# Nonlinear reaction-diffusion systems with a non-constant diffusivity: conditional symmetries in no-go case 

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#### Abstract

$Q$-conditional symmetries (nonclassical symmetries) for a general class of twocomponent reaction-diffusion systems with non-constant diffusivities are studied. The work is a natural continuation of our paper "Conditional symmetries and exact solutions of nonlinear reaction-diffusion systems with non-constant diffusivities"(Cherniha and Davydovych, 2012) [1] in order to extend the results on so-called no-go case. Using the notion of $Q$-conditional symmetries of the first type, an exhaustive list of reactiondiffusion systems admitting such symmetry is derived. The results obtained are compared with those derived earlier. The symmetries for reducing reaction-diffusion systems to twodimensional dynamical systems (ODE systems) and finding exact solutions are applied. As result, multiparameter families of exact solutions in the explicit form for nonlinear reaction-diffusion systems with an arbitrary power-law diffusivity are constructed and their properties for possible applicability are established.


Keywords: nonlinear reaction-diffusion system; Lie symmetry; non-classical symmetry; $Q$-conditional symmetry of the first type; exact solution.

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## 1 Introduction

This work is a natural continuation of our paper [1] and is devoted to the investigation of the two-component reaction-diffusion (RD) systems of the form

$$
\begin{align*}
& U_{t}=\left[D^{1}(U) U_{x}\right]_{x}+F(U, V), \\
& V_{t}=\left[D^{2}(V) V_{x}\right]_{x}+G(U, V), \tag{1}
\end{align*}
$$

where $U=U(t, x)$ and $V=V(t, x)$ are two unknown functions representing, say, the densities of populations (cells, chemicals), $F(U, V)$ and $G(U, V)$ are the given smooth functions describing interaction between them and environment, the functions $D^{1}(U)$ and $D^{2}(V)$ are the relevant diffusivities (hereafter they are positive smooth functions) and the subscripts $t$ and $x$ denote differentiation with respect to these variables. The class of RD systems (1) generalizes many well-known nonlinear second-order models and is used to describe various processes in physics, biology, chemistry and ecology. The relevant examples can be found in the well-known books [2-5] and a wide range of papers.

In paper [1], the $Q$-conditional invariance of these systems in the case when the operator in question has the form

$$
Q=\xi^{0}(t, x, u, v) \partial_{t}+\xi^{1}(t, x, u, v) \partial_{x}+\eta^{1}(t, x, u, v) \partial_{u}+\eta^{2}(t, x, u, v) \partial_{v}
$$

where $\xi^{0} \neq 0$, has been examined and an exhaustive list of reaction-diffusion systems admitting $Q$-conditional symmetries of the first type [6] has been derived. Here we study so-called no-go case when $\xi^{0}=0$, which is thought to be much more difficult and usually is skipped from examination. The additional reason to avoid examination of no-go case was the well-known fact (firstly proved in [7]) that complete description of $Q$-conditional symmetries with $\xi^{0}=0$ for scalar evolution equations is equivalent to solving of the equation in question.

On the other hand, the algorithm of heir-equations was proposed in [8, which allows to construct a hierarchy of the conditional symmetry operators starting from a particular one with $\xi^{0}=0$. This algorithm was successfully applied in order to find exact solutions for some evolution equations (in particular, see its application in the recent paper [9]). It can be also noted the very recent paper [10], in which RD systems were investigated in order to find conditional Lie-Bäcklund symmetry (generalised conditional symmetries in terminology of the pioneering paper [11]). However, this is well-known that deriving a complete classification of generalised conditional symmetries is unrealistic even for classes of scalar PDEs because the relevant systems of determining equations are very complicated.

To the best of our knowledge, there are no papers devoted to search $Q$-conditional symmetry (non-classical symmetry) of the class of systems (1) in the case $\xi^{0}=0$ and application of such symmetries for finding exact solutions of nonlinear reaction-diffusion systems. Here we show that a complete description of $Q$-conditional symmetries of the first type [6] can be derived in no-go case.

The paper is organized as follows. In section 2, basic definitions are presented, the systems of determining equations are derived and the main theorems are proved. In section 3, the $Q$-conditional symmetries obtained for reducing of RD systems to systems of ODEs are applied and multiparameter families of exact solutions are constructed. Moreover, it is shown that the solutions obtained possess attractive properties, which may lead to their possible applications. Finally, we summarize and discuss the results obtained in the last section.

## 2 Conditional symmetry for RD systems

### 2.1 Definitions and preliminary analysis

Following [1], we simplify RD system (1) by applying the Kirchhoff substitution

$$
\begin{equation*}
u=\int D^{1}(U) d U, \quad v=\int D^{1}(V) d V \tag{2}
\end{equation*}
$$

where $u(t, x)$ and $v(t, x)$ are new unknown functions. Hereafter we assume that there exist unique inverse functions to those arising in right-hand-sides of (2). Substituting (2) into (1), one obtains

$$
\begin{align*}
& u_{x x}=d^{1}(u) u_{t}+C^{1}(u, v),  \tag{3}\\
& v_{x x}=d^{2}(v) v_{t}+C^{2}(u, v),
\end{align*}
$$

where the functions $d^{1}, d^{2}, C^{1}$ and $C^{2}$ are uniquely defined via $D^{1}, D^{2}, F$ and $G$ by the formulae

$$
\begin{equation*}
d^{1}(u)=\frac{1}{D^{1}(U)}, d^{2}(v)=\frac{1}{D^{2}(V)}, C^{1}(u, v)=-F(U, V), C^{2}(u, v)=-G(U, V) \tag{4}
\end{equation*}
$$

where $U=D_{*}^{1}(u) \equiv\left(\int D^{1}(u) d u\right)^{-1}, \quad V=D_{*}^{2}(v) \equiv\left(\int D^{2}(v) d v\right)^{-1}$ (the upper subscripts -1 mean inverse functions).

Hereafter we examine class of RD systems (3) instead of (11) because both classes are equivalent. In fact, having any conditional symmetry operator of a RD system of the form (3), one may easily transform those into the relevant operator and a RD system from the class (1) provided the inverse functions in (4) are known.

It is well-known that to find Lie invariance operators, one needs to consider system (3) as the manifold $\mathcal{M}=\left\{S_{1}=0, S_{2}=0\right\}$ where

$$
\begin{aligned}
& S_{1} \equiv u_{x x}-d^{1}(u) u_{t}-C^{1}(u, v), \\
& S_{2} \equiv v_{x x}-d^{2}(v) v_{t}-C^{2}(u, v),
\end{aligned}
$$

in the prolonged space of the variables: $t, x, u, v, u_{t}, v_{t}, u_{x}, v_{x}, u_{x x}, v_{x x}, u_{x t}, v_{x t}, u_{t t}, v_{t t}$. According to the definition, system (3) is invariant under the transformations generated by the infinitesimal
operator

$$
\begin{equation*}
Q=\xi^{0}(t, x, u, v) \partial_{t}+\xi^{1}(t, x, u, v) \partial_{x}+\eta^{1}(t, x, u, v) \partial_{u}+\eta^{2}(t, x, u, v) \partial_{v} \tag{5}
\end{equation*}
$$

if the following invariance conditions are satisfied:

$$
\begin{aligned}
& \left.\underset{2}{Q}\left(S_{1}\right)\right|_{\mathcal{M}}=0, \\
& \left.\underset{2}{Q}\left(S_{2}\right)\right|_{\mathcal{M}}=0 .
\end{aligned}
$$

The operator $\underset{2}{Q}$ is the second prolongation of the operator $Q$, i.e.

$$
\begin{aligned}
\underset{2}{Q}= & Q+\rho_{t}^{1} \frac{\partial}{\partial u_{t}}+\rho_{x}^{1} \frac{\partial}{\partial u_{x}}+\rho_{t}^{2} \frac{\partial}{\partial v_{t}}+\rho_{x}^{2} \frac{\partial}{\partial v_{x}}+ \\
& \sigma_{t t}^{1} \frac{\partial}{\partial u_{t t}}+\sigma_{t x}^{1} \frac{\partial}{\partial u_{t x}}+\sigma_{x x}^{1} \frac{\partial}{\partial u_{x x}}+\sigma_{t t}^{2} \frac{\partial}{\partial v_{t t}}+\sigma_{t x}^{2} \frac{\partial}{\partial v_{t x}}+\sigma_{x x}^{2} \frac{\partial}{\partial v_{x x}},
\end{aligned}
$$

where the coefficients $\rho$ and $\sigma$ with relevant subscripts are expressed via the functions $\xi^{0}, \xi^{1}, \eta^{1}$ and $\eta^{2}$ by well-known formulae (see, e.g., $[12-14]$ ).

The crucial idea used for introducing the notion of $Q$-conditional symmetry (non-classical symmetry) is to change the manifold $\mathcal{M}$, namely: the operator $Q$ is used to reduce $\mathcal{M}$ (see the pioneer paper [15]). However, there are two essentially different possibilities to realize this idea in the case of two-component systems. Moreover, there are many different possibilities in the case of multi-component systems [6]. Following [6], we formulate two definitions in the case of system (3).
Definition 1. Operator (5) is called the $Q$-conditional symmetry of the first type for the RD system (3) if the following invariance conditions are satisfied:

$$
\begin{aligned}
& \left.\underset{2}{Q}\left(S_{1}\right)\right|_{\mathcal{M}_{1}}=0, \\
& \left.\underset{2}{Q}\left(S_{2}\right)\right|_{\mathcal{M}_{1}}=0,
\end{aligned}
$$

where the manifold $\mathcal{M}_{1}$ is either $\left\{S_{1}=0, S_{2}=0, Q(u)=0\right\}$ or $\left\{S_{1}=0, S_{2}=0, Q(v)=0\right\}$.
Definition 2. Operator (5) is called the $Q$-conditional symmetry of the second type, i.e., the standard non-classical symmetry for the RD system (3) if the following invariance conditions are satisfied:

$$
\begin{aligned}
& \left.\underset{2}{Q}\left(S_{1}\right)\right|_{\mathcal{M}_{2}}=0, \\
& \left.\underset{2}{Q}\left(S_{2}\right)\right|_{\mathcal{M}_{2}}=0,
\end{aligned}
$$

where the manifold $\mathcal{M}_{2}=\left\{S_{1}=0, S_{2}=0, Q(u)=0, Q(v)=0\right\}$.

It is well-known that constructing $Q$-conditional symmetries for evolution equations and evolution systems leads to the requirement to analyze two essentially different cases a) $\xi^{0} \neq 0$; b) $\xi^{0}=0, \xi^{1} \neq 0$ (see operator (5)). The case $a$ ) for system (3) was examined in our previous papers [1, 16, 17]. Here we restrict ourselves on the case b), i.e. when operator (5) has the structure

$$
\begin{equation*}
Q=\xi(t, x, u, v) \partial_{x}+\eta^{1}(t, x, u, v) \partial_{u}+\eta^{2}(t, x, u, v) \partial_{v} . \tag{6}
\end{equation*}
$$

In order to find $Q$-conditional symmetries of the second type (non-classical symmetries), one applies Definition 2 to system (3). Direct calculations produce the following system of determining equations:

$$
\begin{align*}
& \left(d^{1}-d^{2}\right)\left(\xi \eta_{v}^{1}-\eta^{1} \xi_{v}\right)=0, \quad\left(d^{1}-d^{2}\right)\left(\xi \eta_{u}^{2}-\eta^{2} \xi_{u}\right)=0 \\
& 2 d^{1}\left(\xi_{x}+\frac{\eta^{1}}{\xi} \xi_{u}+\frac{\eta^{2}}{\xi} \xi_{v}\right)+\eta^{1} d_{u}^{1}=0, \\
& 2 d^{2}\left(\xi_{x}+\frac{\eta^{1}}{\xi} \xi_{u}+\frac{\eta^{2}}{\xi} \xi_{v}\right)+\eta^{2} d_{v}^{2}=0, \\
& \eta^{1} C_{u}^{1}+\eta^{2} C_{v}^{1}+\left(\frac{\eta^{1}}{\xi} \xi_{v}-\eta_{v}^{1}\right) C^{2}+\left(2 \xi_{x}+3 \frac{\eta^{1}}{\xi} \xi_{u}+2 \frac{\eta^{2}}{\xi} \xi_{v}-\eta_{u}^{1}\right) C^{1}+\left(\eta_{t}^{1}-\frac{\eta^{1}}{\xi} \xi_{t}\right) d^{1}- \\
& \eta_{x x}^{1}+\frac{\eta^{1}}{\xi}\left(\xi_{x x}-2 \eta_{x u}^{1}\right)+\left(\frac{\eta^{1}}{\xi}\right)^{2}\left(2 \xi_{x u}-\eta_{u u}^{1}\right)+2 \frac{\eta^{1} \eta^{2}}{\xi^{2}}\left(\xi_{x v}-\eta_{u v}^{1}\right)+  \tag{7}\\
& \left(\frac{\eta^{1}}{\xi}\right)^{3} \xi_{u u}+\frac{\eta^{1}\left(\eta^{2}\right)^{2}}{\xi^{3}} \xi_{v v}+2 \frac{\eta^{2}\left(\eta^{1}\right)^{2}}{\xi^{3}} \xi_{u v}-2 \frac{\eta^{2}}{\xi} \eta_{x v}^{1}-\left(\frac{\eta^{2}}{\xi}\right)^{2} \eta_{v v}^{1}=0 \\
& \eta^{1} C_{u}^{2} \eta^{2} C_{v}^{2}+\left(\frac{\eta^{2}}{\xi} \xi_{u}-\eta_{u}^{2}\right) C^{1}+\left(2 \xi_{x}+3 \frac{\eta^{2}}{\xi} \xi_{v}+2 \frac{\eta^{1}}{\xi} \xi_{u}-\eta_{v}^{2}\right) C^{2}+ \\
& \left(\eta_{t}^{2}-\frac{\eta^{2}}{\xi} \xi_{t}\right) d^{2}-\eta_{x x}^{2}+\frac{\eta^{2}}{\xi}\left(\xi_{x x}-2 \eta_{x v}^{2}\right)+\left(\frac{\eta^{2}}{\xi}\right)^{2}\left(2 \xi_{x v}-\eta_{v v}^{2}\right)+2 \frac{\eta^{1} \eta^{2}}{\xi^{2}}\left(\xi_{x u}-\eta_{u v}^{2}\right)+ \\
& \left(\frac{\eta^{2}}{\xi}\right)^{3} \xi_{v v}+\frac{\eta^{2}\left(\eta^{1}\right)^{2}}{\xi^{3}} \xi_{u u}+2 \frac{\eta^{1}\left(\eta^{2}\right)^{2}}{\xi^{3}} \xi_{u v}-2 \frac{\eta^{1}}{\xi} \eta_{x u}^{2}-\left(\frac{\eta^{1}}{\xi}\right)^{2} \eta_{u u}^{2}=0
\end{align*}
$$

Hereafter the subscripts $t, x, u$ and $v$ denote differentiation with respect to these variables.
Remark 1. One cannot set $\xi=1$ in operator (6) without losing a generality for obtaining the system of determining equations (in contrary to the case of operator (5) with $\xi^{0} \neq 0$ !) because the system obtained immediately leads to the trivial result $Q=\partial_{x}$.

One easily notes that system (7) consists of 6 nonlinear PDEs for 7 unknown functions $d^{1}(u), d^{2}(v), C^{1}(u, v), C^{2}(u, v), \xi(t, x, u, v), \eta^{1}(t, x, u, v)$ and $\eta^{2}(t, x, u, v)$. In order to solve any system of determining equations one needs to establish structure of coefficients of operator (6). In the case of system (7), it means that the subsystem consisting of four PDEs for
$d^{1}(u), d^{2}(v), \xi(t, x, u, v), \eta^{1}(t, x, u, v)$ and $\eta^{2}(t, x, u, v)$ should be examined. Unfortunately, we were unable to solve this subsystem for arbitrary functions $d^{1}(u)$ and $d^{2}(v)$ and believe that it is possible to do for the correctly-specified pairs $\left(d^{1}(u), d^{2}(v)\right)$ only.

Happily our efforts were successful in application of Definition 1. Formally speaking, we should construct systems of DEs using two different manifolds $\mathcal{M}_{1}$ (see Definition 1). However, the class of RD systems (3) is invariant under discrete transformation $u \rightarrow v, v \rightarrow u$. Thus, we can use only one manifold, say, $\left\{S_{1}=0, S_{2}=0, Q(u)=0\right\}$. Having the complete list of the conditional symmetry operators and the relevant forms of RD systems, one may simply extend such list by applying the transformation mentioned above.

Thus, now we present the system of DEs, obtained by direct application of Definition 1 with $\mathcal{M}_{1}=\left\{S_{1}=0, S_{2}=0, Q(u)=0\right\}$, for finding $Q$-conditional symmetry operator of the form (6), namely:

$$
\begin{align*}
& \xi_{u}=\xi_{v}=0,  \tag{8}\\
& \left(d^{1}-d^{2}\right) \eta_{v}^{1}=\left(d^{1}-d^{2}\right) \eta_{u}^{2}=\eta_{v v}^{1}=\eta_{v v}^{2}=0, \quad \eta_{x v}^{1}+\eta_{u v}^{1} \frac{\eta^{1}}{\xi}=0,  \tag{9}\\
& 2 \xi_{x} d^{1}+\eta^{1} d_{u}^{1}=0,  \tag{10}\\
& 2 \xi_{x} d^{2}+\eta^{2} d_{v}^{2}=0,  \tag{11}\\
& \xi_{t} d^{2}+2 \eta_{x v}^{2}-\xi_{x x}+2 \frac{\eta^{1}}{\xi} \eta_{u v}^{2}=0,  \tag{12}\\
& \eta^{1} C_{u}^{1}+\eta^{2} C_{v}^{1}-\eta_{v}^{1} C^{2}+\left(2 \xi_{x}-\eta_{u}^{1}\right) C^{1}+\eta_{t}^{1} d^{1}-\eta_{x x}^{1}- \\
& \quad\left(\frac{\eta^{1}}{\xi}\right)^{2} \eta_{u u}^{1}-\frac{\eta^{1}}{\xi}\left(\xi_{t} d^{1}+2 \eta_{x u}^{1}-\xi_{x x}\right)=0,  \tag{13}\\
& \eta^{1} C_{u}^{2}+\eta^{2} C_{v}^{2}-\eta_{u}^{2} C^{1}+\left(2 \xi_{x}-\eta_{v}^{2}\right) C^{2}+\eta_{t}^{2} d^{2}-\eta_{x x}^{2}- \\
& \quad\left(\frac{\eta^{1}}{\xi}\right)^{2} \eta_{u u}^{2}-2 \frac{\eta^{1}}{\xi} \eta_{x u}^{2}=0 . \tag{14}
\end{align*}
$$

It should be stressed that we are looking for purely conditional symmetry operators, i.e., all the operators, which are equivalent to the Lie symmetries presented in [18] should be excluded.

Having this aim, we use the DEs system to search for Lie symmetry operators of the form (6):

$$
\begin{align*}
& \xi_{u}=\xi_{v}=0, \\
& \eta_{u u}^{1}=\eta_{u v}^{1}=\eta_{v v}^{1}=\eta_{u u}^{2}=\eta_{u v}^{2}=\eta_{v v}^{2}=0, \\
& \left(d^{1}-d^{2}\right) \eta_{v}^{1}=0, \quad\left(d^{1}-d^{2}\right) \eta_{u}^{2}=0, \quad \eta_{x v}^{1}=\eta_{x u}^{2}=0, \\
& 2 \xi_{x} d^{1}+\eta^{1} d_{u}^{1}=0, \\
& 2 \xi_{x} d^{2}+\eta^{2} d_{v}^{2}=0,  \tag{15}\\
& \xi_{t} d^{1}+2 \eta_{x u}^{1}-\xi_{x x}=0, \\
& \xi_{t} d^{2}+2 \eta_{x v}^{2}-\xi_{x x}=0, \\
& \eta^{1} C_{u}^{1}+\eta^{2} C_{v}^{1}-\eta_{v}^{1} C^{2}+\left(2 \xi_{x}-\eta_{u}^{1}\right) C^{1}+\eta_{t}^{1} d^{1}-\eta_{x x}^{1}=0, \\
& \eta^{1} C_{u}^{2}+\eta^{2} C_{v}^{2}-\eta_{u}^{2} C^{1}+\left(2 \xi_{x}-\eta_{v}^{2}\right) C^{2}+\eta_{t}^{2} d^{2}-\eta_{x x}^{2}=0,
\end{align*}
$$

which can be easily extracted from the relevant system derived in [18]. One notes, that systems of DEs (8) $-(14)$ and (15) coincide (in the case $d^{1} \neq d^{2}$ ) if the restrictions

$$
\begin{equation*}
\eta_{u u}^{1}=0, \quad \xi_{t} d^{1}+2 \eta_{x u}^{1}-\xi_{x x}=0 \tag{16}
\end{equation*}
$$

take place. Thus, we take into account only such solutions of (8)-(14), which do not satisfy at least one of the equations from (16). Moreover, since $Q$-conditional symmetry of the first type is automatically one of the second type, we should also check the same for coefficients of the operator obtained by multiplying (5) on any smooth functions. Otherwise the $Q$-conditional symmetry obtained will be equivalent to a Lie symmetry.

### 2.2 The main theorems

It turns out that to solve the DEs system (8)-(14) one needs to examine separately four cases

1. two nonconstant diffusivities $d_{u}^{1} d_{v}^{2} \neq 0$;
2. one nonconstant diffusivity $d_{u}^{1} d_{v}^{2}=0,\left(d_{u}^{1}\right)^{2}+\left(d_{v}^{2}\right)^{2} \neq 0$;
3. two constant diffusivities being different $d^{k}=\lambda_{k}=$ const $(k=1,2), \lambda_{1} \neq \lambda_{2}$;
4. two equal diffusivities $d^{k}=\lambda_{k}=$ const $(k=1,2), \lambda_{1}=\lambda_{2}$.

In fact, a simple analysis of equations (9) -(11) shows that the form and number of equations are different in each of the cases listed above. Similarly to the examination of operator (5) with $\xi^{0}(t, x, u, v) \neq 0$ [1], we consider firstly two most general cases, involving at least one nonconstant diffusivity while other two cases will be treated elsewhere.

Our main result can be formulated in form of two theorems presenting the complete lists of $Q$-conditional operators of the first type and having the form (6), which are admitted by any RD system (3) with a nonconstant diffusivity.

Theorem 1 In the case $d_{u}^{1} d_{v}^{2} \neq 0$ the $R D$ system (3) admits $Q$-conditional operator of the first type (6), up to equivalent transformation

$$
\begin{align*}
& t \rightarrow C_{1} t+C_{2}, \quad x \rightarrow C_{3} x+C_{4}, \\
& u \rightarrow C_{5} u+C_{6}, \quad v \rightarrow C_{7} v+C_{8}, \tag{17}
\end{align*}
$$

(here $C_{l}(l=1, \ldots, 8)$ are constants, $\left.C_{2 k-1} \neq 0, k=1, \ldots, 4\right)$ and to discrete transformation

$$
\begin{equation*}
u \rightarrow v, \quad v \rightarrow u \tag{18}
\end{equation*}
$$

only in three cases:

$$
\begin{align*}
& \text { (I) } \quad u_{x x}=d^{1}(u) u_{t}+\frac{\left(d^{1}\right)^{2}}{d_{u}^{1}} f\left(v^{4} d^{1}\right)+16\left(1-\frac{d^{1} d^{1} u u}{\left(d_{u}^{1}\right)^{2}}\right) \frac{d^{1}}{d_{u}^{1}},  \tag{19}\\
& \\
& v_{x x}=v^{-4} v_{t}+v^{-3} g\left(v^{4} d^{1}\right)+v,  \tag{20}\\
& Q=
\end{align*}
$$

where $d^{1} \neq \delta_{1} u^{-4}$ is an arbitrary function;

$$
\begin{align*}
(I I) \quad u_{x x} & =d^{1}(u) u_{t}+\frac{\left(d^{1}\right)^{2}}{d_{u}^{1}}\left(f\left(v^{4} d^{1}\right)+\frac{4 \mu}{d^{1}}+4 \mu \int \frac{d^{1} u u}{d^{1} d_{u}^{1}} d u\right),  \tag{21}\\
v_{x x} & =v^{-4} v_{t}+v^{-3} g\left(v^{4} d^{1}\right)+\frac{\mu}{4} v, \\
Q=\xi(x) \partial_{x} & -2 \xi_{x} \frac{d^{1}}{d_{u}^{1}} \partial_{u}+\frac{1}{2} \xi_{x} v \partial_{v}, \tag{22}
\end{align*}
$$

where $d^{1}=e^{u}$ or $d^{1}=u^{\beta}(\beta \neq-4)$, while

$$
\xi(x)= \begin{cases}\exp (\sqrt{\mu} x)+\alpha \exp (-\sqrt{\mu} x), & \text { if } \mu>0  \tag{23}\\ \sin \sqrt{-\mu} x, & \text { if } \mu<0\end{cases}
$$

where $\alpha \neq 0$ is an arbitrary constant;

$$
\begin{aligned}
& \text { (III) } \\
& u_{x x}=d^{1}(u) u_{t}+\frac{\left(d^{1}\right)^{2}}{d_{u}^{1}}\left(f\left(v^{4} d^{1}\right)+\frac{3 \mu}{2 d^{1}}+2 \mu \int \frac{d^{1} u u}{d^{1} d_{u}^{1}} d u\right), \\
& v_{x x}=v^{-4} v_{t}+v^{-3} g\left(v^{4} d^{1}\right)+\frac{\mu}{4} v, \\
& Q=\xi(x) \partial_{x}-2 \xi_{x} \frac{d^{1}}{d_{u}^{1}} \partial_{u}+\frac{\xi_{x}}{2} v \partial_{v},
\end{aligned}
$$

where $d^{1} \neq \delta_{1} u^{-4}$ is an arbitrary solution of the equation

$$
8 \frac{d^{1}}{d_{u}^{1}}\left(\frac{d^{1}}{d_{u}^{1}}\right)_{u u}-4\left(\frac{d^{1}}{d_{u}^{1}}\right)_{u}-1=0
$$

while

$$
\xi(x)= \begin{cases}x^{2}, & \text { if } \mu=0  \tag{24}\\ \left(\exp \left(\frac{\sqrt{\mu}}{2} x\right)+\alpha \exp \left(-\frac{\sqrt{\mu}}{2} x\right)\right)^{2}, & \text { if } \mu>0 \\ \sin \sqrt{-\mu} x \pm 1, & \text { if } \mu<0\end{cases}
$$

where $\alpha \neq 0$ is an arbitrary constant. In cases (I)-(III), $f$ and $g$ are arbitrary smooth functions of the argument $v^{4} d^{1}(u)$, while the function $d^{1}(u)$ is described in each case.

Theorem 2 In the case $d_{u}^{1} d_{v}^{2}=0$ and $\left(d_{u}^{1}\right)^{2}+\left(d_{v}^{2}\right)^{2} \neq 0$ the $R D$ system (3) admits $Q$-conditional operator of the first type (6), up to equivalent transformation (17) and to discrete transformation (18), only in such cases:

$$
\begin{align*}
& u_{x x}=d^{1}(u) u_{t}+f(u),  \tag{I}\\
& v_{x x}=v_{t}+v g(u)+\alpha v \ln v
\end{align*}
$$

$$
Q=e^{-\alpha t}\left(2 \partial_{x}+\alpha x v \partial_{v}\right),
$$

$$
\begin{align*}
& u_{x x}=d^{1}(u) u_{t}+f(u),  \tag{II}\\
& v_{x x}=v_{t}+v g(u)
\end{align*}
$$

$$
Q=-2 t \partial_{x}+x v \partial_{v}
$$

$$
\begin{align*}
& u_{x x}=d^{1}(u) u_{t}+f(u), \\
& v_{x x}=v_{t}+e^{v} g(u)+\alpha e^{2 v} \tag{III}
\end{align*}
$$

$$
Q=\partial_{x}+\alpha e^{u} \partial_{u}
$$

where $\alpha \neq 0$. In cases (I)-(III), $f$ and $g$ are arbitrary smooth functions of the argument $u$.
Proofs of Theorems 1 and 2 is based on solving the nonlinear DEs system (8) -(14) under restrictions $d_{u}^{1} d_{v}^{2} \neq 0$ and $d_{u}^{1} d_{v}^{2}=0,\left(d_{u}^{1}\right)^{2}+\left(d_{v}^{2}\right)^{2} \neq 0$, respectively. We consider in details only the proof of Theorem 1 , which is more complicated.

Differentiating equation (11) w.r.t. $v$, one immediately obtains $\xi_{t}=0$, i.e. the function $\xi$ depends only on $x$ (see Eqs. (8)). Now we solve equation (10) and (11) :

$$
\begin{align*}
& \eta^{1}=-2 \xi_{x} \frac{d^{1}}{d_{u}^{1}}  \tag{25}\\
& \eta^{2}=-2 \xi_{x} \frac{d^{2}}{d_{v}^{2}} \tag{26}
\end{align*}
$$

and, taking into account restrictions (16), conclude that $\xi_{x} \neq 0$.
Because $\eta_{v v}^{2}=0$ (see (9)) we obtain from (26) the first-order ODE

$$
\begin{equation*}
\frac{d^{2}}{d_{v}^{2}}=\alpha_{1} v+\alpha_{2} \tag{27}
\end{equation*}
$$

(here $\alpha_{1}$ and $\alpha_{2}$ are arbitrary constants and $\alpha_{1}^{2}+\alpha_{2}^{2} \neq 0$ ) with the general solutions

$$
\begin{align*}
& d^{2}=\delta_{2} \exp \left(\frac{v}{\alpha_{2}}\right), \text { if } \alpha_{1}=0, \quad \alpha_{2} \neq 0  \tag{28}\\
& d^{2}=\delta_{2}\left(\alpha_{1} v+\alpha_{2}\right)^{\frac{1}{\alpha_{1}}}, \text { if } \alpha_{1} \neq 0 \tag{29}
\end{align*}
$$

where $\delta_{2} \neq 0$ is an arbitrary constant.
A further examination of the remaining equations (13)-(14) leads only to Lie symmetries provided the diffusivity $d^{2}$ has the form (28). Thus, we present only details for $d^{2}$ of the form (29).

Substituting (29) and (25) - (27) into the system in question and operator (6), one arrives at the system

$$
\begin{align*}
& u_{x x}=d^{1}(u) u_{t}+C^{1}(u, v), \\
& v_{x x}=\delta_{2}\left(\alpha_{1} v+\alpha_{2}\right)^{\frac{1}{\alpha_{1}}} v_{t}+C^{2}(u, v) \tag{30}
\end{align*}
$$

and the operator

$$
\begin{equation*}
Q=\xi(x) \partial_{x}-2 \xi_{x} \frac{d^{1}}{d_{u}^{1}} \partial_{u}-2 \xi_{x}\left(\alpha_{1} v+\alpha_{2}\right) \partial_{v} . \tag{31}
\end{equation*}
$$

Obviously, system (30) and operator (31) can be simplified using the equivalence transformation (see (17)) $\alpha_{1} v+\alpha_{2} \rightarrow v$ and new notation $\alpha_{1}=\frac{1}{\beta}, \beta \neq 0$, hence one arrives at the system

$$
\begin{aligned}
& u_{x x}=d^{1}(u) u_{t}+C^{1}(u, v), \\
& v_{x x}=\delta_{2} v^{\beta} v_{t}+C^{2}(u, v)
\end{aligned}
$$

and the operator

$$
\begin{equation*}
Q=\xi(x) \partial_{x}-2 \xi_{x} \frac{d^{1}}{d_{u}^{1}} \partial_{u}-\frac{2 \xi_{x}}{\beta} v \partial_{v} . \tag{32}
\end{equation*}
$$

Substituting coefficients of (32) into (12), we obtain

$$
\begin{equation*}
(4+\beta) \xi_{x x}=0 \tag{33}
\end{equation*}
$$

Solving ODE (33) with $\beta \neq-4$, we find easily $\xi$, which must a linear function. However, the straightforward analysis in this case shows that the second equation in (16) is satisfied while equation (13) produces $\eta_{u u}^{1}=0$ (if one takes into account equations (25)-(26) and that the function $\xi$ is linear). Thus, Lie symmetries can be found only if $\beta \neq-4$.

Now we examine the special case $\beta=-4$ when ODE (33) vanishes. Now the classification equations (13)-(14) (all other equations of DEs systems are already solved) take the form

$$
\begin{align*}
-2 h C_{u}^{1}+ & \frac{1}{2} v C_{v}^{1}+2\left(1+h_{u}\right) C^{1}=-2 \frac{\xi_{x x x}}{\xi_{x}} h+ \\
& \frac{2}{\xi} \xi_{x x}\left(4 h_{u}+1\right) h-\frac{8}{\xi^{2}} \xi_{x}^{2} h^{2} h_{u u}  \tag{34}\\
-2 h C_{u}^{2}+ & \frac{1}{2} v C_{v}^{2}+\frac{3}{2} C^{2}=\frac{\xi_{x x x}}{2 \xi_{x}} v, \tag{35}
\end{align*}
$$

where $h(u)=\frac{d^{1}}{d_{u}^{1}}$. The linear first-order PDE (35) has the general solution

$$
C^{2}=v^{-3} g(\omega)+\frac{\mu}{4} v
$$

where $g$ is an arbitrary differentiable function of the argument $\omega=d^{1} v^{4}$. Because the function $C^{2}$ does not depend on $x$ the equations

$$
\begin{equation*}
\frac{\xi_{x x x}}{\xi_{x}}=\mu \tag{36}
\end{equation*}
$$

simultaneously springs up.
In order to solve PDE (34) we find its differential consequence w.r.t. $x$ :

$$
4\left(\frac{\xi_{x}^{2}}{\xi^{2}}\right)_{x} h h_{u u}-\left(\frac{\xi_{x x}}{\xi}\right)_{x}\left(4 h_{u}+1\right)=0
$$

It turns out that $4 h_{u}+1 \neq 0 \Leftrightarrow d^{1} \neq \delta_{1}\left(u+\alpha_{2}\right)^{-4}$ (otherwise restrictions (16)) leading to Lie symmetry are automatically fulfilled), therefore two different cases arise:

$$
\text { (1) }\left(\frac{\xi_{x}^{2}}{\xi^{2}}\right)_{x}=0 \Rightarrow\left(\frac{\xi_{x x}}{\xi}\right)_{x}=0
$$

and

$$
\begin{equation*}
\text { (2) }\left(\frac{\xi_{x}^{2}}{\xi^{2}}\right)_{x} \neq 0 \Rightarrow 4 h h_{u u}-\nu_{1}\left(4 h_{u}+1\right)=0 \tag{37}
\end{equation*}
$$

where $\nu_{1}=\left(\frac{\xi_{x x}}{\xi}\right)_{x}\left[\left(\frac{\xi_{x}^{2}}{\xi^{2}}\right)_{x}\right]^{-1}$.

In case (1) we immediately obtain $\xi=\lambda_{1} \exp (\sqrt{\mu} x)$ with arbitrary $\lambda_{1}$ and $\mu>0$. Substituting this function into $\operatorname{PDE}$ (34), the equation obtained can be solved in quite a similar way as PDE (35). Thus, we arrive at the system

$$
\begin{align*}
& u_{x x}=d^{1}(u) u_{t}+\frac{\left(d^{1}\right)^{2}}{d_{u}^{1}} f\left(v^{4} d^{1}\right)+4 \mu\left(1-\frac{d^{1} d^{1} u u}{\left(d_{u}^{1}\right)^{2}}\right) \frac{d^{1}}{d_{u}^{1}},  \tag{38}\\
& v_{x x}=\delta_{2} v^{-4} v_{t}+v^{-3} g\left(v^{4} d^{1}\right)+\frac{\mu}{4} v,
\end{align*}
$$

which is invariant under $Q$-conditional symmetry operator

$$
\begin{equation*}
Q=\exp (\sqrt{\mu} x)\left(\partial_{x}-2 \sqrt{\mu} \frac{d^{1}}{d_{u}^{1}} \partial_{u}+\frac{\sqrt{\mu}}{2} v \partial_{v}\right), \mu>0 . \tag{39}
\end{equation*}
$$

Using the equivalence transformation (see (17)) $x \rightarrow \frac{2}{\sqrt{\mu}} x, v \rightarrow \sqrt[4]{\delta_{2}} v$ for system (38) and operator (39) we arrive at case (I) of Theorem 1.

Examination of case (2) has been done in a similar way. Taking into account (36) and (37), PDE (34) can be again solved

$$
C^{1}=\frac{\left(d^{1}\right)^{2}}{d_{u}^{1}}\left(f\left(d^{1} v^{4}\right)-\frac{\mu}{d^{1}}+\left(\nu_{1} \frac{\xi_{x}^{2}}{\xi^{2}}-\frac{\xi_{x x}}{\xi}\right) \int\left(\frac{d_{u}^{1}\left(4 h_{u}+1\right)}{\left(d^{1}\right)^{2}}\right) d u\right) .
$$

Because the function $C^{1}$ does not depend on $x$ we obtain ODE

$$
\begin{equation*}
\nu_{1} \frac{\xi_{x}^{2}}{\xi^{2}}-\frac{\xi_{x x}}{\xi}=\nu_{2} \tag{40}
\end{equation*}
$$

( $\nu_{2}$ is an arbitrary constant) for the function $\xi(x)$. Thus, one needs to solve the system of equations (36) and (40) in order to find the function $\xi(x)$. If $\nu_{1}=0$ then the general solution has the form (23) and then the system and the operator arising in case (II) of Theorem 1 are obtained. If $\nu_{1} \neq 0$ then the general solution has the form (24) and then the system and the operator arising in case (III) of Theorem 1 are obtained.

The proof is now completed.
Theorems 1 and 2 present the complete (in no-go case) list of $Q$-conditional symmetries of the first type for the nonlinear RD systems of the form (3). Because each $Q$-conditional symmetry of the first type is also a $Q$-conditional symmetry (non-classical symmetry) the above theorems present also a set (not exhaustive!) of the later symmetries for the corresponding RD systems. We shall return to this issue also in the final section.

## 3 Reduction nonlinear RD system to ODE system and constructing exact solutions

It is well-known that using any $Q$-conditional symmetry (non-classical symmetry), one reduces the given system of PDEs to a system of ODEs via the same procedure as for classical Lie
symmetries. Since any $Q$-conditional symmetry of the first type is automatically one of the second type, i.e., the non-classical symmetry, we apply this procedure for finding exact solutions. Thus, to construct an ansatz corresponding to the given operator $Q$, the system of the linear first-order PDEs

$$
\begin{equation*}
Q(u)=0, \quad Q(v)=0 \tag{41}
\end{equation*}
$$

should be solved. Substituting the ansatz obtained into the RD system with correctly-specified coefficients, one obtains the reduced system of ODEs.

Let us construct exact solutions of the non-linear RD system (19) with $d^{1}=u^{\beta}, \beta \neq-4$ (see a comment about the case $\beta=-4$ below). Thus, the system (19) takes the form

$$
\begin{align*}
& u_{x x}=u^{\beta} u_{t}+f\left(v^{4} u^{\beta}\right) u^{\beta+1}+\frac{16 u}{\beta^{2}}, \\
& v_{x x}=v^{-4} v_{t}+g\left(v^{4} u^{\beta}\right) v^{-3}+v, \tag{42}
\end{align*}
$$

Solving system (41) for operator (20), one constructs ansatz

$$
\begin{align*}
& u(t, x)=\varphi(t) \exp \left(-\frac{4}{\beta} x\right),  \tag{43}\\
& v(t, x)=\psi(t) \exp (x)
\end{align*}
$$

where $\varphi(t)$ and $\psi(t)$ are new unknown functions.
Substituting ansatz (43) into (42), one obtains reduced system of ODEs

$$
\begin{align*}
& \varphi^{\prime}+f\left(\varphi^{\beta} \psi^{4}\right) \varphi=0  \tag{44}\\
& \psi^{\prime}+g\left(\varphi^{\beta} \psi^{4}\right) \psi=0 .
\end{align*}
$$

Because the nonlinear ODEs system (44) is non-integrable in the general case, we noted that the simplest (but still nonlinear!) case leading to the integrable system occurs when

$$
\begin{equation*}
f=\alpha_{1} \varphi^{\beta k} \psi^{4 k}, \quad g=\alpha_{2} \varphi^{\beta k} \psi^{4 k} \tag{45}
\end{equation*}
$$

where $\alpha_{1}, \alpha_{2}$ and $k \neq 0$ are arbitrary constants $(k=0$ leads to solutions of the form $\varphi=$ $\exp \left(-\alpha_{1} t\right), \psi=\exp \left(-\alpha_{2} t\right)$, hence, (43) will be a simple plane-wave solution of the RD system (42) with $f=\alpha_{1}$ and $g=\alpha_{2}$ ).

The general solution of (44) with (45) is

$$
\begin{align*}
& \varphi(t)=\lambda_{1}\left(\left(4 \alpha_{2}+\beta \alpha_{1}\right) k \lambda_{1}^{\beta k}\left(t-t_{0}\right)\right)^{-\frac{\alpha_{1}}{\left(4 \alpha_{2}+\beta \alpha_{1}\right) k}}  \tag{46}\\
& \psi(t)=\left(\left(4 \alpha_{2}+\beta \alpha_{1}\right) k \lambda_{1}^{\beta k}\left(t-t_{0}\right)\right)^{-\frac{\alpha_{2}}{\left(4 \alpha_{2}+\beta \alpha_{1}\right) k}}
\end{align*}
$$

where $\lambda_{1}$ and $t_{0}$ are arbitrary constants and $\alpha_{1} \neq-\frac{4 \alpha_{2}}{\beta}$.
The relation $\alpha_{1}=-\frac{4 \alpha_{2}}{\beta}$ leads again to the plane-wave solution of the form mentioned above.

Thus, substituting (46) into ansatz (43) one arrives at the two-parameter family of exact solutions

$$
\begin{align*}
& u(t, x)=\lambda_{1}\left(\left(4 \alpha_{2}+\beta \alpha_{1}\right) k \lambda_{1}^{\beta k}\left(t-t_{0}\right)\right)^{-\frac{\alpha_{1}}{\left(4 \alpha_{2}+\beta \alpha_{1}\right) k}} \exp \left(-\frac{4}{\beta} x\right) \\
& v(t, x)=\left(\left(4 \alpha_{2}+\beta \alpha_{1}\right) k \lambda_{1}^{\beta k}\left(t-t_{0}\right)\right)^{-\frac{\alpha_{2}}{\left.44 \alpha_{2}+\beta \alpha_{1}\right) k}} \exp (x), \quad \alpha_{1} \neq-\frac{4 \alpha_{2}}{\beta}, k \neq 0 \tag{47}
\end{align*}
$$

of the RD system

$$
\begin{align*}
& u_{x x}=u^{\beta} u_{t}+\alpha_{1} u^{1+\beta(1+k)} v^{4 k}+\frac{16 u}{\beta^{2}}  \tag{48}\\
& v_{x x}=v^{-4} v_{t}+\alpha_{2} u^{\beta k} v^{-3+4 k}+v .
\end{align*}
$$

Now we transform the RD system (48) to the form, which usually arises in applications, using the substitution

$$
\begin{equation*}
u=U^{\frac{1}{\beta+1}}, \quad v=V^{-\frac{1}{3}} \tag{49}
\end{equation*}
$$

by which one takes the form

$$
\begin{align*}
& U_{t}=\left(U^{-\frac{\beta}{\beta+1}} U_{x}\right)_{x}-(\beta+1) U\left(\alpha_{1} U^{\frac{\beta k}{\beta+1}} V^{-\frac{4 k}{3}}+\frac{16}{\beta^{2}} U^{-\frac{\beta}{\beta+1}}\right) \\
& V_{t}=\left(V^{-\frac{4}{3}} V_{x}\right)_{x}+3 V\left(\alpha_{2} U^{\frac{\beta k}{\beta+1}} V^{-\frac{4 k}{3}}+V^{-\frac{4}{3}}\right) \tag{50}
\end{align*}
$$

System (50) can be simplified to

$$
\begin{align*}
& U_{t}=\left(U^{-\kappa} U_{x}\right)_{x}-\alpha_{1}^{*} U\left(U^{\kappa} V^{-\frac{4}{3}}\right)^{k}+\frac{16}{\beta \kappa} U^{1-\kappa}, \\
& V_{t}=\left(V^{-\frac{4}{3}} V_{x}\right)_{x}+\alpha_{2}^{*} V\left(U^{\kappa} V^{-\frac{4}{3}}\right)^{k}+3 V^{-\frac{1}{3}} \tag{51}
\end{align*}
$$

by introducing new notations $\kappa=\frac{\beta}{\beta+1}, \alpha_{1}^{*}=(\beta+1) \alpha_{1}, \alpha_{2}^{*}=3 \alpha_{2}$.
Remark 2. The RD system (51) with $\kappa=\frac{4}{3}$, is a particular case of one (28) [18], hence, the terms containing $U^{-\frac{1}{3}}$ and $V^{-\frac{1}{3}}$ can be removed by the substitution (34) [18]. However, this substitution does not work for $\kappa \neq \frac{4}{3}$.

Now we rewrite (47) using substitution (49) and new notations, hence, the two-parameter family of exact solutions takes the form

$$
\begin{align*}
& U(t, x)=\lambda_{1}\left(\gamma k \lambda_{1}^{\beta k}\left(t-t_{0}\right)\right)_{\alpha_{2}^{*}}^{-\frac{\alpha_{1}^{*}}{\gamma k}} \exp \left(-\frac{4}{\kappa} x\right)  \tag{52}\\
& V(t, x)=\left(\gamma k \lambda_{1}^{\beta k}\left(t-t_{0}\right)\right)^{-\frac{\alpha_{2}^{k}}{\gamma k}} \exp (-3 x)
\end{align*}
$$

where $\gamma k \neq 0, \gamma=\kappa \alpha_{1}^{*}+\frac{4}{3} \alpha_{2}^{*}$.
It should be noted that solutions of the form (52) possess essentially different properties depending on the parameter signs, namely:

Table 1: Reduction of RD systems to the systems of ODEs (case (II) of Theorem 1)

| RD system | $Q$ | Ansatz | System of ODEs |
| :---: | :---: | :---: | :---: |
| $\begin{gathered} u_{x x}=e^{u} u_{t}+e^{u} f(\omega) \\ v_{x x}=v^{-4} v_{t}+v+ \\ v^{-3} g(\omega), \omega=e^{u} v^{4} \end{gathered}$ | $\begin{gathered} \left(e^{2 x}+\alpha e^{-2 x}\right) \partial_{x}- \\ 4\left(e^{2 x}-\alpha e^{-2 x}\right) \partial_{u}+ \\ \left(e^{2 x}-\alpha e^{-2 x}\right) v \partial_{v} \\ \hline \end{gathered}$ | $\begin{gathered} u=\varphi(t)- \\ 2 \ln \left(e^{2 x}+\alpha e^{-2 x}\right) \\ v=\psi(t)\left(e^{2 x}+\alpha e^{-2 x}\right)^{\frac{1}{2}} \\ \hline \end{gathered}$ | $\begin{gathered} e^{\varphi} \varphi^{\prime}+e^{\varphi} f(\chi)+32 \alpha=0 \\ \psi^{\prime}+\psi g(\chi)-4 \alpha \psi^{5}=0 \\ \chi=e^{\varphi} \psi^{4} \\ \hline \end{gathered}$ |
| $\begin{gathered} u_{x x}=e^{u} u_{t}+e^{u} f(\omega) \\ v_{x x}=v^{-4} v_{t}-v+ \\ v^{-3} g(\omega), \omega=e^{u} v^{4} \\ \hline \end{gathered}$ | $\begin{gathered} \sin (2 x) \partial_{x}- \\ 4 \cos (2 x) \partial_{u}+ \\ \cos (2 x) v \partial_{v} \\ \hline \end{gathered}$ | $\begin{gathered} u=\varphi(t)-2 \ln \sin (2 x) \\ v=\psi(t) \sin ^{\frac{1}{2}}(2 x) \end{gathered}$ | $\begin{gathered} e^{\varphi} \varphi^{\prime}+e^{\varphi} f(\chi)-8=0 \\ \psi^{\prime}+\psi g(\chi)+\psi^{5}=0 \\ \chi=e^{\varphi} \psi^{4} \end{gathered}$ |
| $\begin{gathered} u_{x x}=u^{\beta} u_{t}+ \\ u^{\beta+1} f(\omega)+\frac{16}{\beta^{2}} u \\ v_{x x}=v^{-4} v_{t}+v+ \\ v^{-3} g(\omega), \omega=u^{\beta} v^{4} \\ \hline \end{gathered}$ | $\begin{gathered} \left(e^{2 x}+\alpha e^{-2 x}\right) \partial_{x}- \\ \frac{4}{\beta}\left(e^{2 x}-\alpha e^{-2 x}\right) u \partial_{u} \\ \left(e^{2 x}-\alpha e^{-2 x}\right) v \partial_{v} \end{gathered}$ | $\begin{gathered} u=\varphi(t)\left(e^{2 x}+\alpha e^{-2 x}\right)^{-\frac{2}{\beta}} \\ v=\psi(t)\left(e^{2 x}+\alpha e^{-2 x}\right)^{\frac{1}{2}} \end{gathered}$ | $\begin{gathered} \varphi^{\prime}+\varphi f(\chi)+ \\ \frac{32 \alpha(2+\beta)}{\beta^{2}} \varphi^{1-\beta}=0 \\ \psi^{\prime}+\psi g(\chi)-4 \alpha \psi^{5}=0 \\ \chi=\varphi^{\beta} \psi^{4} \end{gathered}$ |
| $\begin{gathered} u_{x x}=u^{\beta} u_{t}+ \\ u^{\beta+1} f(\omega)-\frac{16}{\beta^{2}} u \\ v_{x x}=v^{-4} v_{t}-v+ \\ v^{-3} g(\omega), \omega=u^{\beta} v^{4} \end{gathered}$ | $\begin{gathered} \sin (2 x) \partial_{x}- \\ \frac{4}{\beta} \cos (2 x) u \partial_{u}+ \\ \cos (2 x) v \partial_{v} \end{gathered}$ | $\begin{gathered} u=\varphi(t) \sin ^{-\frac{2}{\beta}}(2 x) \\ v=\psi(t) \sin ^{\frac{1}{2}}(2 x) \end{gathered}$ | $\begin{gathered} \varphi^{\prime}+\varphi f(\chi)- \\ \frac{8(2+\beta)}{\beta^{2}} \varphi^{1-\beta}=0 \\ \psi^{\prime}+\psi g(\chi)+\psi^{5}=0 \\ \chi=\varphi^{\beta} \psi^{4} \end{gathered}$ |

1. each solution $(U, V)$ blows up for the finite time $t_{0}>0$ provided $\alpha_{1}^{*} \gamma k>0$ and $\alpha_{2}^{*} \gamma k>0$;
2. the component $U$ blows up for the finite time $t_{0}>0$ while the component $V$ vanishes provided $\alpha_{1}^{*} \gamma k>0$ and $\alpha_{2}^{*} \gamma k<0$ (and vice versa);
3. each solution $(U, V)$ vanishes for the finite time $t_{0}>0$ provided $\alpha_{1}^{*} \gamma k<0$ and $\alpha_{2}^{*} \gamma k<0$;
4. if $t_{0}<0$ then all solutions belonging to the family (52) are global (in time) and, depending on the parameters and the space domain, can be either bounded or unbounded.

Thus, the solutions obtained may be used for describing a wide range of different processes arising in applications (examples lie outside the scope of the current work). However, it should be stressed that each solution of the form (52) is a non-Lie solution, i.e. one is not obtainable via Lie symmetry operators. In fact, the nonlinear RD system (51) under the restrictions listed above admits only the trivial Lie algebra generated by time and space translation operators (see Theorem 1 in [18]). It means that plane-wave solutions only are obtainable via the Lie method.

Exact solutions of non-linear RD systems arising in cases (II) and (III) of Theorem 1 can be constructed in a similar way. In particular, using operator (22), four different reductions were obtained for the RD systems of the form (21), which are listed in Table 1 (the parameter $\mu$ can be reduced to $\pm 4$ without losing a generality). One observes that structures of solutions essentially depend on diffusivity in the first equation and on the linear term sign in the second equation. Because the ODE systems obtained have rather complicated structures, one needs to specify correctly the functions $f$ and $g$ in order to find exact solutions in an explicit form (otherwise numerical simulations should be applied).

## 4 Discussion

In this paper, $Q$-conditional symmetries of the form (6) for the RD systems belonging to the class (3) and their application are studied. The main result is presented in Theorems 1 and 2. Thus, taking into account Theorem 1 [1], one may claim that $Q$-conditional symmetries of the first type of the RD systems with $\left(d_{u}^{1}\right)^{2}+\left(d_{v}^{2}\right)^{2} \neq 0$ are completely described. In particular, three new subclasses of RD systems with non-constant diffusivities are found (see Theorem 1), which do not coincide with those presented in [1] (see Table 1). It is interesting to note that all the RD systems listed in Theorem 1 contain the second RD equation involving the powerlaw diffusivity with exponent -4 . This exponent corresponds to the known critical exponent $-\frac{4}{3}$ arising in RD systems of the form (1) with the widest Lie symmetry [18, 19] (note that the list of the RD systems possessing non-trivial Lie symmetry was essentially reduced in [18] comparing with earlier paper [19] in order to obtain really in-equivalent systems). Moreover, setting formally the diffusivity with the same exponent (i.e. $d^{1}=u^{-4}$ ) in the first equation, one arrives at the known (from the Lie symmetry point of view!) systems. Thus, the systems
obtained here are generalizations of those with the widest Lie symmetry in order to obtain conditional symmetry instead of classical symmetry.

It is well-known that the differential consequences of the operator in question can be used in order to extend non-classical symmetry (see, e.g., [7, 12]), hence one may reformulate Definition 2 in a such way that the manifold
$\mathcal{M}_{2}^{*}=\left\{S_{1}=0, S_{2}=0, Q(u)=0, Q(v)=0, \frac{\partial}{\partial t} Q(u)=0, \frac{\partial}{\partial x} Q(u)=0, \frac{\partial}{\partial t} Q(v)=0, \frac{\partial}{\partial x} Q(v)=0\right\}$
can be used instead of $\mathcal{M}_{2}$. However, the definition obtained does not lead to any new conditional symmetries of system (3) if one is looking for operators of the form (5) with $\xi^{0} \neq 0$ [6] (the detailed proof is presented in [20]). In the case $\xi^{0}=0, \xi^{1} \neq 0$ the situation is essentially different. As it is shown above, Definition 2 produces the system of determining equations (7) (this is still a challenge to solve one without any restrictions). On the other hand, application of this definition with the manifold $\mathcal{M}_{2}^{*}$ leads to a system of determining equations, which consists of two equations only, namely (in this case we can put $\xi^{1}=1$ without losing a generality):

$$
\begin{gather*}
\eta^{1} C_{u}^{1}+\eta^{2} C_{v}^{1}+\eta^{1} \frac{d_{u}^{1}}{d^{1}}\left(\eta_{x}^{1}+\eta^{1} \eta_{u}^{1}+\eta^{2} \eta_{v}^{1}-C^{1}\right)-\frac{d^{1}}{d^{2}} \eta_{v}^{1} C^{2}-\eta_{u}^{1} C^{1}+d^{1} \eta_{t}^{1}-\eta_{x x}^{1}+ \\
\left(\frac{d^{1}}{d^{2}}-1\right) \eta_{v}^{1}\left(\eta_{x}^{2}+\eta^{1} \eta_{u}^{2}+\eta^{2} \eta_{v}^{2}\right)-2 \eta^{1} \eta_{x u}^{1}-2 \eta^{2} \eta_{x v}^{1}-\left(\eta^{1}\right)^{2} \eta_{u u}^{1}- \\
2 \eta^{1} \eta^{2} \eta_{u v}^{1}-\left(\eta^{2}\right)^{2} \eta_{v v}^{1}=0  \tag{53}\\
\eta^{1} C_{u}^{2}+\eta^{2} C_{v}^{2}+\eta^{2} \frac{d_{v}^{2}}{d^{2}}\left(\eta_{x}^{2}+\eta^{1} \eta_{u}^{2}+\eta^{2} \eta_{v}^{2}-C^{2}\right)-\frac{d^{2}}{d^{1}} \eta_{u}^{2} C^{1}-\eta_{v}^{2} C^{2}+d^{2} \eta_{t}^{2}-\eta_{x x}^{2}+ \\
\left(\frac{d^{2}}{d^{1}}-1\right) \eta_{u}^{2}\left(\eta_{x}^{1}+\eta^{1} \eta_{u}^{1}+\eta^{2} \eta_{v}^{1}\right)-2 \eta^{1} \eta_{x u}^{2}-2 \eta^{2} \eta_{x v}^{2}-\left(\eta^{1}\right)^{2} \eta_{u u}^{2}- \\
2 \eta^{1} \eta^{2} \eta_{u v}^{2}-\left(\eta^{2}\right)^{2} \eta_{v v}^{2}=0 .
\end{gather*}
$$

According to the so called no-go theorem [7] (its generalisation on systems of evolution equations is straightforward), system (531) is reducible to the initial RD system (5), i.e. cannot be solved in the general case. In other words, the amended definition is not applicable for constructing new $Q$-conditional symmetries because the problem is reduced to solving the reaction-diffusion system in question. However, we have shown in this paper that using Definition 1 a special subset of such symmetries can be completely described and found in explicit form.

Generally speaking, the notion of $Q$-conditional symmetry of the $p$-th type [6], which is successfully applied here to the nonlinear RD systems of the form (3), may be thought as a further development of the concept of conditional invariance proposed in [12, Section 5.7] (see also highly nontrivial examples in [21]). It is important because a list of successful applications of this concept for nonlinear systems of evolution equations is relatively short.

It should be also noted that symmetry based methods for solving nonlinear PDEs have clear connection to the method of differential constraints, which has been formulated in [22] (see also
the later monograph [23]). In fact, a common property that underlies all the symmetry based methods can be described as follows: in order to find exact solutions one solves a nonlinear PDE (system of PDEs) together with the differential constraint(s) generated by a symmetry operator. The corresponding symmetry can be of different types: Lie symmetry, $Q$-conditional symmetry, generalised conditional symmetry etc. Because the over-determined system consisting of the given PDE and the differential constraint is compatible one can find its solutions in a much simpler way. It means that the main problem of the method of differential constraints, how to define suitable constraint(s) for the given PDE in a such way that the over-determined system obtained will be compatible, is automatically solved. Of course, one may try to find the suitable differential constraints by other methods (see, e.g., [24, 25] and the papers cited therein), i.e. without using any symmetry based approach. An overview of possible approaches with attempt to create the general algorithm of integrating over-determined systems is presented in [23].

In order to demonstrate the applicability of the derived symmetries, we used those for reducing the nonlinear RD systems to the relevant ODEs systems and constructing exact solutions. In particular $Q$-conditional operators arising in cases (I) and (II) of Theorem 1 were used in order to construct non-Lie ansätze and to reduce the relevant RD systems to the corresponding ODE systems, which are presented in formulae (43)-(44) and Table 1. As result, multi-parameter families of exact solutions in the explicit form (52) were constructed for the RD system (51) with an arbitrary power-law diffusivity. Moreover, we have shown that the solutions obtained possess attractive properties, hence can describe different phenomena arising in applications.

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