

# Numerical methods of solutions of boundary value problems for the multi-term variable-distributed order diffusion equation

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## Abstract

Solutions of the Dirichlet and Robin boundary value problems for the multi-term variable-distributed order diffusion equation are studied. A priori estimates for the corresponding differential and difference problems are obtained by using the method of the energy inequalities. The stability and convergence of the difference schemes follow from these a priori estimates. The credibility of the obtained results is verified by performing numerical calculations for test problems.

*Keywords:* fractional order diffusion equation, fractional derivative, a priori estimate, difference scheme, stability and convergence

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## 1. Introduction

Differential equations with fractional order derivatives provide a powerful mathematical tool for accurate and realistic description of physical and chemical processes proceeding in media with fractal geometry [1, 2, 3, 4, 5]. It is known that the order of a fractional derivative depends on the fractal dimension of medium [6, 7]. It is therefore reasonable to construct mathematical models based on partial differential equations with the variable and distributed order derivatives [1, 8, 9, 10, 11, 12, 13, 14]. Analytical methods for solving such equations are scarcely effective, so that the development of the corresponding numerical methods is very important.

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The initial-boundary-value problems for the generalized multi-term time fractional diffusion equation over an open bounded domain  $G \times (0, T)$ ,  $G \in \mathbb{R}^n$  were considered [15]. Multi-term linear and non-linear diffusion-wave equations of fractional order were solved in [16] using the Adomian decomposition method. Applications of the homotopy analysis and new modified homotopy perturbation methods to solutions of multi-term linear and nonlinear diffusion-wave equations of fractional order are discussed in [17, 18]. In the papers [19, 20] analytical solutions for the multi-term time-fractional diffusion-wave and the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain are studied. The fundamental solution of the multi-term diffusion equation with the Dzharbashyan-Nersesyan fractional differentiation operator with respect to the time variables is constructed in [21].

Several methods for solving variable and distributed order fractional differential equations with various kinds of the variable and distributed fractional derivative have been proposed [11, 22, 23, 24, 25, 26, 27, 28]. A priori estimates for the difference problems obtained in [29, 30, 31] by using the maximum principle imply the stability and convergence of the considered difference schemes. Using the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the fractional and variable order diffusion equation with Caputo fractional derivative have been obtained [32], [33].

## 2. Boundary value problems in differential setting

### 2.1. The Dirichlet boundary value problem

In rectangle  $\bar{Q}_T = \{(x, t) : 0 \leq x \leq l, 0 \leq t \leq T\}$  let us study the boundary value problem

$$\mathbb{P}_{(\omega)}^{(\theta)}(\partial_{0t})u(x, t) = \frac{\partial}{\partial x} \left( k(x, t) \frac{\partial u}{\partial x} \right) - q(x, t)u + f(x, t), \quad 0 < x < l, \quad 0 < t \leq T, \quad (1)$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad 0 \leq t \leq T, \quad (2)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq l, \quad (3)$$

where

$$\mathbb{P}_{(\omega)}^{(\theta)}(\partial_{0t})u(x, t) = \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \omega_r(x, \gamma) \partial_{0t}^{\theta_r(x, \gamma)} u(x, t),$$

$\alpha < \beta$ ,  $0 < \theta_r(x, \gamma) < 1$ ,  $\omega_r(x, \gamma) \geq 0$ ,  $r = 1, 2, \dots, m$ , for all

$$(x, \gamma) \in [0, l] \times [\alpha, \beta], \quad \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \omega_r(x, \gamma) > 0, \quad \theta_r(x, \gamma) \in C[0, l] \times [\alpha, \beta],$$

$$0 < c_1 \leq k(x, t) \leq c_2, \quad q(x, t) \geq 0,$$

$\partial_{0t}^{\theta_r(x, \gamma)} u(x, \eta) = \int_0^t u_\eta(x, \eta) (t - \eta)^{-\theta_r(x, \gamma)} d\eta / \Gamma(1 - \theta_r(x, \gamma))$  is a Caputo fractional derivative of order  $\theta_r(x, \gamma)$  [34, 35].

The existence of the solution for the initial boundary value problem of fractional, multi-term and distributed order diffusion equation has been proven in [12, 19, 36, 37, 38, 39].

Let us assume further the existence of a solution  $u(x, t) \in C^{2,1}(\bar{Q}_T)$  for the problems (1)–(3), where  $C^{m,n}$  is the class of functions, continuous together with their partial derivatives of the order  $m$  with respect to  $x$  and order  $n$  with respect to  $t$  on  $\bar{Q}_T$ .

**Lemma 1.** For any functions  $v(t)$  and  $w(t)$  absolutely continuous on  $[0, T]$ , one has the equality:

$$\begin{aligned} & v(t) \mathbb{P}_{(\bar{\omega})}^{(\bar{\theta})} (\partial_{0t}) w(t) + w(t) \mathbb{P}_{(\bar{\omega})}^{(\bar{\theta})} (\partial_{0t}) v(t) = \mathbb{P}_{(\bar{\omega})}^{(\bar{\theta})} (\partial_{0t}) (v(t)w(t)) + \\ & + \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \frac{\bar{\omega}_r(\gamma) \bar{\theta}_r(\gamma)}{\Gamma(1 - \bar{\theta}_r(\gamma))} \int_0^t \frac{d\xi}{(t - \xi)^{1 - \bar{\theta}_r(\gamma)}} \int_0^{\xi} \frac{v'(\eta) d\eta}{(t - \eta)^{\bar{\theta}_r(\gamma)}} \int_0^{\xi} \frac{w'(s) ds}{(t - s)^{\bar{\theta}_r(\gamma)}}, \quad (4) \end{aligned}$$

where  $\bar{\omega}_r(\gamma) \geq 0$ ,  $0 < \bar{\theta}_r(\gamma) < 1$ , for all  $\gamma \in [\alpha, \beta]$ ,  $\int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \bar{\omega}_r(\gamma) > 0$ .

**Proof.** For any fixed  $\gamma \in [\alpha, \beta]$  and  $r \in \{1, 2, \dots, m\}$ , relying on lemma 1 [33] one finds the following equality

$$\begin{aligned} & v(t) \partial_{0t}^{\bar{\theta}_r(\gamma)} w(t) + w(t) \partial_{0t}^{\bar{\theta}_r(\gamma)} v(t) = \partial_{0t}^{\bar{\theta}_r(\gamma)} (v(t)w(t)) + \\ & + \frac{\bar{\theta}_r(\gamma)}{\Gamma(1 - \bar{\theta}_r(\gamma))} \int_0^t \frac{d\xi}{(t - \xi)^{1 - \bar{\theta}_r(\gamma)}} \int_0^{\xi} \frac{v'(\eta) d\eta}{(t - \eta)^{\bar{\theta}_r(\gamma)}} \int_0^{\xi} \frac{w'(s) ds}{(t - s)^{\bar{\theta}_r(\gamma)}}. \quad (5) \end{aligned}$$

Multiplying (5) by  $\bar{\omega}_r(\gamma)$  and summing the result over  $r$  from 1 to  $m$ , then integrating over  $\gamma$  from  $\alpha$  to  $\beta$  one obtains (4). The proof of the lemma 1 is complete.

**Corollary.** For any function  $v(t)$  absolutely continuous on  $[0, T]$ , the following equality takes place:

$$v(t)\mathbb{P}_{(\bar{\omega})}^{(\bar{\theta})}(\partial_{0t})v(t) = \frac{1}{2}\mathbb{P}_{(\bar{\omega})}^{(\bar{\theta})}(\partial_{0t})v^2(t) + \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \frac{\bar{\omega}_r(\gamma)\bar{\theta}_r(\gamma)}{2\Gamma(1-\bar{\theta}_r(\gamma))} \int_0^t \frac{d\xi}{(t-\xi)^{1-\bar{\theta}_r(\gamma)}} \left( \int_0^{\xi} \frac{v'(\eta)d\eta}{(t-\eta)^{\bar{\theta}_r(\gamma)}} \right)^2, \quad (6)$$

where  $\bar{\omega}_r(\gamma) \geq 0$ ,  $0 < \bar{\theta}_r(\gamma) < 1$ , for all  $\gamma \in [\alpha, \beta]$ ,  $\int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \bar{\omega}_r(\gamma) > 0$ .

Let us use the following notation:  $\|u\|_0^2 = \int_0^l u^2(x, t)dx$ ,  $D_{0t}^{-\nu}u(x, t) = \int_0^t (t-s)^{\nu-1}u(x, s)ds/\Gamma(\nu)$  is a fractional Riemann-Liouville integral of order  $\nu > 0$ .

**Theorem 1.** If  $k(x, t) \in C^{1,0}(\bar{Q}_T)$ ,  $q(x, t), f(x, t) \in C(\bar{Q}_T)$ ,  $k(x, t) \geq c_1 > 0$ ,  $q(x, t) \geq 0$  everywhere on  $\bar{Q}_T$ , then the solution  $u(x, t)$  of the problem (1)–(3) satisfies the a priori estimate:

$$\begin{aligned} & \int_0^l \mathbb{P}_{(\omega)}^{(\theta-1)}(D_{0t})u^2(x, t)dx + c_1 \int_0^t \|u_x(x, s)\|_0^2 ds \leq \\ & \leq \frac{l^2}{2c_1} \int_0^t \|f(x, s)\|_0^2 ds + \int_0^l u_0^2(x)dx \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \frac{\omega_r(x, \gamma)t^{1-\theta_r(x, \gamma)}}{\Gamma(2-\theta_r(x, \gamma))}, \quad (7) \end{aligned}$$

where  $\mathbb{P}_{(\omega)}^{(\theta-1)}(D_{0t}) = \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \omega_r(x, \gamma)D_{0t}^{\theta_r(x, \gamma)-1}$ .

**Proof.** Let us multiply equation (1) by  $u(x, t)$  and integrate the resulting relation over  $x$  from 0 to  $l$ :

$$\int_0^l u(x, t)\mathbb{P}_{(\omega)}^{(\theta)}(\partial_{0t})u(x, t)dx - \int_0^l u(x, t)(k(x, t)u_x(x, t))_x dx +$$

$$+ \int_0^l q(x, t) u^2(x, t) dx = \int_0^l u(x, t) f(x, t) dx. \quad (8)$$

Then transform the terms in identity (8) as

$$- \int_0^l u(x, t) (k(x, t) u_x(x, t))_x dx = \int_0^l k(x, t) u_x^2(x, t) dx \geq c_1 \|u_x(x, t)\|_0^2, \quad (9)$$

$$\left| \int_0^l u(x, t) f(x, t) dx \right| \leq \varepsilon \|u(x, t)\|_0^2 + \frac{1}{4\varepsilon} \|f(x, t)\|_0^2, \quad \varepsilon > 0. \quad (10)$$

Using the equality (6) one obtains

$$\int_0^l u(x, t) \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u(x, t) dx \geq \frac{1}{2} \int_0^l \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u^2(x, t) dx. \quad (11)$$

Taking into account the above performed transformations, from the identity (8) one arrives at the inequality

$$\frac{1}{2} \int_0^l \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u^2(x, t) dx + c_1 \|u_x(x, t)\|_0^2 \leq \varepsilon \|u(x, t)\|_0^2 + \frac{1}{4\varepsilon} \|f(x, t)\|_0^2. \quad (12)$$

Using the inequality  $\|u(x, t)\|_0^2 \leq (l^2/2) \|u_x(x, t)\|_0^2$ , from the inequality (12) at  $\varepsilon = c_1/l^2$ , one obtains

$$\int_0^l \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u^2(x, t) dx + c_1 \|u_x(x, t)\|_0^2 \leq \frac{l^2}{2c_1} \|f(x, t)\|_0^2. \quad (13)$$

Changing the variable  $t$  by  $s$  in inequality (13) and integrating it over  $s$  from 0 to  $t$ , one obtains the a priori estimate (7).

The uniqueness and the continuous dependence of the solution of the problem (1)–(3) on the input data follow from the a priori estimate (7).

2.2. *The Robin boundary value problem.*

In the problem (1)–(3) we replace the boundary conditions (2) with

$$\begin{cases} k(0, t)u_x(0, t) = \beta_1(t)u(0, t) - \mu_1(t), \\ -k(l, t)u_x(l, t) = \beta_2(t)u(l, t) - \mu_2(t). \end{cases} \quad (14)$$

In the rectangle  $\bar{Q}_T$  we consider the Robin boundary value problem (1), (3), (14).

**Theorem 2.** If  $k(x, t) \in C^{1,0}(\bar{Q}_T)$ ,  $q(x, t), f(x, t) \in C(\bar{Q}_T)$ ,  $k(x, t) \geq c_1 > 0$ ,  $q(x, t) \geq 0$  everywhere on  $\bar{Q}_T$ ,  $\beta_i(t), \mu_i(t) \in C[0, T]$ ,  $\beta_i(t) \geq \beta_0 > 0$ , for all  $t \in [0, T]$ ,  $i = 1, 2$ , then the solution  $u(x, t)$  of the problem (1), (3), (14) satisfies the a priori estimate:

$$\begin{aligned} & \int_0^l \mathbb{P}_{(\omega)}^{(\theta-1)} (D_{0t}) u^2(x, t) dx + \gamma_1 \left( \int_0^t (\|u_x(x, s)\|_0^2 + u^2(0, s) + u^2(l, s)) ds \right) \leq \\ & \leq \frac{\delta_1}{\gamma_1} \left( \int_0^t (\|f(x, s)\|_0^2 + \mu_1^2(s) + \mu_2^2(s)) ds \right) + \\ & + \int_0^l u_0^2(x) dx \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \frac{\omega_r(x, \gamma) t^{1-\theta_r(x, \gamma)}}{\Gamma(2 - \theta_r(x, \gamma))}, \end{aligned} \quad (15)$$

where  $\gamma_1 = \min\{c_1, \beta_0\}$ ,  $\delta_1 = \max\{1 + l, l^2\}$ .

**Proof.** Let us multiply the equation (1) by  $u(x, t)$  and integrate the resulting relation over  $x$  from 0 to  $l$ :

$$\begin{aligned} & \int_0^l u(x, t) \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u(x, t) dx - \int_0^l u(x, t) (k(x, t) u_x(x, t))_x dx + \\ & + \int_0^l q(x, t) u^2(x, t) dx = \int_0^l u(x, t) f(x, t) dx. \end{aligned} \quad (16)$$

Now we transform the terms of the identity (16):

$$\int_0^l u(x, t) \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u(x, t) dx \geq \frac{1}{2} \int_0^l \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u^2(x, t) dx.$$

$$-\int_0^l u(ku_x)_x dx = \beta_1(t)u^2(0,t) + \beta_2(t)u^2(l,t) - \mu_1(t)u(0,t) - \mu_2(t)u(l,t) + \int_0^l ku_x^2 dx,$$

$$\left| \int_0^l u f dx \right| \leq \varepsilon \|u\|_0^2 + \frac{1}{4\varepsilon} \|f\|_0^2, \quad \varepsilon > 0.$$

From (16), taking into account the transformations performed, one arrives at the inequality

$$\begin{aligned} \frac{1}{2} \int_0^l \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u^2(x,t) dx + c_1 \|u_x(x,t)\|_0^2 + \beta_0 u^2(0,t) + \beta_0 u^2(l,t) \leq \\ \leq \mu_1(t)u(0,t) + \mu_2(t)u(l,t) + \varepsilon \|u\|_0^2 + \frac{1}{4\varepsilon} \|f\|_0^2. \end{aligned} \quad (17)$$

Using the inequalities  $\mu_1(t)u(0,t) \leq \varepsilon u^2(0,t) + (4\varepsilon)^{-1} \mu_1^2(t)$ ,  $\mu_2(t)u(l,t) \leq \varepsilon u^2(l,t) + (4\varepsilon)^{-1} \mu_2^2(t)$ ,  $\varepsilon > 0$ ;  $\|u(x,t)\|_0^2 \leq l^2 \|u_x(x,t)\|_0^2 + l(u^2(0,t) + u^2(l,t))$  with  $\varepsilon = \gamma_1/(2\delta_1)$ , from (17) one has the following inequality

$$\begin{aligned} \int_0^l \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u^2(x,t) dx + \gamma_1 (\|u_x(x,t)\|_0^2 + u^2(0,t) + u^2(l,t)) \leq \\ \leq \frac{\delta_1}{\gamma_1} (\|f(x,t)\|_0^2 + \mu_1^2(t) + \mu_2^2(t)). \end{aligned} \quad (18)$$

Changing the variable  $t$  by  $s$  in inequality (18) and integrating it over  $s$  from 0 to  $t$ , we obtain the a priori estimate (15).

The uniqueness and the continuous dependence of the solution of problem (1), (3), (14) on the input data follow from the a priori estimate (15).

### 3. Boundary value problems in difference setting

Suppose that a solution  $u(x,t) \in C^{4,3}(Q_T)$  of the problem (1)–(3) exists, and the coefficients of the equation (1) and the functions  $f(x,t)$ ,  $u_0(x)$  satisfy the smoothness conditions, required for the construction of difference schemes with the order of approximation  $O(\tau^{2-\theta_{\max}} + h^2)$ , where  $\theta_{\max} = \max_{r,\gamma} \theta_r(x,\gamma)$ .

In the rectangle  $\bar{Q}_T$  we introduce the grid  $\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$ , where  $\bar{\omega}_h = \{x_i = ih, i = 0, 1, \dots, N, hN = l\}$ ,  $\bar{\omega}_\tau = \{t_j = j\tau, j = 0, 1, \dots, j_0, \tau j_0 = T\}$ .

Before to turn to the approximation of the problem (1)–(3), let us find the discrete analog of the  $\mathbb{P}_{(\omega)}^{(\theta)}(\partial_{0t})u(x, t)$ . For any fixed  $\gamma \in [\alpha, \beta]$  and  $r \in \{1, 2, \dots, m\}$ , the following equality takes place

$$\begin{aligned}
\partial_{0t_{j+1}}^{\theta_r(x_i, \gamma)} u(x_i, \eta) &= \frac{1}{\Gamma(1 - \theta_r(x_i, \gamma))} \int_0^{t_{j+1}} \frac{\partial}{\partial \eta} u(x_i, \eta) d\eta = \\
&= \frac{1}{\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \int_{t_s}^{t_{s+1}} \frac{\partial}{\partial \eta} u(x_i, \eta) d\eta = \\
&= \frac{1}{\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \int_{t_s}^{t_{s+1}} \frac{\partial}{\partial \eta} u(x_i, \eta) \Big|_{\eta=t_{s+1/2}} d\eta + \\
&= \frac{1}{\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \int_{t_s}^{t_{s+1}} \frac{\partial^2}{\partial \eta^2} u(x_i, \eta) \Big|_{\eta=t_{s+1/2}} \frac{(\eta - t_{s+1/2})}{(t_{j+1} - \eta)^{\theta_r(x_i, \gamma)}} d\eta + O(\tau^2) = \\
&= \frac{1}{\Gamma(2 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \left( t_{j-s+1}^{1-\theta_r(x_i, \gamma)} - t_{j-s}^{1-\theta_r(x_i, \gamma)} \right) \frac{u(x_i, t_{s+1}) - u(x_i, t_s)}{\tau} + \\
&+ \frac{1}{\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \frac{\partial^2}{\partial \eta^2} u(x_i, \eta) \Big|_{\eta=t_{s+1/2}} \int_{t_s}^{t_{s+1}} \frac{(\eta - t_{s+1/2}) d\eta}{(t_{j+1} - \eta)^{\theta_r(x_i, \gamma)}} + O(\tau^2). \quad (19)
\end{aligned}$$

Since

$$\begin{aligned}
&\left| \frac{1}{\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \frac{\partial^2}{\partial \eta^2} u(x_i, \eta) \Big|_{\eta=t_{s+1/2}} \int_{t_s}^{t_{s+1}} \frac{(\eta - t_{s+1/2}) d\eta}{(t_{j+1} - \eta)^{\theta_r(x_i, \gamma)}} \right| \leq \\
&\leq \frac{M}{\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \left| \int_{t_s}^{t_{s+1}} \frac{(\eta - t_{s+1/2}) d\eta}{(t_{j+1} - \eta)^{\theta_r(x_i, \gamma)}} \right| =
\end{aligned}$$

$$\begin{aligned}
&= \frac{M}{\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \left| \int_{t_{s+1/2}}^{t_{s+1}} \frac{(\eta - t_{s+1/2}) d\eta}{(t_{j+1} - \eta)^{\theta_r(x_i, \gamma)}} - \int_{t_s}^{t_{s+1/2}} \frac{(t_{s+1/2} - \eta) d\eta}{(t_{j+1} - \eta)^{\theta_r(x_i, \gamma)}} \right| = \\
&= \frac{2^{\theta_r(x_i, \gamma)} M \tau^{2 - \theta_r(x_i, \gamma)}}{4\Gamma(1 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \left( \int_0^1 \frac{z dz}{(2s+1-z)^{\theta_r(x_i, \gamma)}} - \int_0^1 \frac{z dz}{(2s+1+z)^{\theta_r(x_i, \gamma)}} \right) = \\
&= \frac{2^{\theta_r(x_i, \gamma)} M \tau^{2 - \theta_r(x_i, \gamma)}}{4\Gamma(1 - \theta_r(x_i, \gamma))} \int_0^1 z \sum_{s=0}^j \left( \frac{1}{(2s+1-z)^{\theta_r(x_i, \gamma)}} - \frac{1}{(2s+1+z)^{\theta_r(x_i, \gamma)}} \right) dz = \\
&= \frac{2^{\theta_r(x_i, \gamma)} M \tau^{2 - \theta_r(x_i, \gamma)}}{4\Gamma(1 - \theta_r(x_i, \gamma))} \int_0^1 z \left( \frac{1}{(1-z)^{\theta_r(x_i, \gamma)}} - \frac{1}{(2j+1+z)^{\theta_r(x_i, \gamma)}} \right) dz - \\
&- \frac{2^{\theta_r(x_i, \gamma)} M \tau^{2 - \theta_r(x_i, \gamma)}}{4\Gamma(1 - \theta_r(x_i, \gamma))} \int_0^1 z \sum_{s=1}^j \left( \frac{1}{(2s-1+z)^{\theta_r(x_i, \gamma)}} - \frac{1}{(2s+1-z)^{\theta_r(x_i, \gamma)}} \right) dz \leq \\
&\leq \frac{2^{\theta_r(x_i, \gamma)} M \tau^{2 - \theta_r(x_i, \gamma)}}{4\Gamma(1 - \theta_r(x_i, \gamma))} \int_0^1 \frac{z dz}{(1-z)^{\theta_r(x_i, \gamma)}} = \frac{2^{\theta_r(x_i, \gamma)} M \tau^{2 - \theta_r(x_i, \gamma)}}{4\Gamma(3 - \theta_r(x_i, \gamma))} \leq \frac{M \tau^{2 - \theta_{\max}}}{2}
\end{aligned}$$

with  $M = \max_{(x,t) \in \bar{Q}_T} |\frac{\partial^2}{\partial t^2} u(x,t)|$ , then multiplying (19) by  $\omega_r(x_i, \gamma)$  and summing the result over  $r$  from 1 to  $m$ , then integrating over  $\gamma$  from  $\alpha$  to  $\beta$  one finds

$$\mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t_{j+1}}) u(x_i, t) = \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) u(x_i, t) + O(\tau^{2 - \theta_{\max}}), \quad (20)$$

where

$$\begin{aligned}
\mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) u(x_i, t) &= \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \omega_r(x_i, \gamma) \Delta_{0t_{j+1}}^{\theta_r(x_i, \gamma)} u(x_i, t), \\
\Delta_{0t_{j+1}}^{\theta_r(x_i, \gamma)} u(x_i, t) &= \frac{1}{\Gamma(2 - \theta_r(x_i, \gamma))} \sum_{s=0}^j \left( t_{j-s+1}^{1 - \theta_r(x_i, \gamma)} - t_{j-s}^{1 - \theta_r(x_i, \gamma)} \right) \frac{u(x_i, t_{s+1}) - u(x_i, t_s)}{\tau}.
\end{aligned}$$

**Lemma 2.** For any function  $v(t)$  defined on the grid  $\bar{\omega}_\tau$  one has the inequalities

$$v^{j+1} \mathbb{P}_{(\bar{\omega})}^{(\bar{\theta})} (\Delta_{0t_{j+1}}) v \geq \frac{1}{2} \mathbb{P}_{(\bar{\omega})}^{(\bar{\theta})} (\Delta_{0t_{j+1}}) v^2. \quad (21)$$

**Proof.** For any fixed  $\gamma \in [\alpha, \beta]$  and  $r \in \{1, 2, \dots, m\}$ , relying on lemma 2 [33] one finds the following inequality

$$v^{j+1} \Delta_{0t_{j+1}}^{\bar{\theta}_r(\gamma)} v \geq \frac{1}{2} \Delta_{0t_{j+1}}^{\bar{\theta}_r(\gamma)} v^2 + \frac{\tau^2 \Gamma(2 - \bar{\theta}_r(\gamma))}{2} \left( \Delta_{0t_{j+1}}^{\bar{\theta}_r(\gamma)} v \right)^2 \geq \frac{1}{2} \Delta_{0t_{j+1}}^{\bar{\theta}_r(\gamma)} v^2. \quad (22)$$

Multiplying (22) by  $\omega_1(\gamma)$  and summing the result over  $r$  from 1 to  $m$ , then integrating over  $\gamma$  from  $\alpha$  to  $\beta$  one arrives at (21). The proof of the lemma 2 is complete.

### 3.1. The Dirichlet boundary value problem

To problem (1)–(3), we assign the difference scheme:

$$\mathbb{P}_{(\omega)}^{(\theta)} \left( \Delta_{0t_{j+1}} \right) y_i = \Lambda y_i^{j+1} + \varphi_i^{j+1}, \quad i = 1, 2, \dots, N-1, \quad j = 0, 1, \dots, j_0 - 1, \quad (23)$$

$$y(0, t) = 0, \quad y(l, t) = 0, \quad j = 0, 1, \dots, j_0, \quad (24)$$

$$y(x, 0) = u_0(x), \quad i = 0, 1, \dots, N, \quad (25)$$

where  $\Lambda y = (ay_{\bar{x}})_x - dy$ ,  $v_{\bar{x},i} = (v_i - v_{i-1})/h$ ,  $v_{x,i} = (v_{i+1} - v_i)/h$ ,  $a_i^{j+1} = k(x_{i-1/2}, t_{j+1})$ ,  $d_i^{j+1} = q(x_i, t_{j+1})$ ,  $\varphi_i^{j+1} = f(x_i, t_{j+1})$ ,  $\Delta_{0t_j}^{\theta_r(x_i, \gamma)} y_i = \sum_{s=0}^j (t_{j-s+1}^{1-\theta_r(x_i, \gamma)} - t_{j-s}^{1-\theta_r(x_i, \gamma)}) y_{t,i}^s / \Gamma(2 - \theta_r(x_i, \gamma))$  is the difference analogue of the Caputo fractional derivative of order  $\theta_r(x_i, \gamma)$ ,  $y_{t,i}^s = (y_i^{s+1} - y_i^s) / \tau$ .

According to [40] and the formula (20), the order of the approximation of the difference scheme (23)–(25) is  $O(\tau^{2-\theta_{\max}} + h^2)$ .

**Theorem 3.** The difference scheme (23)–(25) is absolutely stable and its solution satisfies the following a priori estimate:

$$\begin{aligned} & \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \left( \frac{\omega_r(x_i, \gamma)}{\Gamma(2 - \theta_r(x_i, \gamma))}, \sum_{s=0}^j (t_{j-s+1}^{1-\theta_r(x_i, \gamma)} - t_{j-s}^{1-\theta_r(x_i, \gamma)}) (y_i^{s+1})^2 \right) + \\ & + c_1 \sum_{s=0}^j \|y_{\bar{x}}^{j+1}\|_0^2 \tau \leq \frac{l^2}{2c_1} \sum_{s=0}^j \|\varphi^s\|_0^2 \tau + \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \left( \frac{\omega_r(x_i, \gamma) t_{j+1}^{1-\theta_r(x_i, \gamma)}}{\Gamma(2 - \theta_r(x_i, \gamma))}, u_0^2(x_i) \right), \end{aligned} \quad (26)$$

where  $(y, v) = \sum_{i=1}^{N-1} y_i v_i h$ ,  $(y, v) = \sum_{i=1}^N y_i v_i h$ ,  $\|y\|_0^2 = (y, y)$ ,  $\|y\|_0^2 = (y, y)$ .

**Proof.** Let us multiply scalarly equation (23) by  $y_i^{j+1}$ :

$$\left( y^{j+1}, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i \right) - (\Lambda y^{j+1}, y^{j+1}) = (\varphi^j, y^{j+1}). \quad (27)$$

Let us transform the terms in identity (27):

$$- (\Lambda y^{j+1}, y^{j+1}) = (a, (y_{\bar{x}}^{j+1})^2) + (d, (y^{j+1})^2) \geq c_1 \|y_{\bar{x}}^{j+1}\|_0^2, \quad (28)$$

$$|(\varphi^{j+1}, y^{j+1})| \leq \varepsilon \|y^{j+1}\|_0^2 + \frac{1}{4\varepsilon} \|\varphi^{j+1}\|_0^2 \leq \frac{\varepsilon l^2}{2} \|y_{\bar{x}}^{j+1}\|_0^2 + \frac{1}{4\varepsilon} \|\varphi^{j+1}\|_0^2, \quad \varepsilon > 0. \quad (29)$$

Relying on lemma 2 one has

$$\left( y^{j+1}, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i \right) \geq \frac{1}{2} \left( 1, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i^2 \right). \quad (30)$$

From (27) with taking into account (28), (29) and (30), it follows that

$$\frac{1}{2} \left( 1, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i^2 \right) + c_1 \|y_{\bar{x}}^{j+1}\|_0^2 \leq \frac{\varepsilon l^2}{2} \|y_{\bar{x}}^{j+1}\|_0^2 + \frac{1}{4\varepsilon} \|\varphi\|_0^2. \quad (31)$$

Multiplying the inequality (31) at  $\varepsilon = c_1/l^2$ , by  $\tau$  and summing over  $s$  from 0 to  $j$ , one obtains the a priori estimate (26).

The stability and convergence of the difference scheme (23)–(25) follow from the a priori estimate (26).

Here the results are obtained for the homogeneous boundary conditions  $u(0, t) = 0$ ,  $u(l, t) = 0$ . In the case of inhomogeneous boundary conditions  $u(0, t) = \mu_1(t)$ ,  $u(l, t) = \mu_2(t)$  the boundary conditions for the difference problem will have the following form:

$$y(0, t) = \mu_1(t), \quad y(l, t) = \mu_2(t). \quad (32)$$

Convergence of the difference scheme (23), (25), (32) follows from the results obtained above. Actually, let us introduce the notation  $y = z + u$ . Then the error  $z = y - u$  is a solution of the following problem:

$$\mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) z_i = \Lambda z^{j+1} + \psi^{j+1}, \quad i = 1, \dots, N-1, \quad j = 0, 1, \dots, j_0-1, \quad (33)$$

$$z(0, t) = 0, \quad z(l, t) = 0, \quad j = 0, \dots, j_0, \quad (34)$$

$$z(x, 0) = 0, \quad i = 0, \dots, N, \quad (35)$$

where  $\psi \equiv \Lambda u^{j+1} - \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) u_i + \varphi^{j+1} = O(\tau^{2-\theta_{\max}} + h^2)$ .

The solution of the problem (33)–(35) satisfies the estimation (26) so that the solution of the difference scheme (23), (25), (32) converges to the solution of the corresponding differential problem with order  $O(\tau^{2-\theta_{\max}} + h^2)$ .

### 3.2. Numerical results

In this section, the following multi-term variable-distributed order time fractional diffusion equation is considered:

$$\begin{cases} \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u(x, t) = \frac{\partial}{\partial x} (k(x, t) \frac{\partial u}{\partial x}) - q(x, t)u + f(x, t), \\ u(0, t) = \mu_1(t), \quad u(l, t) = \mu_2(t), \quad 0 \leq t \leq 1, \\ u(x, 0) = u_0(x), \quad 0 \leq x \leq l, \end{cases} \quad (36)$$

where

$$\mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u(x, t) = \int_0^1 d\gamma \sum_{r=1}^5 \omega_r(x, \gamma) \partial_{0t}^{\theta_r(x, \gamma)} u(x, t),$$

$$\begin{aligned} \theta_r(x, \gamma) &= \frac{1+(rx+1)\gamma-\cos(rx\gamma)}{r+4}, \quad \theta_{\max} \approx 0.856, \quad \omega_r(x, \gamma) = (rx+1+rx \sin(rx\gamma)) \frac{\Gamma(3-\theta_r(x, \gamma))}{2r+8}, \\ k(x, t) &= \frac{8+\sin(t)}{3x^2+1}, \quad q(x, t) = 1-\sin(xt), \quad f(x, t) = \sum_{r=1}^5 (t^{2-\theta_r(x, 0)} - t^{2-\theta_r(x, 1)}) \frac{x^3+x+1}{\ln t} + \\ &(x^3 + x + 1)(t^2 + 1)(1 - \sin(xt)), \quad \mu_1(t) = t^2 + 1, \quad \mu_2(t) = 3(t^2 + 1), \\ u_0(x) &= x^3 + x + 1. \end{aligned}$$

The exact solution is  $u(x, t) = (x^3 + x + 1)(t^2 + 1)$ .

A comparison of the numerical solution and exact solution is provided in **Table 1**.

**Table 2** shows that when we take a fixed value  $h = 0.001$ , then as the number of time of our approximate scheme is decreased, a redaction in the maximum error is observed, as expected and the convergence order of time is  $O(\tau^{2-\theta_{\max}}) \approx O(\tau^{1.144})$ , where the convergence order is calculated by the following formula: Convergence order =  $\log_{\frac{\tau_1}{\tau_2}} \frac{e_1}{e_2}$ .

**Table 3** shows that when we take  $h^2 = \tau^{2-\theta_{\max}} \approx \tau^{1.144}$ , as the number as spatial subintervals/time steps is decreased, a reduction in the maximum error is observed, es expected the convergence order of the approximate scheme is  $O(h^2)$ , where the convergence order is calculated by the following formula: Convergence order =  $\log_{\frac{h_1}{h_2}} \frac{e_1}{e_2}$ .

**Table 1**The error, numerical solution and exact solution, when  $t = 0.99$ ,  $h = 0.1$ ,  $\tau = 0.01$ .

Space( $x_i$ )	Numerical solution	Exact solution	Error
0.0000	1.9801000	1.9801000	0.0000000
0.1000	2.1798574	2.1800901	0.0002337
0.2000	2.3915106	2.3919608	0.0004502
0.3000	2.6269549	2.6275927	0.0006378
0.4000	2.8980853	2.8988664	0.0007811
0.5000	3.2167962	3.2176625	0.0008663
0.6000	3.5949810	3.5958616	0.0008806
0.7000	4.0445325	4.0453443	0.0008118
0.8000	4.5773422	4.5779912	0.0006490
0.9000	5.2053012	5.2056829	0.0003817
1.0000	5.9403000	5.9403000	0.0000000

**Table 2**Maximum error behavior versus time grid size reduction at  $t = 0.99$  when  $h = 0.001$ .

$\tau$	Maximum error	Convergence order
0.99/10	0.0006796	
0.99/20	0.0002474	1.458
0.99/40	0.0000907	1.448

**Table 3**Maximum error behavior versus grid size reduction at  $t = 0.99$  when  $h^2 = \tau^{1.144}$ .

$h$	Maximum error	Convergence order
1/10	0.0008536	
1/20	0.0002178	1.970
1/40	0.0000550	1.985
1/80	0.0000138	1.995

### 3.3. The Robin boundary value problem.

To the differential problem (1), (3), (14) we assign the following difference scheme:

$$\mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i = \Lambda y_i^{j+1} + \varphi^{j+1}, \quad i = 0, \dots, N, j = 0, 1, \dots, j_0 - 1, \quad (37)$$

$$y(x, 0) = u_0(x), \quad i = 0, \dots, N, \quad (38)$$

where  $\Lambda y = (a_1 y_x - \tilde{\beta}_1 y)/(0.5h)$ ,  $i = 0$ ,  $\Lambda y = (a y_{\bar{x}})_x - dy$ ,  $i = 1, \dots, N-1$ ,  $\Lambda y = (-a_N y_{\bar{x}} - \tilde{\beta}_2 y)/(0.5h)$ ,  $i = N$ ,  $\varphi_0 = (2\tilde{\mu}_1)/h$ ,  $\varphi_N = (2\tilde{\mu}_2)/h$ ,  $\tilde{\beta}_1 = \beta_1 + 0.5hd_0$ ,  $\tilde{\beta}_2 = \beta_2 + 0.5hd_N$ ,  $\tilde{\mu}_1 = \mu_1 + 0.5hf_0$ ,  $\tilde{\mu}_2 = \mu_2 + 0.5hf_N$ . The difference scheme (37)–(38) has the order of approximation  $O(\tau^{2-\theta_{\max}} + h^2)$ .

**Theorem 4.** The difference scheme (37)–(38) is absolutely stable and its solution satisfies the following a priori estimate:

$$\begin{aligned}
& \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \left[ \frac{\omega_r(x_i, \gamma)}{\Gamma(2 - \theta_r(x_i, \gamma))}, \sum_{s=0}^j (t_{j-s+1}^{1-\theta_r(x_i, \gamma)} - t_{j-s}^{1-\theta_r(x_i, \gamma)}) (y_i^{s+1})^2 \right] + \\
& \quad + \gamma_1 \sum_{s=0}^j (\| (y_{\bar{x}}^{s+1}) \|_0^2 + (y_0^{s+1})^2 + (y_N^{s+1})^2) \tau \leq \\
& \leq \frac{\delta_1}{\gamma_1} \sum_{s=0}^j ((\tilde{\mu}_1^{s+1})^2 + (\tilde{\mu}_2^{s+1})^2 + \|\varphi^{s+1}\|_0^2) \tau + \\
& \quad + \int_{\alpha}^{\beta} d\gamma \sum_{r=1}^m \left[ \frac{\omega_r(x_i, \gamma) t_{j+1}^{1-\theta_r(x_i, \gamma)}}{\Gamma(2 - \theta_r(x_i, \gamma))}, u_0^2(x_i) \right], \tag{39}
\end{aligned}$$

where  $\gamma_1 = \min\{c_1, \beta_0\}$ ,  $\delta_1 = \max\{1 + l, l^2\}$ ,  $[y, v] = \sum_{i=1}^{N-1} y_i v_i h + 0.5y_0 v_0 h + 0.5y_N v_N h$ ,  $\| [y] \|_0^2 = [y, y]$ .

**Proof.** Let us multiply scalarly equation (37) by  $y^{j+1}$ :

$$\left[ y^{j+1}, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i \right] - [\Lambda y^{j+1}, y^{j+1}] = [\varphi^{j+1}, y^{j+1}], \tag{40}$$

Let us transform the terms occurring in identity (40) as

$$\begin{aligned}
& \left[ y^{j+1}, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i \right] \geq \frac{1}{2} \left[ 1, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i^2 \right], \\
& -[\Lambda y^{j+1}, y^{j+1}] = \tilde{\beta}_1 (y_0^{j+1})^2 + \tilde{\beta}_2 (y_N^{j+1})^2 + (a, (y_{\bar{x}}^{j+1})^2) + [d, (y^{j+1})^2], \\
& \| [\varphi, y^{j+1}] \| \leq \varepsilon \| y^{j+1} \|_0^2 + \tilde{\mu}_1 y_0^{j+1} + \tilde{\mu}_2 y_N^{j+1} + \frac{1}{4\varepsilon} \|\varphi\|_0^2, \quad \varepsilon > 0.
\end{aligned}$$

Taking into account the above performed transformations, from identity (40) one arrives at the inequality

$$\begin{aligned} & \frac{1}{2} \left[ 1, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i^2 \right] + c_1 \|y_{\bar{x}}^{j+1}\|_0^2 + \beta_0 ((y_0^{j+1})^2 + (y_N^{j+1})^2) \leq \\ & \leq \varepsilon (\|y^{(\sigma)}\|_0^2 + (y_0^{j+1})^2 + (y_N^{j+1})^2) + \frac{1}{4\varepsilon} (\tilde{\mu}_1^2 + \tilde{\mu}_2^2 + \|\varphi\|_0^2). \end{aligned} \quad (41)$$

From (41) at  $\varepsilon = \gamma_1/(2\delta_1)$ , using that  $\|y\|_0^2 \leq l^2 \|y_{\bar{x}}\|_0^2 + l(y_0^2 + y_N^2)$ , one has the following inequality:

$$\begin{aligned} & \left[ 1, \mathbb{P}_{(\omega)}^{(\theta)} (\Delta_{0t_{j+1}}) y_i^2 \right] + \gamma_1 (\|y_{\bar{x}}^{j+1}\|_0^2 + (y_0^{j+1})^2 + (y_N^{j+1})^2) \leq \\ & \leq \frac{\delta_1}{\gamma_1} (\tilde{\mu}_1^2 + \tilde{\mu}_2^2 + \|\varphi\|_0^2). \end{aligned} \quad (42)$$

Multiplying inequality (42) by  $\tau$  and summing over  $s$  from 0 to  $j$ , one obtains a priori estimate (39). The stability and convergence of the difference scheme (37)–(38) follow from the a priori estimate (39).

#### 3.4. Numerical results

In this section, the following multi-term variable-distributed order diffusion equation is considered:

$$\begin{cases} \mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u(x, t) = \frac{\partial}{\partial x} (k(x, t) \frac{\partial u}{\partial x}) - q(x, t)u + f(x, t), \\ k(0, t)u_x(0, t) = \beta_1(t)u(0, t) - \mu_1(t), \\ -k(1, t)u_x(1, t) = \beta_2(t)u(1, t) - \mu_2(t), \quad 0 \leq t \leq 1, \\ u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \end{cases} \quad (43)$$

where

$$\mathbb{P}_{(\omega)}^{(\theta)} (\partial_{0t}) u(x, t) = \int_{-2}^3 d\gamma \sum_{r=1}^9 \omega_r(x, \gamma) \partial_{0t}^{\theta_r(x, \gamma)} u(x, t),$$

$$\begin{aligned} \theta_r(x, \gamma) &= \frac{3+\gamma+e^{x(\gamma-3)}}{rx+14}, \quad \theta_{\max} = 0.5, \quad \omega_r(x, \gamma) = \frac{(1+xe^{x(\gamma-3)})\Gamma(4-\theta_r(x, z))}{6rx+84}, \quad k(x, t) = \\ &= \frac{10+\cos(2t)}{5x^4+1}, \quad q(x, t) = 1 - \cos(2xt), \quad f(x, t) = \sum_{r=1}^9 (t^{2-\theta_r(x, -2)} - t^{2-\theta_r(x, 3)}) \frac{x^5+x+1}{\ln t} + \\ & (x^5+x+1)(t^3+1)(1 - \cos(2xt)), \quad \beta_1(t) = 5 + \cos(2t), \quad \beta_2(t) = 1 - \cos(2t)/3, \\ \mu_1(t) &= -5(t^2+1), \quad \mu_2(t) = 13(t^3+1), \quad u_0(x) = x^5+x+1. \end{aligned}$$

The exact solution is  $u(x, t) = (x^5 + x + 1)(t^3 + 1)$ .

A comparison of the numerical solution and exact solution is provided in **Table 4**.

**Table 5** shows that when we take a fixed value  $h = 0.01$ , then as the number of time of our approximate scheme is decreased, a reduction in the maximum error is observed, as expected and the convergence order of time is  $O(\tau^{2-\theta_{\max}}) = O(\tau^{1.5})$ , where the convergence order is calculated by the following formula: Convergence order =  $\log_{\frac{\tau_1}{\tau_2}} \frac{e_1}{e_2}$ .

**Table 6** shows that when we take  $h^2 = \tau^{1.5}$ , as the number as spatial subintervals/time steps is decreased, a reduction in the maximum error is observed, es expected the convergence order of the approximate scheme is  $O(h^2 + \tau^{1.5}) = O(h^2)$ , where the convergence order is calculated by the following formula: Convergence order =  $\log_{\frac{h_1}{h_2}} \frac{e_1}{e_2}$ .

**Table 4**

The error, numerical solution and exact solution, when  $t = 0.99$ ,  $h = 0.1$ ,  $\tau = 0.045$ .

Space( $x_i$ )	Numerical solution	Exact solution	Error
0.0000	1.9743424	1.9702990	0.0040434
0.1000	2.1715724	2.1673486	0.0042238
0.2000	2.3692967	2.3649893	0.0043074
0.3000	2.5703757	2.5661765	0.0041992
0.4000	2.7824031	2.7785945	0.0038086
0.5000	3.0200717	3.0170203	0.0030514
0.6000	3.3075397	3.3056889	0.0018508
0.7000	3.6807935	3.6806565	0.0001370
0.8000	4.1900054	4.1921658	0.0021604
0.9000	4.9018802	4.9070100	0.0051298
1.0000	5.9019826	5.9108970	0.0089144

**Table 5**

Maximum error behavior versus time grid size reduction at  $t = 0.99$  when  $h = 0.002$ .

$\tau$	Maximum error	Convergence order
0.99/10	0.0051283	
0.99/20	0.0017203	1.576
0.99/40	0.0005658	1.604
0.99/60	0.0001834	1.625

**Table 6**Maximum error behavior versus grid size reduction at  $t = 0.99$  when  $h^2 = \tau^{1.5}$ .

$h$	Maximum error	Convergence order
1/10	0.0089152	
1/20	0.0022597	1.980
1/40	0.0005727	1.980

#### 4. Conclusion

Solutions of the Dirichlet and Robin boundary value problems for the multi-term variable-distributed order diffusion equation are studied. A priori estimates for the corresponding differential and difference problems are obtained by using the method of the energy inequalities. The stability and convergence of the difference schemes follow from these a priori estimates. The credibility of the obtained results is verified by performing numerical calculations for test problems.

The method of the energy inequalities proposed in the present paper can be used to find a priori estimates for solutions of a wide class of boundary value problems for the multi-term variable-distributed order diffusion equation to which the maximum principle is not applicable (for example, problems considered in [41, 42, 43, 44]).

It should be emphasized that 1) from the considered equation at  $m = 1$ ,  $\theta_1(x, \gamma) = \gamma$ ,  $\omega_1(x, \gamma) = \omega(\gamma)$  one obtains the distributed order time-fractional diffusion equation, 2) setting  $\theta_r(x, \gamma) = \theta_r = const$ ,  $\int_{\alpha}^{\beta} \omega_r(x, \gamma) d\gamma = \lambda_r(x)$  yields the multi-term time-fractional diffusion equation.

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