks

In our present investigation, we have proposed the summation formulas (31) and (32) for the Clausenian hypergeometric function  ${}_{3}F_{2}$  with unit argument and arbitrary negative integral parameter differences in two pairs of the upper and lower parameters. Our formulas are alternative representations of two special cases of more general reduction formulas derived in 1974 by Karlsson [31] in a form of multiple finite sums. The manifestly symmetric three-term relations for  ${}_{3}F_{2}$  functions with unit argument recorded in (31) and (32) are evidently more advantageous than some double-sum representations quoted in (7). They can be easily transformed, by using standard relations for  ${}_{3}F_{2}$  functions, according to the specific needs of certain calculations.

As discussed already in Section 1, the present summation formulas are of interest from the point of view of practical applications in field theories in *d*-dimensional spaces  $\mathbb{R}^d = \mathbb{R}^D \oplus \mathbb{R}^m$ , where a global rotational O(d) symmetry no more exists (see, for details, [32, 33, 34, 35, 36, 44]) or in relativistic field theories with broken Lorentz invariance of the space-time, the so-called Lorentz violating theories. A short review of both statistical and high-energy physics realizations of such theories can be found in the introduction of [36].

Some implications from the purely mathematical side are mentioned in our Remarks 1 to 6.

Of course, it would be of great interest to derive representations of similar kind for the z-dependent functions  ${}_{3}F_{2}$  starting from (3) or by some other means. A specimen of such relations is given in [41, Entry (7.4.1.5)], it is obtained by trivially applying the result (3) for the case when m = n = 0:

$${}_{3}F_{2}\begin{bmatrix}a, b, c;\\b+1, c+1;\end{bmatrix} = \frac{1}{c-b}\left(c {}_{2}F_{1}\begin{bmatrix}a, b;\\b+1;\end{bmatrix} - b {}_{2}F_{1}\begin{bmatrix}a, c;\\z\\c+1;\end{bmatrix}\right).$$

Also, it should be interesting to write down relations analogous to (31) involving both positive and negative integers m and n and to derive counterparts of equations (31) and (32) for *bilateral* hypergeometric series [45].

Finally, as suggested by the anonymous referee, similar problems might be considered for certain special cases of Kampé de Fériet functions, which would be related to earlier works of Srivastava [46] and Karlsson [47].

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