STANCU TYPE GENERALIZATION OF THE q-FAVARD-SZÀSZ OPERATORS

ALİ KARAİSA, DURHASAN TURGUT TOLLU AND YASİN ASAR

ABSTRACT. In this paper, we introduce a Stancu type generalization of the q-Favard-Szàsz operators, estimate the rates of statistical convergence and study the local approximation properties of these operators.

1. Introduction

In [10], Jakimovski and Leviatan introduced a Favard Szàsz type operator, by using Appell polynomials $p_k(x)$, $k \ge 0$, defined by

$$g(u) e^{-ux} = \sum_{k=0}^{\infty} p_k(x) u^k,$$

where $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in the disc |z| < R, R > 1 and $g(1) \neq 0$, and they established several approximation properties of these operators. Ciupa [8], by defined the following operators

$$P_{n,t}\left(f;x\right) = \frac{e^{-nx}}{g\left(1\right)} \sum_{k=0}^{\infty} p_k\left(nt\right) f\left(x + \frac{k}{n}\right)$$

and investigated the approximation properties and the rate of convergence of these operators via the modulus of continuity.

In [6], Atakut and Büyükyazıcı studied some approximation properties of the operators

$$P_{n,t}^{\alpha,\beta}\left(f;x\right) = \frac{e^{-nx}}{g\left(1\right)} \sum_{k=0}^{\infty} p_k\left(nt\right) f\left(x + \frac{k+\alpha}{n+\beta}\right)$$

where $p_k(nt)$ is an Appell polynomial in nt which is a Stancu type generalization of the classical Favard-Szàsz operators. Moreover, In [5], the same authors established the approximation properties of the operators

$$L_{n}(f;q,x) = \frac{E^{-[n]_{q}t}}{A(1)} \sum_{k=0}^{\infty} \frac{P_{k}(q;[n]_{q}t)}{[k]_{q}!} f\left(x + \frac{[k]_{q}}{[n]_{q}}\right),$$

which is a q-analogue of the classical Favard-Szàsz operators related to the q-Appell polinomials. They also estimated the rate of convergence of these operators.

Now, let us define Stancu type generalization of the q-Favard-Szàsz operators as follows:

$$T_{n,t}^{\alpha,\beta}\left(f;q;x\right) = \frac{E_{q}^{-\left[n\right]_{q}t}}{A\left(1\right)} \sum_{k=0}^{\infty} \frac{P_{k}\left(q;\left[n\right]_{q}t\right)}{\left[k\right]_{q}!} f\left(x + \frac{\left[k\right]_{q} + \alpha}{\left[n\right]_{q} + \beta}\right),$$

where $\{P_k(q;.)\}_{k\geqslant 0}$ is a q-Appell polynomial set which is generated by

(1.2)
$$A(u) e_q^{[n]_q t u} = \sum_{k=0}^{\infty} \frac{P_k(q; [n]_q t) u^k}{[k]_q!}$$

and A(t) is defined by

$$A(u) = \sum_{k=0}^{\infty} a_k u^k.$$

In this work, we investigate a Korovkin theorem and the rate of statistical convergence by using modulus of continuity of (1.1). We also obtain some local approximation results of these new operators.

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Let us recall some definitions and notations regarding the concept of q-calculus. Further results can be found in [7]. In the sequel, q is a real number satisfying 0 < q < 1. For $n \in \mathbb{N}$, the q-integer $[n]_q$ is defined by

$$[n]_q := \frac{1 - q^n}{1 - q}$$

and the q-factorial $[n]_q!$ is defined as following

$$[n]_q! := \left\{ \begin{array}{l} [n]_q [n-1]_q \cdots [1]_q, & n \in \mathbb{N} \\ 1, & n = 0 \end{array} \right.$$

The q-binomial coefficients are given by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q! \left[n-k\right]_q!}, 0 \leqslant k \leqslant n.$$

The q-derivative $D_q f$ of a function f is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ x \neq 0.$$

Also, if there exists $\frac{df}{dx}(0)$, then $(D_q f)(0) = \frac{df}{dx}(0)$. The following q-derivatives of the product of the functions f(x) and g(x) are equivalent:

$$D_{a}(f(x)g(x)) = f(qx) D_{a}g(x) + g(x)D_{a}f(x)$$

and

$$D_q(f(x)g(x)) = f(x) D_q g(x) + g(qx) D_q f(x).$$

The q-analogues of the exponential function are given by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$

and

$$E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}.$$

The exponential functions have the following properties:

$$D_{q}\left(e_{q}^{ax}\right)=ae_{q}^{ax},\ D_{q}\left(E_{q}^{ax}\right)=aE_{q}^{aqx},\ e_{q}^{x}E_{q}^{-x}=E_{q}^{x}e_{q}^{-x}=1.$$

2. STATISTICAL APPROXIMATION PROPERTIES

Before proceeding further, let us give basic definition and notation on the concept of the statistical convergence which was introduced by Fast [11]. Let K be a subset of \mathbb{N} , the set of natural numbers. Then, $K_n = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n} |K_n|$ provided that the limit exists, where $|K_n|$ denotes the cardinality of the set K_n . A sequence $x = (x_k)$ is called statistically convergent to the number $\ell \in \mathbb{R}$, denoted by $st - \lim x = \ell$. For each $\epsilon > 0$, the set $K_{\epsilon} = \{k \in \mathbb{N} : |x_k - \ell| \geq \epsilon\}$ has a natural density zero, that is

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leqslant n : |x_k - \ell| \geqslant \epsilon\}| = 0.$$

It is well know that every statistically convergence sequence is ordinary convergent, but the converse is not true. The concept of statistical convergence was firstly used in approximation theory by Gadjiev and Orhan [12]. They proved the Bohman-Korovkin type approximation theorem for statistical convergence. For further information related to the statistical approximation of the operators, the followings are remarkable among others: [13–17].

Now, we may begin the following lemma which is needed proving our main result.

Lemma 2.1. For $n \in \mathbb{N}$, $x \in [0, \infty)$ and 0 < q < 1, we have

$$(2.1) \quad T_{n,t}^{\alpha,\beta}(e_{0}(t);q;x) = 1,$$

$$(2.2) \quad T_{n,t}^{\alpha,\beta}(e_{1}(t);q;x) = x + \frac{[n]_{q}t}{[n]_{q}+\beta} + \frac{\alpha}{[n]_{q}+\beta} + \frac{RD_{q}(A(1))}{[n]_{q}+\beta},$$

$$T_{n,t}^{\alpha,\beta}(e_{2}(t);q;x) = \left(x + \frac{[n]_{q}t}{[n]_{q}+\beta}\right)^{2} + \frac{2x[\alpha + RD_{q}(A(1))]}{[n]_{q}+\beta}$$

$$+ \frac{\alpha^{2} + 2R\alpha D_{q}(A(1) + D_{q}^{2}(A(1))}{\left([n]_{q}+\beta\right)^{2}} + \frac{[n]_{q}t[2\alpha + D_{q}(A(1))]}{\left([n]_{q}+\beta\right)^{2}} + \frac{q[n]_{q}t}{\left([n]_{q}+\beta\right)^{2}},$$

where
$$R = \frac{e_q^{q[n]_q t} E_q^{-[n]_q t}}{A(1)}$$
, and $e_v(t) = (x+t)^v$, $v = 0, 1, 2$.

Proof. If we consider (1.2), we can show the following

(2.4)
$$\sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t \right)}{[k]_q!} = A(1) e_q^{[n]_q t},$$

(2.5)
$$\sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t\right)}{[k]_q!} [k]_q = A(1) [n]_q t e_q^{[n]_q t} + e_q^{q[n]_q t} D_q A(1)$$

and

$$(2.6) \quad \sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t\right)}{\left[k\right]_q!} \left[k\right]_q^2 = D_q^2 \left(A\left(1\right)\right) e_q^{[n]_q t} + D_q(A\left(1\right)) \left[n\right]_q t e_q^{[n]_q t} + \left[n\right]_q t D_q A\left(q\right) e_q^{q[n]_q t} + A(1) \left[n\right]_q^2 t^2 e_q^{[n]_q t}.$$

By using the relations (2.4)-(2.6), from (1.1), we obtain the results

$$T_{n,t}^{\alpha,\beta}\left(e_{0}(t);q;x\right) = \frac{E_{q}^{-[n]_{q}t}}{A\left(1\right)} \sum_{k=0}^{\infty} \frac{p_{k}\left(q;[n]_{q}t\right)}{[k]_{q}!} = \frac{E_{q}^{-[n]_{q}t}}{A\left(1\right)} A\left(1\right) e_{q}^{[n]_{q}t} = 1,$$

$$\begin{split} T_{n,t}^{\alpha,\beta}\left(e_{1}(t);q;x\right) &= \frac{E_{q}^{-[n]_{q}t}}{A\left(1\right)}\sum_{k=0}^{\infty}\frac{p_{k}\left(q;[n]_{q}t\right)}{\left[k\right]_{q}!}\left(x+\frac{\left[k\right]_{q}+\alpha}{\left[n\right]_{q}+\beta}\right) \\ &= x+\frac{\alpha E_{q}^{-[n]_{q}t}}{A\left(1\right)\left(\left[n\right]_{q}+\beta\right)}\sum_{k=0}^{\infty}\frac{p_{k}\left(q;[n]_{q}t\right)}{\left[k\right]_{q}!}+\frac{E_{q}^{-[n]_{q}t}}{A\left(1\right)\left(\left[n\right]_{q}+\beta\right)}\sum_{k=0}^{\infty}\frac{p_{k}\left(q;[n]_{q}t\right)}{\left[k\right]_{q}!}\left[k\right]_{q} \\ &= x+\frac{\alpha}{\left[n\right]_{q}+\beta}+\frac{E_{q}^{-[n]_{q}t}}{A\left(1\right)\left(\left[n\right]_{q}+\beta\right)}\left(A\left(1\right)\left[n\right]_{q}te_{q}^{\left[n\right]_{q}t}+e_{q}^{q\left[n\right]_{q}t}D_{q}A\left(1\right)\right) \\ &= x+\frac{\left[n\right]_{q}t}{\left[n\right]_{q}+\beta}+\frac{\alpha}{\left[n\right]_{q}+\beta}+\frac{D_{q}(A(1))R}{\left[n\right]_{q}+\beta} \end{split}$$

and

$$\begin{split} T_{n,t}^{\alpha,\beta}\left(e_{2}(t);q;x\right) &= \frac{E_{q}^{-[n]_{q}t}}{A\left(1\right)}\sum_{k=0}^{\infty}\frac{p_{k}\left(q;[n]_{q}t\right)}{[k]_{q}!}\left(x+\frac{[k]_{q}+\alpha}{[n]_{q}+\beta}\right)^{2} \\ &= x^{2}+\frac{2\alpha x}{[n]_{q}+\beta}+\frac{\alpha^{2}}{\left([n]_{q}+\beta\right)^{2}}+\frac{2xE_{q}^{-[n]_{q}t}}{A\left(1\right)\left([n]_{q}+\beta\right)}\sum_{k=0}^{\infty}\frac{p_{k}\left(q;[n]_{q}t\right)}{[k]_{q}!}\left[k\right]_{q} \\ &+\frac{2\alpha E_{q}^{-[n]_{q}t}}{A\left(1\right)\left([n]_{q}+\beta\right)^{2}}\sum_{k=0}^{\infty}\frac{p_{k}\left(q;[n]_{q}t\right)}{[k]_{q}!}\left[k\right]_{q} \\ &+\frac{E_{q}^{-[n]_{q}t}}{A\left(1\right)\left([n]_{q}+\beta\right)^{2}}\sum_{k=0}^{\infty}\frac{p_{k}\left(q;[n]_{q}t\right)}{[k]_{q}!}\left[k\right]_{q}^{2} \\ &=\left(x+\frac{[n]_{q}t}{[n]_{q}+\beta}\right)^{2}+\frac{2x[\alpha+RD_{q}(A(1))]}{[n]_{q}+\beta} \\ &+\frac{\alpha^{2}+2R\alpha D_{q}(A(1)+D_{q}^{2}(A(1))}{\left([n]_{q}+\beta\right)^{2}}+\frac{[n]_{q}t[2\alpha+D_{q}(A(1))]}{\left([n]_{q}+\beta\right)^{2}}+\frac{q\left[n\right]_{q}t}{\left([n]_{q}+\beta\right)^{2}}. \end{split}$$

Hence, the proof is completed.

Theorem 2.2. Assume that $q := (q_n)$, $0 < q_n < 1$ be a sequence satisfying the following conditions:

(2.7)
$$st - \lim_{n} q_n = 1, \quad st - \lim_{n} q_n^n = b, \quad b < 1$$

Then, if f is any monotone increasing continuous function defined on [0,a], we have the following:

$$st - \lim_{n} || T_{n,t}^{\alpha,\beta}(f, q_n; .) - f ||_{C[0,a]} = 0.$$

Proof. It is enough to prove that

$$st - \lim_{n} \| T_{n,t}^{\alpha,\beta}(e_v(t), q_n; .) - e_v(t) \|_{C[0,a]} = 0$$

where v = 0, 1, 2.

From the equation (2.1), it is easy to obtain that

$$st - \lim_{n,t} || T_{n,t}^{\alpha,\beta}(e_0(t), q_n; .) - e_0(t) ||_{C[0,a]} = 0.$$

By taking $\sup_{x,t\in[0,a]}$ in (2.2), we get

$$||T_{n,t}^{\alpha,\beta}(e_1(t),q_n;.)-e_1(t)||_{C[0,a]} \le \frac{\alpha+RD_q(A(1))}{[n]_q+\beta}+\frac{\beta a}{[n]_q+\beta}.$$

Now, let $\epsilon > 0$ be given, define the following sets:

$$K := \left\{ k : \parallel T_{n,t}^{\alpha,\beta}(e_1(t), q_k; .) - e_1(t) \parallel \geqslant \epsilon \right\},$$

$$K_1 := \left\{ k : \frac{\beta a}{[n]_{q_k} + \beta} \geqslant \frac{\epsilon}{2} \right\},$$

$$K_2 := \left\{ k : \frac{RD_q(A(1)) + \alpha}{[n]_{q_k} + \beta} \geqslant \frac{\epsilon}{2} \right\},$$

such that $K \subseteq K_1 \cup K_2$.

From (2.7), one can see that

$$\delta\left\{k\leqslant n: \parallel T_{n,t}^{\alpha,\beta}(e_1(t),q_n;.)-e_1(t)\parallel_{C[0,a]}\geqslant \epsilon\right\}\leqslant \delta\left\{k\leqslant n: \frac{\beta a}{[n]_{q_k}+\beta}\geqslant \frac{\epsilon}{2}\right\}+\delta\left\{k\leqslant n: \frac{R+\alpha}{[n]_{q_k}+\beta}\geqslant \frac{\epsilon}{2}\right\}.$$

By (2.7), it is obvious that,

$$(2.8) st - \lim_{n} \left(\frac{1}{[n]_{\alpha} + \beta} \right) = 0.$$

Thus, we have

(2.9)
$$st - \lim_{n} \| T_{n,t}^{\alpha,\beta}(e_1(t), q_n; .) - e_1 \|_{C[0,a]} = 0.$$

By taking $\sup_{x,t\in[0,a]}$ in (2.3), one can write the following

$$\| T_{n,t}^{\alpha,\beta}(e_2(t),q_n;.) - e_2(t) \|_{C[0,a]} \le \frac{\beta^2 a^2 [n]_q}{[n]_q + \beta} + \frac{2a[\alpha + RD_q(A(1))]}{[n]_q + \beta} + \frac{\alpha^2 + 2R\alpha D_q(A(1)) + D_q^2(A(1))}{\left([n]_q + \beta\right)^2} + \frac{[n]_q a[2\alpha + D_q(A(1))]}{\left([n]_q + \beta\right)^2} + \frac{q [n]_q a}{\left([n]_q + \beta\right)^2}.$$

It is obvious that

$$st - \lim_{n} \left(\frac{1}{([n]_{q_n} + \beta)^2} \right) = 0, \ st - \lim_{n} \left(\frac{[n]_{q_n}}{([n]_{q_n} + \beta)^2} \right) = 0 \text{ and } st - \lim_{n} \left(\frac{q_n[n]_{q_n}}{([n]_{q_n} + \beta)^2} \right) = 0.$$

Now, let $\epsilon > 0$ be given, we define the following sets:

$$V := \left\{ k : \| T_{n,t}^{\alpha,\beta}(e_2(t), q_k; .) - e_2(t) \| \ge \epsilon \right\},$$

$$V_1 := \left\{ k : \frac{2a \left[\alpha + RD_q(A(1)) \right]}{\left[n \right]_{q_k} + \beta} \ge \frac{\epsilon}{4} \right\},$$

$$V_2 := \left\{ k : \frac{\alpha^2 + 2R\alpha D_q((A(1)) + D_q^2(A(1)) + \beta^2 a^2}{(\left[n \right]_{q_k} + \beta)^2} \ge \frac{\epsilon}{4} \right\},$$

$$V_3 := \left\{ k : \frac{a \left[n \right]_{q_k} \left[2\alpha + D_q(A(1)) \right]}{(\left[n \right]_{q_k} + \beta)^2} \ge \frac{\epsilon}{4} \right\},$$

$$V_4 := \left\{ k : \frac{a q_k \left[n \right]_{q_k}}{(\left[n \right]_{q_k} + \beta)^2} \ge \frac{\epsilon}{4} \right\}$$

such that $V \subseteq V_1 \cup V_2 \cup V_3 \cup V_4$.

Thus, we obtain

$$\delta \left\{ k \leqslant n : \| T_{n,t}^{\alpha,\beta}(e_{2}(t), q_{n}; .) - e_{2}(t) \|_{C[0,a]} \geqslant \epsilon \right\} \leqslant \delta \left\{ k \leqslant n : \frac{2a \left[\alpha + D_{q}(A(1))\right]}{[n]_{q_{k}} + \beta} \geqslant \frac{\epsilon}{4} \right\} \\
+ \delta \left\{ k \leqslant n : \frac{\alpha^{2} + 2R\alpha D_{q}((A(1)) + D_{q}^{2}(A(1)) + \beta^{2}a^{2}}{([n]_{q_{k}} + \beta)^{2}} \geqslant \frac{\epsilon}{4} \right\} \\
+ \delta \left\{ k \leqslant n : \frac{a[n]_{q_{k}}[2\alpha + D_{q}(A(1))]}{([n]_{q_{k}} + \beta)^{2}} \geqslant \frac{\epsilon}{4} \right\} \\
+ \delta \left\{ k \leqslant n : \frac{aq_{k}[n]_{q_{k}}}{([n]_{q_{k}} + \beta)^{2}} \geqslant \frac{\epsilon}{4} \right\}.$$
(2.10)

Hence, (2.8), (2) and (2.10) imply that

(2.11)
$$st - \lim_{n} \| T_{n,t}^{\alpha,\beta}(e_2(t), q_n; .) - e_2(t) \|_{C[0,a]} = 0.$$

3. Rates of statistical convergence

In this section, we give the rates of statistical convergence of the operators $T_{n,t}^{\alpha,\beta}(f;q;x)$ by means of modulus of continuity with the help of functions from Lipschitz class.

The modulus of continuity of f, $\omega(f,\delta)$ is defined by

$$\omega(f, \delta) = \sup_{\substack{|x-y| \leq \delta \\ x, y \in [0, a]}} |f(x) - f(y)|.$$

It is well-known that for a function $f \in C[0, a]$,

$$\lim_{n \to 0^+} \omega(f, \delta) = 0$$

for any $\delta > 0$

$$(3.1) |f(x) - f(y)| \leq \omega(f, \delta) \left(\frac{|x - y|}{\delta} + 1\right).$$

Now, we prove the following theorem for the rate of pointwise convergence of the operators $T_{n,t}^{\alpha,\beta}(f;q;x)$ to the function f(x+t) by means of modulus of continuity.

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Theorem 3.1. If the sequence $q := (q_n)$ satisfies the condition (2.7), $x \in [0, \infty)$ and $t \ge 0$, then we have

$$|T_{n,t}^{\alpha,\beta}(f;q;x) - f(x)| \leq 2\omega(f,\sqrt{\delta_{n,t}}),$$

for all $f \in C^*[0,\infty)$, where $\delta_{n,t} = T_{n,t}^{\alpha,\beta} \left((s - e_1(t))^2; q_n; x \right)$

Proof. To prove the theorem, we will use the linearity and positivity of the operators $T_{n,t}^{\alpha,\beta}(f;q;x)$. By (3.1), we have

$$|T_{n,t}^{\alpha,\beta}(f;q;x) - f(x+t)| \leq \frac{E_q^{-[n]_q t}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t\right)}{[k]_q !} \left| f\left(x + \frac{[k]_q + \alpha}{[n]_q + \beta}\right) - f(x+t) \right|$$

$$\leq \frac{E_q^{-[n]_q t}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t\right)}{[k]_q !} \omega(f, \delta) \left\{ \frac{1}{\delta} \left| \frac{[k]_q + \alpha}{[n]_q + \beta} - t \right| + 1 \right\}$$

$$= \left\{ \frac{1}{\delta} \frac{E_q^{-[n]_q t}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t\right)}{[k]_q !} \left| \frac{[k]_q + \alpha}{[n]_q + \beta} - t \right| + 1 \right\} \omega(f, \delta).$$

If we apply Cauchy-Schwarz inequality for sums, we obtain

$$\sum_{k=0}^{\infty} \frac{p_k\left(q;\left[n\right]_q t\right)}{\left[k\right]_q!} \left| \frac{\left[k\right]_q + \alpha}{\left[n\right]_q + \beta} - t \right|^2 \leqslant \left(\sum_{k=0}^{\infty} \frac{p_k\left(q;\left[n\right]_q t\right)}{\left[k\right]_q!}\right)^{1/2} \left(\sum_{k=0}^{\infty} \frac{p_k\left(q;\left[n\right]_q t\right)}{\left[k\right]_q!} \left(\frac{\left[k\right]_q + \alpha}{\left[n\right]_q + \beta} - t\right)^2\right)^{1/2}.$$

Using above inequality and by (2.1), we have

$$|T_{n,t}^{\alpha,\beta}(f;q;x) - f(x+t)| \leq \omega(f,\delta) \left\{ 1 + \frac{1}{\delta} \left[\frac{E_q^{-[n]_q t}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t \right)}{[k]_q!} \left(\frac{[k]_q + \alpha}{[n]_q + \beta} - t \right)^2 \right]^{1/2} \right\}$$

$$= \omega(f,\delta) \left\{ 1 + \frac{1}{\delta} \left[\frac{E_q^{-[n]_q t}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k \left(q; [n]_q t \right)}{[k]_q!} \left(\frac{[k]_q + \alpha}{[n]_q + \beta} - t \right)^2 \right]^{1/2} \right\}$$

$$= \omega(f,\delta) \left\{ 1 + \frac{1}{\delta} \left[T_{n,t}^{\alpha,\beta} \left(s - e_1(t) \right)^2; q; x \right]^{1/2} \right\}$$

such that we choose $\delta = \sqrt{\delta_{n,t}} = T_{n,t}^{\alpha,\beta} \left((s - e_1(t))^2; q_n; x \right)$. This step concludes the proof.

Theorem 3.2. If the sequence $q := (q_n)$ satisfies the condition (2.7) and $f \in C[0, a]$, then we have

$$||T_{n,t}^{\alpha,\beta}(f;q_n;.) - f(.)||_{C[0,a]} \le 2\omega(f,\sqrt{\delta_n})$$

where

$$\delta_{n} = \frac{\beta^{2}t^{2}\left[n\right]_{q}}{\left(\left[n\right]_{q} + \beta\right)^{2}} + \frac{\alpha^{2} + 2R\alpha D_{q}(A(1)) + D_{q}^{2}(A(1))}{\left(\left[n\right]_{q} + \beta\right)^{2}} + \frac{\left[n\right]_{q}t\left[2\alpha + D_{q}(A(1))\right]}{\left(\left[n\right]_{q} + \beta\right)^{2}} + \frac{q\left[n\right]_{q}t}{\left(\left[n\right]_{q} + \beta\right)^{2}}.$$

Proof. Now, let us estimate the second moment of the operators $T_{n,t}^{\alpha,\beta}(f;q;x)$. From (2.1)-(2.3), we get

$$(3.2) T_{n,t}^{\alpha,\beta} \left((s - e_1(t))^2; q; x \right) = \frac{t^2 \left[n \right]_q^2}{\left([n]_q + \beta \right)^2} + \frac{\alpha^2 + 2R\alpha D_q(A(1)) + D_q^2(A(1))}{\left([n]_q + \beta \right)^2} + \frac{\left[n \right]_q t \left[2\alpha + D_q(A(1)) \right]}{\left([n]_q + \beta \right)^2} + \frac{q \left[n \right]_q t}{\left([n]_q + \beta \right)^2}.$$

By Theorem 3.1, we have

$$|T_{n,t}^{\alpha,\beta}(f;q;x) - f(x+t)| \leq \omega(f,\delta) \left\{ 1 + \frac{1}{\delta} \left[T_{n,t}^{\alpha,\beta} \left(s - e_1(t) \right)^2; q; x \right] \right]^{1/2} \right\}$$

Substituting (3.2) into (3.3) and letting $\delta = \delta_n$ in (3.3), we obtain

$$||T_{n,t}^{\alpha,\beta}(f;q_n;.) - f(.)||_{C[0,a]} \le 2\omega(f,\sqrt{\delta_n}).$$

Thus, we get the desired result.

Now, we give the rate of convergence of the operators $T_{n,t}^{\alpha,\beta}$ with the help of functions from Lipschitz class $Lip_M(\alpha)$ where M>0 and $0<\alpha\leqslant 1$. A function f is an element of $Lip_M(\alpha)$ if

$$|f(x) - f(y)| \le M|x - y|^{\alpha} \ (x, y \in [a, b]).$$

Theorem 3.3. Let $q := (q_n)$ be a sequence satisfies the condition (2.7) and $f \in Lip_M(\alpha)$, $0 < \alpha \le 1$, then we have

$$|T_{n,t}^{\alpha,\beta}(f;q_n;.)-f(.)| \leq M\delta_n^{\alpha}$$

where $\delta_n = (T_{n,t}^{\alpha,\beta}(e_1(t) - s)^2, q; x)^{1/2}$.

Proof. Since $T_{n,t}^{\alpha,\beta}$ is linear and positive and by (3.4), we obtain

$$|T_{n,t}^{\alpha,\beta}(f;q_n;x) - f(x)| \leqslant T_{n,t}^{\alpha,\beta}(|f(t) - f(x)|, q; x)$$

$$\leqslant MT_{n,t}^{\alpha,\beta}(|t - x|^{\alpha}, q; x).$$

Assuming $p = \frac{1}{\alpha}$, $q = \frac{\alpha}{2-\alpha}$ and applying the Hölder inequality, we get

$$|T_{n,t}^{\alpha,\beta}(f;q_n;x) - f(x)| \leq T_{n,t}^{\alpha,\beta}(|f(t) - f(x)|, q; x)$$

$$\leq M\{T_{n,t}^{\alpha,\beta}(e_1 - x)^2, q; x)\}^{\alpha/2}.$$

Taking $\delta_{n,t} = (T_{n,t}^{\alpha,\beta}(e_1 - s)^2, q; x)^{1/2}$. We get the desired result.

4. Local Approximation

In this section, we state the local approximation theorem of the operators $T_{n,t}^{\alpha,\beta}(f;q;x)$. Let $C_B[0,\infty)$ be the space of all real valued continuous bounded functions f on $[0,\infty)$ with the norm $||f|| = \sup\{|f(x)| : x \in [0,\infty)\}$. The K-functional of f is defined by

$$K_2(f;\delta) = \inf_{g \in W^2} \{ \|f - g\| + \delta \|g''\| \},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By Devore-Lorentz [1, p. 177], there exists an absolute constant C > 0 such that

(4.1)
$$K_2(f,\delta) \leqslant C\omega_2(f,\sqrt{\delta})$$

where

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0 < h \leqslant 0} \sup_{x \in [0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of f. Moreover,

$$\omega(f,\delta) = \sup_{0 < h \leqslant 0} \sup_{x \in [0,\infty)} |f(x+h) - f(x)|$$

denotes the modulus of continuity of f.

Now, we give the direct local approximation theorem for the operators $T_{n,t}^{\alpha,\beta}(f;q;x)$.

Theorem 4.1. Let $q \in (0,1)$. We have

$$|T_{n,t}^{\alpha,\beta}(f;q;x) - f(x+t)| \le C\omega_2(f,\delta_n) + \omega\left(f,\frac{\alpha + D_q(A(1))R - \beta t}{[n]_q + \beta}\right)$$

 $\forall x \in [0, \infty), f \in C_B[0, \infty), \text{ where } C \text{ is a positive constant.}$

Proof. Let us define the following operators

(4.2)
$$\widetilde{T}_{n,t}^{\alpha,\beta}(f;q;x) = T_{n,t}^{\alpha,\beta}(f;q;x) - f\left(x + \frac{[n]_q t + \alpha + D_q(A(1))R}{[n]_q + \beta}\right) + f(x+t),$$

 $x \in [0, \infty)$. The operators $\widetilde{T}_{n,t}^{\alpha,\beta}(f;q;x)$ are linear. Thus, we have the following:

$$\widetilde{T}_{n,t}^{\alpha,\beta}\left(s-(x+t);q;x\right)=0,$$

(see Lemma 2.1). Let $g \in W^2$, from Taylor's expansion

$$g(s) = g(x+t) + g'(x+t)(s - (x+t)) + \int_{x+t}^{s} (s-u)g''(x)du,$$

 $s \in [0, \infty)$ and (4.3) we obtain

$$\widetilde{T}_{n,t}^{\alpha,\beta}(g;q;x) = g(x+t) + \widetilde{T}_{n,t}^{\alpha,\beta}\left(\int_{x+t}^{s} (s-u)g''(x)du\right).$$

By (4.2), we have the following

$$\begin{split} |\widetilde{T}_{n,t}^{\alpha,\beta}\left(g;q;x\right) - g(x+t)| & \leqslant \left| T_{n,t}^{\alpha,\beta}\left(\int_{x+t}^{s}(s-u)g''(u)du\right) \right| \\ & + \left| \int_{x+t}^{x+\frac{[n]_qt+\alpha+D_q(A(1))R}{[n]_q+\beta}} \left(x + \frac{[n]_qt+\alpha+D_q(A(1))R}{[n]_q+\beta} - u\right)g''(u)du \right| \\ & \leqslant T_{n,t}^{\alpha,\beta}\left(\left| \int_{x+t}^{s}(s-u)g''(u)du \right|, x \right) + \int_{x+t}^{x+\frac{[n]_qt+\alpha+D_q(A(1))R}{[n]_q+\beta}} \left| x + \frac{[n]_qt+\alpha+R}{[n]_q+\beta} - u \right| |g''(u)| \\ & \leqslant \left(T_{n,t}^{\alpha,\beta}\left(s - (x+t)\right)^2 + \left(x + \frac{[n]_qt+\alpha+D_q(A(1))R}{[n]_q+\beta}\right)^2 \right) \|g''\| \end{split}$$

Using (3.2), we get

$$T_{n,t}^{\alpha,\beta}\left((s-(x+t))^2;q;x\right) + \left(x + \frac{[n]_q t + \alpha + D_q(A(1))R}{[n]_q + \beta}\right)^2 \leqslant \delta_n + \left(x + \frac{[n]_q t + \alpha + D_q(A(1))R}{[n]_q + \beta}\right)^2.$$

Thus, by (4.4), we obtain

$$|\widetilde{T}_{n,t}^{\alpha,\beta}(g;q;x) - g(x+t)| \leq \delta_n + \left(x + \frac{[n]_q t + \alpha + D_q(A(1))R}{[n]_q + \beta}\right)^2.$$

By (1.1), (2.1) and (4.2), we get

$$|\widetilde{T}_{n,t}^{\alpha,\beta}(f;q;x)| \leqslant T_{n,t}^{\alpha,\beta}(f;q;x) + 2||f||$$

$$\leqslant ||f||T_{n,t}^{\alpha,\beta}(1;q;x) + 2||f||$$

$$\leqslant 3||f||.$$

Now, by (4.2), (4.5) and (4.6)

$$\begin{aligned} |T_{n,t}^{\alpha,\beta}\left(f;q;x\right) - f(x+t)| & \leqslant & |\widetilde{T}_{n,t}^{\alpha,\beta}\left(f - g;q;x\right) - (f - g)(x+t)| + |\widetilde{T}_{n,t}^{\alpha,\beta}\left(g;q;x\right) - g(x+t)| \\ & + \left| f\left(x + \frac{[n]_q t + \alpha + D_q(A(1))R}{[n]_q + \beta}\right) - f(x+t) \right| \\ & \leqslant & 4\|f - g\| + \delta_n)\|g''\| \end{aligned}$$

In view of (4.1), $\forall q \in (0,1)$ we get

$$|T_{n,t}^{\alpha,\beta}(f;q;x) - f(x+t)| \leqslant C\omega_2(f,\delta_n) + \omega\left(\frac{\alpha + D_q(A(1))R - \beta t}{[n]_q + \beta}\right)$$

and this concludes the proof.

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- (A. Karaisa) DEPARTMENT OF MATHEMATICS-COMPUTER SCIENCE, FACULTY OF SCIENCES, NECMETTİN ERBAKAN UNIVERSITY, MERAM YERLEŞKESİ, 42090 MERAM, KONYA, TURKEY E-mail address, A. Karaisa: alikaraisa@hotmail.com, akaraisa@konya.edu.tr
- (D.T. Tollu) DEPARTMENT OF MATHEMATICS-COMPUTER SCIENCE, FACULTY OF SCIENCES, NECMETTİN ERBAKAN UNIVERSITY, MERAM YERLEŞKESİ , 42090 MERAM, KONYA, TURKEY *E-mail address*, D.T. Tollu: dttollu@konya.edu.tr
- (Y. Asar) DEPARTMENT OF MATHEMATICS-COMPUTER SCIENCE, FACULTY OF SCIENCES, NECMETTİN ERBAKAN UNIVERSITY, MERAM YERLEŞKESİ , 42090 MERAM, KONYA, TURKEY *E-mail address*, Y. Asar: yasar@konya.edu.tr