Calculation of the Number of all Pairs of Disjoint S-permutation Matrices

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Abstract: *The concept of S-permutation matrix is considered. A general formula for counting all disjoint pairs of $n^2 \times n^2$ S-permutation matrices as a function of the positive integer n is formulated and proven in this paper. To do that, the graph theory techniques have been used. It has been shown that to count the number of disjoint pairs of $n^2 \times n^2$ S-permutation matrices, it is sufficient to obtain some numerical characteristics of all $n \times n$ bipartite graphs.*

Keyword: Binary matrix; S-permutation matrix; Sudoku matrix; Disjoint matrices; Bipartite graph

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1 Introduction and notation

The text of the present paper is essentially extended and modified to the previous articles of the author [8, 9], while we have corrected all inaccuracies and calculation errors.

Let n be a positive integer. By $[n]$ we denote the set $[n] = \{1, 2, \ldots, n\}$.

A *binary* (or *boolean*, or (0,1)-*matrix*) is a matrix all of whose elements belong to the set $\mathfrak{B} = \{0, 1\}$. With $\mathfrak{B}_n$ we will denote the set of all $n \times n$ binary matrices.

Let n be a positive integer and let $A \in \mathfrak{B}_{n^2}$ be a $n^2 \times n^2$ binary matrix. With the help of $n-1$ horizontal lines and $n-1$ vertical lines A has been separated into n^2 of number non-intersecting $n \times n$ square sub-matrices A_{kl}, $1 \le k, l \le n$, e.i.

$$A = \left[\begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{array} \right]. \qquad (1)$$

The sub-matrices A_{kl}, $1 \le k, l \le n$ will be called *blocks*.

A matrix $A \in \mathfrak{B}_{n^2}$ is called an *S-permutation* if in each row, each column, and each block of A there is exactly one 1. Let the set of all $n^2 \times n^2$ S-permutation matrices be denoted by Σ_{n^2}.

As it is proved in [2] the cardinality of the set of all S-permutation matrices is equal to

$$|\Sigma_{n^2}| = (n!)^{2n} . \tag{2}$$

Two binary matrices $A = (a_{ij}) \in \mathfrak{B}_m$ and $B = (b_{ij}) \in \mathfrak{B}_m$ will be called *disjoint* if there are not elements with one and the same indices a_{ij} and b_{ij} such that $a_{ij} = b_{ij} = 1$, i.e. if $a_{ij} = 1$ then $b_{ij} = 0$ and if $b_{ij} = 1$ then $a_{ij} = 0$, $1 \le i, j \le m$.

The concept of S-permutation matrix was introduced by Geir Dahl [2] in relation to the popular Sudoku puzzle. Sudoku is a very popular game. On the other hand, it is well known that Sudoku matrices are special cases of Latin squares in the class of gerechte designs [1].

Obviously a square $n^2 \times n^2$ matrix M with elements of $[n^2] = \{1, 2, \dots, n^2\}$ is a Sudoku matrix if and only if there are matrices $A_1, A_2, \dots, A_{n^2} \in \Sigma_{n^2}$, each two of them are disjoint and such that P can be given in the following way:

$$M = 1 \cdot A_1 + 2 \cdot A_2 + \cdots + n^2 \cdot A_{n^2} \tag{3}$$

Some algorithms for obtaining random Sudoku matrices and their valuation are described in detail in [7].

In [5] Roberto Fontana offers an algorithm which randomly gets a family of $n^2 \times n^2$ mutually disjoint S-permutation matrices, where $n = 2, 3$. In $n = 3$ he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then using (3) he obtained $9! \cdot 105 = 38\ 102\ 400$ Sudoku matrices.

But it is known [4] that the total number of 9×9 Sudoku matrices is

$$9! \cdot 72^2 \cdot 2^7 \cdot 27\ 704\ 267\ 971 = 6\ 670\ 903\ 752\ 021\ 072\ 936\ 960$$

Thus, in relation with Fontana's algorithm, it looks useful to calculate the probability of two randomly generated S-permutation matrices to be disjoint. So the question of enumerating all disjoint pairs of S-permutation matrices naturally arises. This work is devoted to this task.

Bipartite graph is the ordered triplet

$$g = \langle R_g \cup C_g, E_g \rangle,$$

where R_g and C_g are non-empty sets such that $R_g \cap C_g = \emptyset$, the elements of which will be called *vertices*. $E_g \subseteq R_g \times C_g = \{\langle r, c\rangle \mid r \in R_g, c \in C_g\}$ - the set of *edges*. Multiple edges are not allowed in our considerations.

For more details on graph theory see [3, 6].

Let n and k be integers such that $0 \le k \le n^2$. Let us denote with $\mathfrak{G}_{n,k}$ the set of all bipartite graphs without multiple edges of the type $g = \langle R_g, C_g, E_g \rangle$, such that $|R_g| = |C_g| = n$ and $|E_g| = k$. With $\mathfrak{G}_n = \displaystyle\bigcup_{k=0}^{n^2} \mathfrak{G}_{n,k}$ we denote the set of all bipartite graphs without multiple edges of the type $g = \langle R_g, C_g, E_g \rangle$, such that $|R_g| = |C_g| = n$ and irrespective of the number of the edges.

We will not take into consideration the nature of the vertices of the bipartite graphs, i.e. we will consider that for each of the examined bipartite graphs $g \in \mathfrak{G}_{n,k}$, the vertices are respectively

$$R_g = \{r_1, r_2, \ldots, r_n\}$$

and

$$C_g = \{c_1, c_2, \ldots, c_n\}.$$

Let $g' = \langle R_{g'} \cup C_{g'}, E_{g'} \rangle$ and $g'' = \langle R_{g''} \cup C_{g''}, E_{g''} \rangle$ be two bipartite graphs. We will say that g' and g'' are *isomorphic* and we will write $g' \cong g''$, if $|R_{g'}| = |R_{g''}|$, $|C_{g'}| = |C_{g''}|$ and there exist bijections $\rho : R_{g'} \to R_{g''}$ and $\sigma : C_{g'} \to C_{g''}$, such that $\langle r, c \rangle \in E_{g'} \Longleftrightarrow \langle \rho(r), \sigma(c) \rangle \in E_{g''}$.

Obviously the so entered relation $\cong$ is an equivalence relation and with $\overline{\mathfrak{G}}_{n,k}$ we will denote the factor set (the set of the equivalence classes)

$$\overline{\mathfrak{G}}_{n,k} = \mathfrak{G}_{n,k_{/\cong}}$$

Let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{n,k}$ for some natural numbers n and k. Then with $\overline{g}$ we will denote the set

$$\overline{g} = \{h \in \mathfrak{G}_{n,k} \mid h \cong g\} \in \overline{\mathfrak{G}}_{n,k}$$

of all isomorphic to g bipartite graphs, and with $|\overline{g}|$ their number. i.e. the cardinality of the set $\overline{g}$.

Let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{n,k}$ for some natural numbers n and k and let $v \in V_g = R_g \cup C_g$. With $\gamma(v)$ we will denote the set of all vertices from V_g, adjacent to v, i.e. $u \in \gamma(v)$ if and only if there exists an edge in E_g which joins u and v. If v is an isolated vertex (i.e. there is no edge incident to v), then by definition $\gamma(v) = \emptyset$ and $|\gamma(v)| = 0$. Obviously if $v \in R_g$, then $\gamma(v) \subseteq C_g$, and if $v \in C_g$, then $\gamma(v) \subseteq R_g$.

Obviously

$$\sum_{v \in V_g} |\gamma(v)| = 2n.$$

Let n and k be positive integers and let $g \in \mathfrak{G}_{n,k}$. We examine the ordered $(n+1)$-tuple

$$\langle \psi \rangle(g) = \langle \psi_0(g), \psi_1(g), \ldots, \psi_n(g) \rangle,$$

where $\psi_i(g)$, $i = 0, 1, \ldots, n$ is equal to the number of the vertices of g, incident to exactly i number of edges. It is easy to see that for each $g \in \mathfrak{G}_{n,k}$ the equalities

$$\sum_{i=1}^{n} \psi_i(g) = 2n, \quad \sum_{i=1}^{n} i\psi_i(g) = 2k.$$

have been executed.

For two bipartite graphs g and h, if $g \cong h$, then obviously

$$\langle\psi\rangle(g) = \langle\psi\rangle(h) = \langle\psi\rangle(\overline{g}) = \langle\psi\rangle(\overline{h}).$$

In the set $\mathfrak{B}_n$ we enter equivalence relation " $\sim$ ", such that if $A, B \in \mathfrak{B}_n$ then $A \sim B$, if B is obtained from A after dislocation of some of the rows and/or columns of A.

Let $g = \langle R_g, C_g, E_g\rangle \in \mathfrak{G}_n$, where $R_g = \{r_1, r_2, \ldots, r_n\}$ and $C_g = \{c_1, c_2, \ldots, c_n\}$. Then we build the matrix $A = [a_{ij}] \in \mathfrak{B}_n$, such that $a_{ij} = 1$ if and only if $\langle r_i, c_j\rangle \in E_g$.

Inversely, let $A = [a_{ij}] \in \mathfrak{B}_n$. We denote the i-th row of A with r_i, while the j-th column of A with c_j. Then we build the bipartite graph $g = \langle R_g, C_g, E_g\rangle \in \mathfrak{G}_n$, where $R_g = \{r_1, r_2, \ldots, r_n\}$, $C_g = \{c_1, c_2, \ldots, c_n\}$ and there exists an edge from the vertex r_i to the vertex c_j if and only if $a_{ij} = 1$.

Thus we showed the following obvious relation between the bipartite graphs and the binary matrices:

Proposition 1 *There exists one-to-one mapping*

$$\varphi : \mathfrak{G}_n \to \mathfrak{B}_n$$

between the elements of $\mathfrak{G}_n$ and $\mathfrak{B}_n$, such that if $g, h \in \mathfrak{G}_n$, then

$$g \cong h \Longleftrightarrow \varphi(g) \sim \varphi(h).$$

2 A representation of S-permutation matrices

If $z_1\ z_2\ \ldots\ z_n$ is a permutation of the elements of the set $[n] = \{1, 2, \ldots, n\}$ and we shortly denote ρ this permutation, then in this case we denote by $\rho(i)$ the i-th element of this permutation, i.e. $\rho(i) = z_i$, $i = 1, 2, \ldots, n$.

Let Π_n denotes the set of all $n \times n$ matrices, constructed such that $\pi \in \Pi_n$ if and only if the following three conditions are true:

i) the elements of π are ordered pairs of numbers $\langle i, j\rangle$, where $1 \le i, j \le n$;

ii) if

$$[\langle a_1, b_1\rangle \quad \langle a_2, b_2\rangle \quad \cdots \quad \langle a_n, b_n\rangle]$$

is the i-th row of π for any $i \in [n] = \{1, 2, \ldots, n\}$, then $a_1\ a_2\ \ldots\ a_n$ in this order is a permutation of the elements of the set $[n]$

iii) if

$$\begin{bmatrix} \langle a_1, b_1\rangle \\ \langle a_2, b_2\rangle \\ \vdots \\ \langle a_n, b_n\rangle \end{bmatrix}$$

is the j-th column of π for any $j \in [n]$, then $b_1, b_2, \ldots, b_n$ in this order is a permutation of the elements of the set $[n]$.

From the definition it follows that every row and every column of any matrix of the set Π_n can be identified with permutation of elements of the set $[n]$. Conversely for every $(2n)$-tuple $\langle\langle\rho_1,\rho_2,\ldots,\rho_n\rangle,\langle\sigma_1,\sigma_2,\ldots,\sigma_n\rangle\rangle$, where $\rho_i = \rho_i(1)\ \rho_i(2)\ \ldots\ \rho_i(n)$, $\sigma_j = \sigma_j(1)\ \sigma_j(2)\ \ldots\ \sigma_j(n)$, $1 \le i,j \le n$ are permutations of elements of $[n]$, then the matrix

$$\pi = \begin{bmatrix} \langle\rho_1(1),\sigma_1(1)\rangle & \langle\rho_1(2),\sigma_2(1)\rangle & \cdots & \langle\rho_1(n),\sigma_n(1)\rangle \\ \langle\rho_2(1),\sigma_1(2)\rangle & \langle\rho_2(2),\sigma_2(2)\rangle & \cdots & \langle\rho_2(n),\sigma_n(2)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\rho_n(1),\sigma_1(n)\rangle & \langle\rho_n(2),\sigma_2(n)\rangle & \cdots & \langle\rho_n(n),\sigma_n(n)\rangle \end{bmatrix}$$

is matrix of Π_n. Hence

$$|\Pi_n| = (n!)^{2n} \tag{4}$$

Theorem 1 *Let n be a positive integer, $n \ge 2$. Then there is one to one correspondence between the sets Σ_{n^2} and Π_n.*

Proof. Let $A \in \Sigma_{n^2}$. Then A is constructed with the help of formula (1) and for every $i,j \in [n]$ in the block A_{ij} there is only one 1 and let this 1 has coordinates (a_i,b_j). For every $i,j \in [n]$ we obtain ordered pairs of numbers $\langle a_i,b_j\rangle$ corresponding to these coordinates. As in every row and every column of A there is only one 1, then the matrix $\left[\alpha_{ij}\right]_{n\times n}$, where $\alpha_{ij} = \langle a_i,b_j\rangle$, $1 \le i,j \le n$, which is obtained by the ordered pairs of numbers is matrix of Π_n, i.e. matrix for which the conditions i), ii) and iii) are true.

Conversely, let $\left[\alpha_{ij}\right]_{n\times n} \in \Pi_n$, where $\alpha_{ij} = \langle a_i,b_j\rangle$, $i,j \in [n]$, $a_i,b_j \in [n]$. Then for every $i,j \in [n]$ we construct binary $n \times n$ matrices A_{ij} with only one 1 with coordinates (a_i,b_j). Then we obtain the matrix of type: (1). According to the properties i), ii) and iii), it is obvious that the obtained matrix is S-permutation matrix.

$\square$

From Theorem 1 and formula (4) it follows the proof of Proposition 3 in [2], i.e. formula (2).

Definition 1 *We say that matrices $\pi',\pi'' \in \Pi_n$, where $\pi' = \left[{p'}_{ij}\right]_{n\times n}$, $\pi'' = \left[{p''}_{ij}\right]_{n\times n}$ are* disjoint*, if ${p'}_{ij} \ne {p''}_{ij}$ for every pair of indices $i,j \in [n]$.*

Proposition 2 *The number of all pairs of disjoint matrices of Σ_{n^2} is equal to the number of all pairs of disjoint matrices of Π_n.*

Proof. It is easy to see that with respect of the described in Theorem 1 one to one correspondence, every pair of disjoint matrices of Σ_{n^2} will correspond to a pair of disjoint matrices of Π_n and conversely every pair of disjoint matrices of Π_n will correspond to a pair of disjoint matrices of Σ_{n^2}.

$\square$

Definition 2 *Let $\pi',\pi'' \in \Pi_n$, $\pi' = \left[{p'}_{ij}\right]_{n\times n}$, $\pi'' = \left[{p''}_{ij}\right]_{n\times n}$ and let the integers $i,j \in [n]$ are such that ${p'}_{ij} = {p''}_{ij}$. In this case we will say that ${p'}_{ij}$ and ${p''}_{ij}$ are* component-wise equal elements*.*

Obviously two Π_n-matrices are disjoint if and only if they do not have component-wise equal elements.

Example 1 We consider the following Π_3-matrices:

$$\pi' = \left[p'_{ij}\right] = \begin{bmatrix} \langle 1,2\rangle & \langle 3,1\rangle & \langle 2,3\rangle \\ \langle 2,1\rangle & \langle 3,3\rangle & \langle 1,2\rangle \\ \langle 3,3\rangle & \langle 1,2\rangle & \langle 2,1\rangle \end{bmatrix}$$

$$\pi'' = \left[p''_{ij}\right] = \begin{bmatrix} \langle 1,3\rangle & \langle 3,2\rangle & \langle 2,1\rangle \\ \langle 3,1\rangle & \langle 1,1\rangle & \langle 2,2\rangle \\ \langle 3,2\rangle & \langle 1,3\rangle & \langle 2,3\rangle \end{bmatrix}$$

$$\pi''' = \left[p'''_{ij}\right] = \begin{bmatrix} \langle 1,2\rangle & \langle 3,3\rangle & \langle 2,1\rangle \\ \langle 2,1\rangle & \langle 3,2\rangle & \langle 1,2\rangle \\ \langle 3,3\rangle & \langle 1,1\rangle & \langle 2,3\rangle \end{bmatrix}$$

Matrices π' and π'' are disjoint, because they do not have component-wise equal elements.

Matrices π'' and π''' are not disjoint, because they have two component-wise equal elements: $p'_{13} = p'''_{13} = \langle 2,1\rangle$ and $p''_{33} = p'''_{33} = \langle 2,3\rangle$.

Matrices π' and π''' are not disjoint, because they have four component-wise equal elements: $p'_{11} = p'''_{11} = \langle 1,2\rangle$, $p'_{21} = p'''_{21} = \langle 2,1\rangle$, $p'_{23} = p'''_{23} = \langle 1,2\rangle$ and $p'_{31} = p'''_{31} = \langle 3,3\rangle$.

3 A formula for counting disjoint pairs of $n^2 \times n^2$ S-permutation matrices

Lemma 1 *Let $\pi \in \Pi_n$. Then the number $q(n,k)$ of all matrices $\pi' \in \Pi_n$, having at least k, $k = 0,1,\dots,n^2$ component-wise equal elements to the matrix π is equal to*

$$q(n,k) = \sum_{\overline{g} \in \overline{\mathfrak{G}}_{n,k}} |\overline{g}| \left(\prod_{i=0}^{n-2} \left[(n-i)!\right]^{\psi_i(\overline{g})} \right) \tag{5}$$

Proof. Let $\pi = [p_{ij}]_{n\times n}$, $\pi' = \left[p'_{ij}\right]_{n\times n} \in \Pi_n$ and let π and π' have exactly k component-wise equal elements. Then we uniquely obtain the binary $n \times n$ matrix $A = [a_{ij}]_{n\times n}$, such that $a_{ij} = 1$ if and only if $p_{ij} = p'_{ij}$, $i,j \in [n]$. Let graph $g \in \mathfrak{G}_{n,k}$ be such that $g = \varphi^{-1}(A)$, where $\varphi : \mathfrak{B}_n \to \mathfrak{G}_n$ is the one-to-one mapping which gave us the grounds to formulate Proposition 1. This graph g identically corresponds to the ordered pair of matrices $\langle \pi, \pi' \rangle \in \Pi \times \Pi$.

Inversely, let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{n,k}$, $V_g = R_g \cup C_g$, $A = [a_{ij}]_{n\times n} = \varphi(g)$ and let $\pi = [p_{ij}]_{n\times n}$ be an arbitrary matrix from Π_n. We search for the number $h(\pi, A)$ of all matrices $\pi' = [p'_{ij}]_{n\times n} \in \Pi_n$, such that $p'_{ij} = p_{ij}$, if $a_{ij} = 1$. (It is assumed that there exist $s,t \in [n]$ such that $a_{st} = 0$ and $p'_{st} = p_{st}$.) Let the i-th row, $i = 1,2,\dots,n$ of π correspond to the permutation ρ_i of the elements of $[n]$

and let the i-th row of the matrix A correspond to the vertex $r_i \in R_g$ of graph . Then there exist $(n - |\gamma(r_i)|)!$ permutations ρ'_i of the elements of $[n]$, such that if $a_{it} = 1$, then $\rho_i(t) = \rho'_i(t)$, $t \in [n]$. Likewise we also prove the respective statement for the columns of π. Therefore

$$h(\pi, A) = \prod_{v \in V_g} (n - |\gamma(v)|)!.$$

From everything said so far it follows that for each $\pi \in \Pi_n$ there exist

$$q(n,k) = \sum_{g \in \mathfrak{G}_{n,k}} \left(\prod_{v \in V_g} (n - |\gamma(v)|)! \right)$$

matrices from Π_n, which have at least k elements that are component-wise equal to the respective elements of π.

But obviously, if $h \in \mathfrak{G}_{n,k}$ is such a bipartite graph that $h \cong g$, then

$$\left(\prod_{v \in V_g} (n - |\gamma(g)|)! \right) = \left(\prod_{u \in V_h} (n - |\gamma(h)|)! \right) = \left(\prod_{i=0}^{n} [(n-i)!]^{\psi_i(\overline{g})} \right),$$

whence follows the equality

$$q(n,k) = \sum_{\overline{g} \in \overline{\mathfrak{G}}_{n,k}} |\overline{g}| \left(\prod_{v \in R_g \cup C_g} (n - |\gamma(v)|)! \right) = \sum_{\overline{g} \in \overline{\mathfrak{G}}_{n,k}} |\overline{g}| \left(\prod_{i=0}^{n} [(n-i)!]^{\psi_i(\overline{g})} \right)$$

And since $(n-n)! = 0! = 1$ and $[n-(n-1)]! = 1! = 1$, then we finally obtain formula (5).

□

Theorem 2 *Let $A \in \Sigma_{n^2}$. Then the number ξ_n of all matrices $B \in \Sigma_{n^2}$ which are disjoint with A is equal to*

$$\xi_n = (n!)^{2n} + \sum_{k=1}^{n^2} (-1)^k \left(\sum_{\overline{g} \in \overline{\mathfrak{G}}_{n,k}} |\overline{g}| \left(\prod_{i=0}^{n-2} [(n-i)!]^{\psi_i(\overline{g})} \right) \right) \tag{6}$$

Proof. Let $n \geq 2$ be an integer. Then applying theorem 1, lemma 1 and the principle of inclusion and exclusion we obtain that the number ξ_n of all matrices $B \in \Sigma_{n^2}$ which are disjoint with A is equal to

$$\xi_n = |\Pi_n| + \sum_{k=1}^{n^2} (-1)^k q(n,k),$$

where the function $q(n,k)$ is calculated with the help of formula (5), while $|\Pi_n|$ with the help of formula (4). Thus we obtain the proof to formula (6).

□

Corollary 1 *The cardinality η_n of the set of all disjoint non-ordered pairs of $n^2 \times n^2$ S-permutation matrices is equal to*

$$\eta_n = \frac{(n!)^{2n}}{2} \xi_n \tag{7}$$

where ξ_n is described using formula 6.

Proof. follows directly from formula (2) and having in mind that the "disjoint" relation is symmetric and antireflexive.

□

Corollary 2 *The probability $p(n)$ of two randomly generated $n^2 \times n^2$ S-permutation matrices to be disjoint is equal to*

$$p(n) = \frac{\xi_n}{(n!)^{2n} - 1}, \tag{8}$$

where ξ_n is described using formula 6.

Proof. Applying Corollary 1 and formula (2), we obtain:

$$p(n) = \frac{\eta_n}{\binom{|\Sigma_{n^2}|}{2}} = \frac{\frac{(n!)^{2n}}{2} \xi_n}{\frac{(n!)^{2n}\left((n!)^{2n} - 1\right)}{2}} = $$

$$= \frac{\xi_n}{(n!)^{2n} - 1}.$$

□

4 Calculation of the Number of the Disjoint Pairs of S-permutation matrices when $n = 2$ and $n = 3$

4.1 Consider $n = 2$

When $n = 2$, $\overline{\mathfrak{G}}_2$ is composed of seven equivalence classes also including the graph without edges ($k = 0$), which does not participate in our calculations. When $k = 1, 2, 3, 4$, we have depicted one representative g_1, g_2, g_3, g_4, g_5 and g_6 from each equivalence class respectively on figures 1, 2, 3 and 4.

It is not difficult to notice that
$|\overline{g}_1| = 4, \quad \langle\psi\rangle(\overline{g}_1) = \langle 2, 2, 0\rangle, \quad k = 1$
$|\overline{g}_2| = 2, \quad \langle\psi\rangle(\overline{g}_2) = \langle 0, 4, 0\rangle, \quad k = 2$
$|\overline{g}_3| = |\overline{g}_4| = 2, \quad \langle\psi\rangle(\overline{g}_3) = \langle\psi\rangle(\overline{g}_4) = \langle 1, 2, 1\rangle, \quad k = 2$

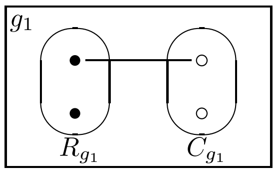

Figure 1: $n = 2$, $k = 1$

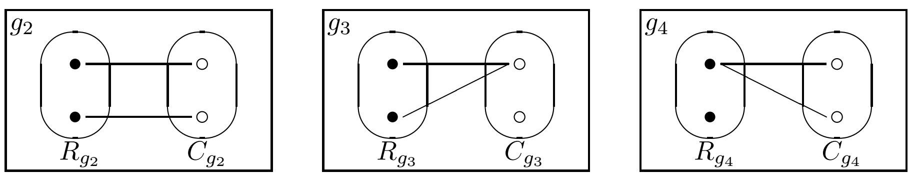

Figure 2: $n = 2$, $k = 2$

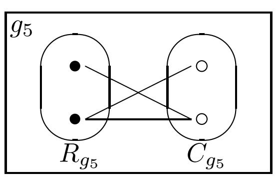

Figure 3: $n = 2$, $k = 3$

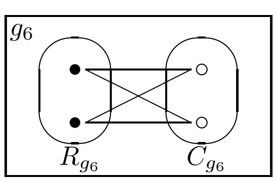

Figure 4: $n = 2$, $k = 4$

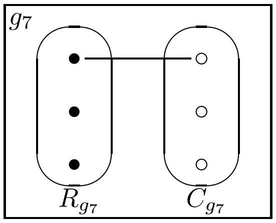

Figure 5: $n = 3$, $k = 1$

$|\overline{g}_5| = 4, \quad \langle\psi\rangle(\overline{g}_5) = \langle 0,2,2\rangle, \quad k = 3$
$|\overline{g}_6| = 1, \quad \langle\psi\rangle(\overline{g}_6) = \langle 0,0,4\rangle, \quad k = 4$

Let $A \in \Sigma_4$. Then the number ξ_2 of all matrices $B \in \Sigma_4$ which are disjoint with A is equal to

$$\xi_2 = (2!)^4 + \sum_{k=1}^{4} (-1)^k \left(\sum_{\overline{g} \in \overline{\mathfrak{G}}_{n,k}} |\overline{g}| \cdot 2^{\psi_0(\overline{g})} \right) =$$

$$= 16 - 4 \cdot 2^2 + \left(2 \cdot 2^0 + 2 \cdot 2^1 + 2 \cdot 2^1\right) - 4 \cdot 2^0 + 1 \cdot 2^0 = 7$$

$$\eta_2 = \frac{2^4}{2}\xi_2 = 56$$

$$p(2) = \frac{\xi_2}{2^4 - 1} = \frac{7}{15}$$

4.2 Consider $n = 3$

On figures from 5 to 13 one representative from each equivalence class of the factor sets $\overline{\mathfrak{G}}_{3,k}$, $k = 1, 2, \ldots, 9$ has been depicted, and we have numbered these graphs from 7 to 41.

When $n = 3$ formula (6) has the form of

$$\xi_3 = 6^6 + \sum_{i=7}^{41} |\overline{g}_i| 6^{\psi_0(\overline{g}_i)} 2^{\psi_1(\overline{g}_i)} (-1)^{\kappa(\overline{g}_i)} \ ,$$

where with $\kappa(\overline{g}_i)$ we have denoted the number of the edges of the graphs from the equivalence class $\overline{g}_i$.

Below we enumerate the examined numerical characteristics of the respective equivalence classes of bipartite graphs. Their calculation is trivial.

$|\overline{g}_7| = \binom{3}{1}^2 = 9, \quad \langle\psi\rangle(\overline{g}_7) = \langle 4,2,0,0\rangle, \quad k = 1$
$|\overline{g}_8| = \binom{3}{1}^2 |\overline{g}_2| = 18, \quad \langle\psi\rangle(\overline{g}_8) = \langle 2,4,0,0\rangle, \quad k = 2$

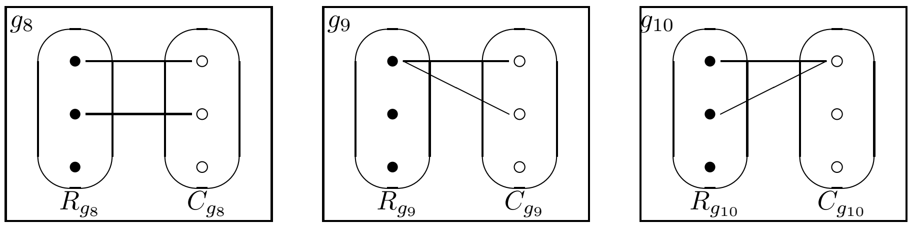

Figure 6: $n = 3$, $k = 2$

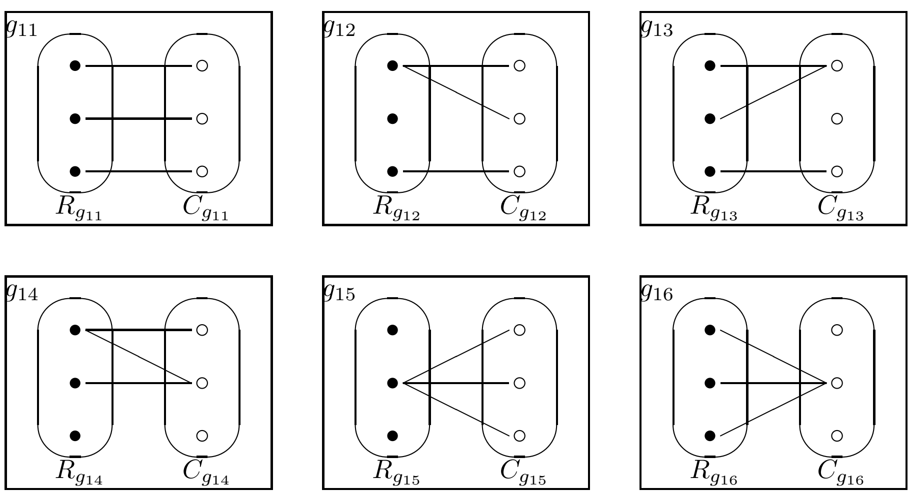

Figure 7: $n = 3$, $k = 3$

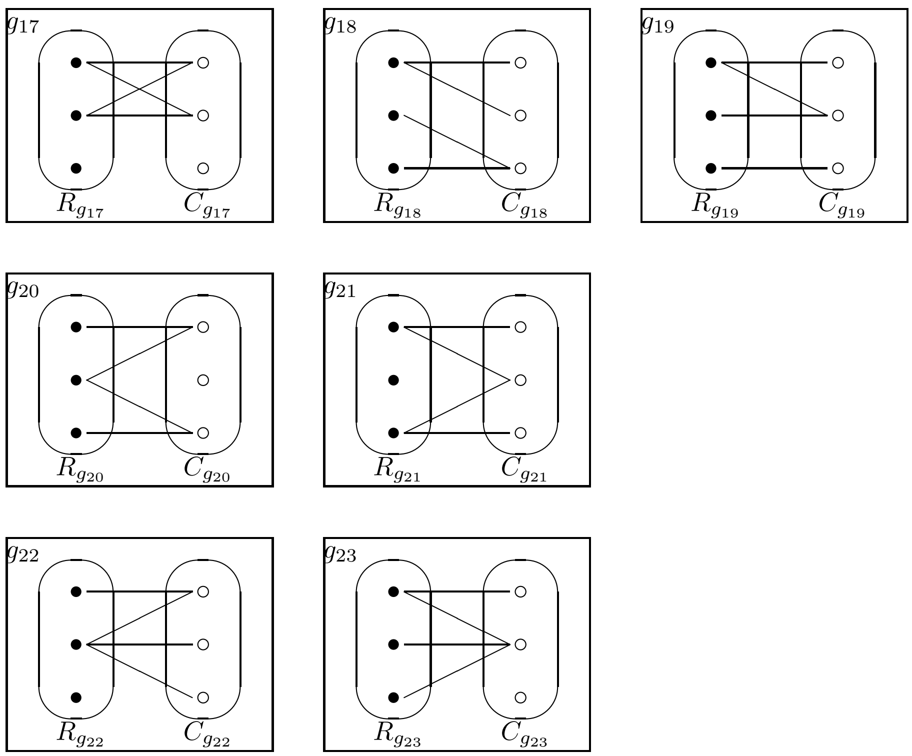

Figure 8: $n = 3$, $k = 4$

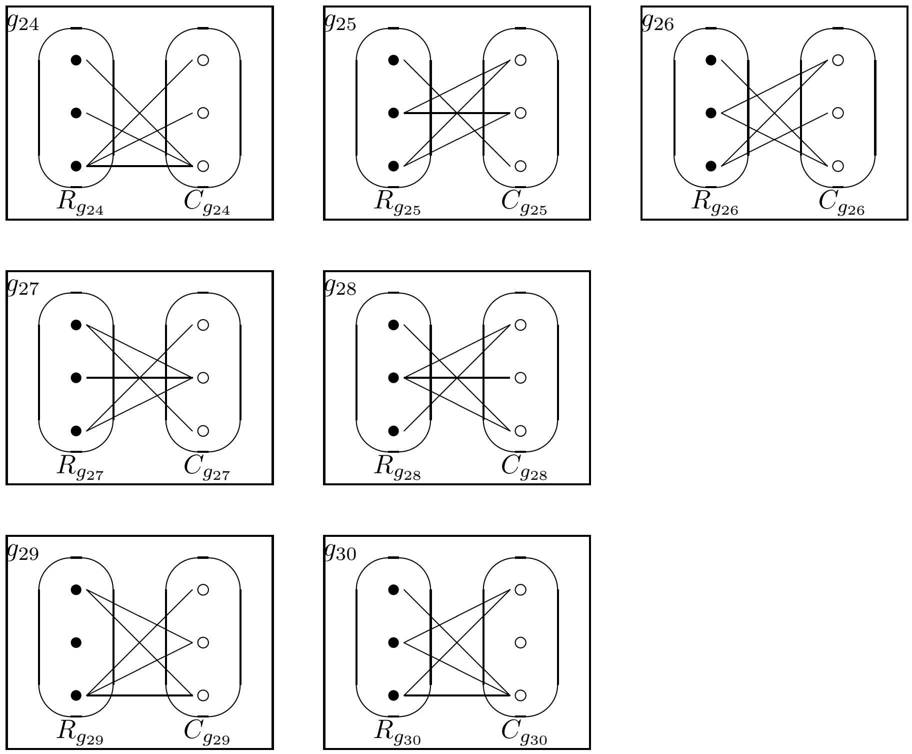

Figure 9: $n = 3$, $k = 5$

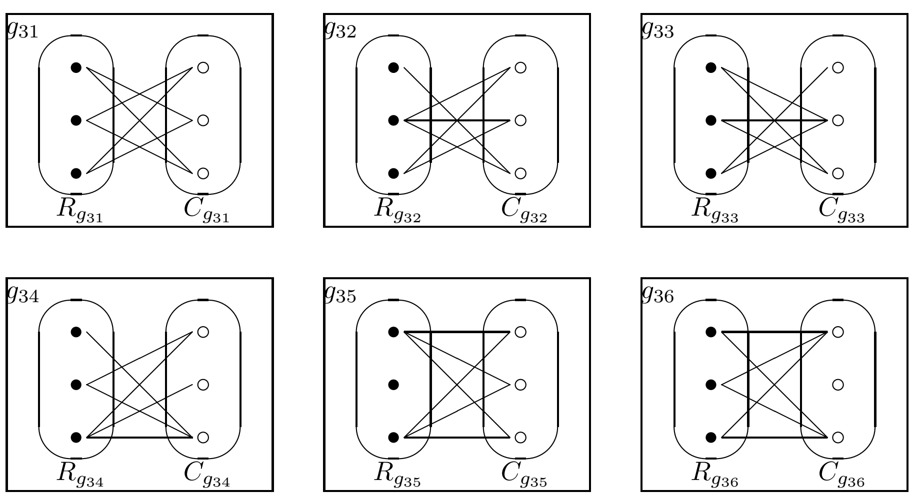

Figure 10: $n = 3$, $k = 6$

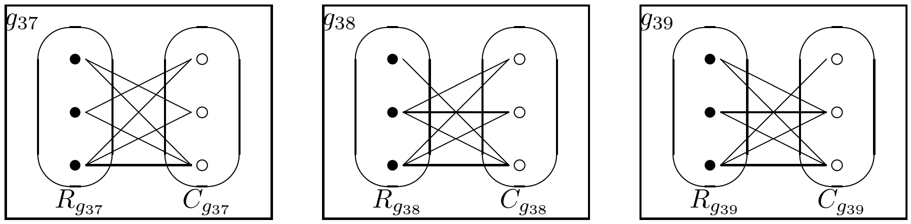

Figure 11: $n = 3$, $k = 7$

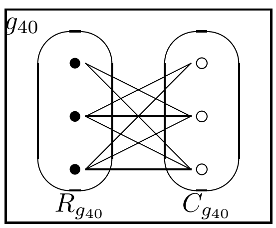

Figure 12: $n = 3$, $k = 8$

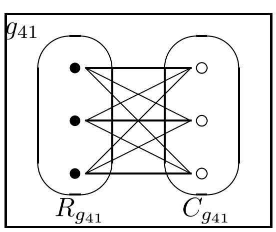

Figure 13: $n = 3$, $k = 9$

$$|\overline{g}_9| = |\overline{g}_{10}| = \binom{3}{1}\binom{3}{2} = 9, \quad \langle\psi\rangle(\overline{g}_9) = \langle\psi\rangle(\overline{g}_{10}) = \langle 3,2,1,0\rangle, \quad k = 2$$
$$|\overline{g}_{11}| = |\mathcal{S}_3| = 3! = 6, \quad \langle\psi\rangle(\overline{g}_{11}) = \langle 0,6,0,0\rangle, \quad k = 3$$
$$|\overline{g}_{12}| = |\overline{g}_{13}| = \binom{3}{1}^2 |\overline{g}_3| = 18, \quad \langle\psi\rangle(\overline{g}_{12}) = \langle\psi\rangle(\overline{g}_{13}) = \langle 1,4,1,0\rangle, \quad k = 3$$
$$|\overline{g}_{14}| = \binom{3}{1}^2 |\overline{g}_5| = 36, \quad \langle\psi\rangle(\overline{g}_{14}) = \langle 2,2,2,0\rangle, \quad k = 3$$
$$|\overline{g}_{15}| = |\overline{g}_{16}| = \binom{3}{1} = 3, \quad \langle\psi\rangle(\overline{g}_{15}) = \langle\psi\rangle(\overline{g}_{16}) = \langle 2,3,0,1\rangle, \quad k = 3$$
$$|\overline{g}_{17}| = \binom{3}{1}^2 = 9, \quad \langle\psi\rangle(\overline{g}_{17}) = \langle 2,0,4,0\rangle, \quad k = 4$$
$$|\overline{g}_{18}| = \binom{3}{1}\binom{3}{2} = 9, \quad \langle\psi\rangle(\overline{g}_{18}) = \langle 0,4,2,0\rangle, \quad k = 4$$
$$|\overline{g}_{19}| = \binom{3}{1}^2 |\overline{g}_5| = 36, \quad \langle\psi\rangle(g_{19}) = \langle 0,4,2,0\rangle, \quad k = 4$$
$$|\overline{g}_{20}| = |\overline{g}_{21}| = \binom{3}{1}^2 |\overline{g}_2| = 18, \quad \langle\psi\rangle(\overline{g}_{20}) = \langle\psi\rangle(\overline{g}_{21}) = \langle 1,2,3,0\rangle, \quad k = 4$$
$$|\overline{g}_{22}| = |\overline{g}_{23}| = \binom{3}{1}^2 \binom{2}{1} = 18, \quad \langle\psi\rangle(\overline{g}_{22}) = \langle\psi\rangle(\overline{g}_{23}) = \langle 1,3,1,1\rangle, \; k = 4$$
$$|\overline{g}_{24}| = \binom{3}{1}^2 = 9, \quad \langle\psi\rangle(\overline{g}_{24}) = \langle 0,4,0,2\rangle, \quad k = 5$$
$$|\overline{g}_{25}| = \binom{3}{1}^2 = 9, \quad \langle\psi\rangle(g_{25}) = \langle 0,2,4,0\rangle, \quad k = 5$$
$$|\overline{g}_{26}| = \binom{3}{1}^2 \binom{2}{1}^2 = 36, \quad \langle\psi\rangle(\overline{g}_{26}) = \langle 0,2,4,0\rangle, \quad k = 5$$
$$|\overline{g}_{27}| = |\overline{g}_{28}| = \binom{3}{1}^2 |\overline{g}_2| = 18, \quad \langle\psi\rangle(g_{27}) = \langle\psi\rangle(g_{28}) = \langle 0,3,2,1\rangle, \quad k = 5$$
$$|\overline{g}_{29}| = |\overline{g}_{30}| = 3!\binom{3}{1} = 18, \quad \langle\psi\rangle(\overline{g}_{29}) = \langle\psi\rangle(\overline{g}_{30}) = \langle 1,1,3,1\rangle, \quad k = 5$$
$$|\overline{g}_{31}| = |\overline{g}_{11}| = 6, \quad \langle\psi\rangle(\overline{g}_{31}) = \langle 0,0,6,0\rangle, \quad k = 6$$
$$|\overline{g}_{32}| = |\overline{g}_{33}| = |\overline{g}_{12}| = 18, \quad \langle\psi\rangle(\overline{g}_{32}) = \langle\psi\rangle(\overline{g}_{33}) = \langle 0,1,4,1\rangle, \quad k = 6$$
$$|\overline{g}_{34}| = |\overline{g}_{14}| = 36, \quad \langle\psi\rangle(\overline{g}_{34}) = \langle 0,2,2,2\rangle, \quad k = 6$$
$$|\overline{g}_{35}| = |\overline{g}_{36}| = |\overline{g}_{15}| = 3, \quad \langle\psi\rangle(g_{35}) = \langle\psi\rangle(g_{36}) = \langle 1,0,3,2\rangle, \quad k = 6$$
$$|\overline{g}_{37}| = |\overline{g}_8| = 18, \quad \langle\psi\rangle(g_{37}) = \langle 0,0,4,2\rangle, \quad k = 7$$
$$|\overline{g}_{38}| = |\overline{g}_{39}| = |\overline{g}_9| = 9, \quad \langle\psi\rangle(g_{38}) = \langle\psi\rangle(g_{39}) = \langle 0,1,2,3\rangle, \quad k = 7$$
$$|\overline{g}_{40}| = \binom{3}{1}^2 = 9, \quad \langle\psi\rangle(g_{40}) = \langle 0,0,2,4\rangle, \quad k = 8$$
$$|\overline{g}_{41}| = 1, \quad \langle\psi\rangle(g_{41}) = \langle 0,0,0,6\rangle, \quad k = 9$$

In order to calculate ξ_3 we used the next computer programme written in programming language C++.

```
int main() {
```

```
double xi_3 =pow(6,6);
int g[][4] = {
9,4,2,1,
18,2,4,2,
9,3,2,2,
9,3,2,2,
6,0,6,3,
18,1,4,3,
18,1,4,3,
36,2,2,3,
3,2,3,3,
3,2,3,3,
9,2,0,4,
9,0,4,4,
36,0,4,4,
18,1,2,4,
18,1,2,4,
18,1,3,4,
18,1,3,4,
9,0,4,5,
9,0,2,5,
36,0,2,5,
18,0,3,5,
18,0,3,5,
18,1,1,5,
18,1,1,5,
6,0,0,6,
18,0,1,6,
18,0,1,6,
36,0,2,6,
3,1,0,6,
3,1,0,6,
18,0,0,7,
9,0,1,7,
9,0,1,7,
9,0,0,8,
1,0,0,9
};
for (int i=0; i<35; i++)
xi_3 += g[i][0] * pow(6,g[i][1]) * pow(2,g[i][2]) * (g[i][3]%2 ? -1 : 1);
double p3;
long eta_3;
eta_3 = (long) (pow(6,6) /2) * xi_3;
p3 =xi_3 / (pow(6,6)-1);
cout<<"xi_3 = "<<xi_3<<endl;
cout<<"eta_3 = "<<eta_3<<endl;
```

```
cout<<"p(3) = "<<p3<<endl;
return 0;
}
```

After its work we obtain that

$$\xi_3 = 17\ 972$$

Then according to formula (7), the number of all disjoint nonordered pairs of 9×9 S-permutation matrices is equal to

$$\eta_3 = \frac{(3!)^6}{2}\xi_3 = 419\ 250\ 816.$$

The probability $p(3)$ for two randomly obtained 9×9 S-permutation matrices to be disjoint, according to formula (8) is equal to

$$p(3) = \frac{\xi_3}{(3!)^6 - 1} = 0.385211$$

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