Sharp bounds for ordinary and signless Laplacian spectral radii of uniform hypergraphs

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Abstract

We give sharp upper bounds for the ordinary spectral radius and signless Laplacian spectral radius of a uniform hypergraph in terms of the average 2-degrees or degrees of vertices, respectively, and we also give a lower bound for the ordinary spectral radius. We also compare these bounds with known ones.

Key words: tensor, eigenvalues of tensors, uniform hypergraph, average 2-degree, adjacency tensor, signless Laplacian tensor

1 Introduction

For positive integers k and n with $k \leq n$, a tensor $\mathcal{T} = (T_{i_1...i_k})$ of order k and dimension n refers to a multidimensional array with complex entries $T_{i_1...i_k}$ for $i_j \in [n] := \{1, ..., n\}$ and $j \in [k]$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2.

Let \mathcal{M} be a tensor of order $s \geq 2$ and dimension n, and \mathcal{N} a tensor of order $k \geq 1$ and dimension n. The product \mathcal{MN} is the tensor of order (s-1)(k-1)+1 and dimension n with entries [10]

$$(\mathcal{M}\mathcal{N})_{ij_1\dots j_{s-1}} = \sum_{i_2,\dots,i_s \in [n]} M_{ii_2\dots i_s} N_{i_2j_1} \cdots N_{i_sj_{s-1}},$$

with $i \in [n]$ and $j_1, ..., j_{s-1} \in [n]^{k-1}$.

For a tensor \mathcal{T} of order $k \geq 2$ and dimension n and a vector $x = (x_1, \ldots, x_n)^{\top}$, $\mathcal{T}x$ is an n-dimensional vector whose i-th entry is

$$(\mathcal{T}x)_i = \sum_{i_2,\dots,i_k \in [n]} T_{ii_2\dots i_k} x_{i_2} \cdots x_{i_k}.$$

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where $i \in [n]$. Let $x^{[r]} = (x_1^r, \ldots, x_n^r)^{\top}$. For some complex ρ , if there is a nonzero *n*-dimensional vector x such that

$$\mathcal{T}x = \rho x^{[k-1]}.$$

then ρ is called an eigenvalue of \mathcal{T} , and x an eigenvector of \mathcal{T} corresponding to ρ , see [7, 8]. Let $\rho(\mathcal{T})$ be the largest modulus of the eigenvalues of \mathcal{T} .

Let \mathcal{G} be a hypergraph with vertex set $V(\mathcal{G}) = [n]$ and edge set $E(\mathcal{G})$, see [1]. If every edge of \mathcal{G} has cardinality k, then we say that \mathcal{G} is a k-uniform hypergraph. Throughout this paper, we consider k-uniform hypergraphs on n vertices with $2 \leq k \leq n$. A uniform hypergraph is a hypergraph that is k-uniform for some k. For $i \in [n]$, E_i denotes the set of edges of \mathcal{G} containing i. The degree of a vertex i in \mathcal{G} is defined as $d_i = |E_i|$. If $d_i = d$ for $i \in V(\mathcal{G})$, then \mathcal{G} is called a regular hypergraph (of degree d). For $i, j \in V(\mathcal{G})$, if there is a sequence of edges e_1, \ldots, e_r such that $i \in e_1, j \in e_r$ and $e_s \cap e_{s+1} \neq \emptyset$ for all $s \in [r-1]$, then we say that i and j are connected. A hypergraph is connected if every pair of different vertices of \mathcal{G} is connected.

The adjacency tensor of a k-uniform hypergraph \mathcal{G} on n vertices is defined as the tensor $\mathcal{A}(\mathcal{G})$ of order k and dimension n whose $(i_1 \dots i_k)$ -entry is

$$A_{i_1\dots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1,\dots,i_k\} \in E(\mathcal{G}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(\mathcal{G})$ be the diagonal tensor of order k and dimension n with its diagonal entry $D_{i...i}$ the degree of vertex i for $i \in [n]$. Then $\mathcal{Q}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) + \mathcal{A}(\mathcal{G})$ is the signless Laplacian tensor of \mathcal{G} . We call $\rho(\mathcal{A}(\mathcal{G}))$ the (ordinary) spectral radius of \mathcal{G} , which is denoted by $\rho(\mathcal{G})$, and $\rho(\mathcal{Q}(\mathcal{G}))$ the signless Laplacian spectral radius of \mathcal{G} , which is denoted by $\mu(\mathcal{G})$.

For a nonnegative tensor \mathcal{T} of order $k \geq 2$ and dimension n, the *i*-th row sum of \mathcal{T} is $r_i(\mathcal{T}) = \sum_{i_2,...,i_k \in [n]} T_{ii_2...i_k}$. If $r_i(\mathcal{T}) > 0$, then the *i*-th average 2-row sum of \mathcal{T} is defined as

$$m_i(\mathcal{T}) = \frac{\sum_{i_2,\dots,i_k \in [n]} T_{ii_2\dots i_k} r_{i_2}(\mathcal{T}) \cdots r_{i_k}(\mathcal{T})}{r_i^{k-1}(\mathcal{T})}.$$

Let \mathcal{G} be a k-uniform hypergraph on n vertices. Let $\mathcal{A} = \mathcal{A}(\mathcal{G})$. For $i \in V(\mathcal{G})$ with $d_i > 0$,

$$m_{i}(\mathcal{A}) = \frac{\sum_{i_{2},...,i_{k} \in [n]} A_{ii_{2}...i_{k}} r_{i_{2}}(\mathcal{A}) \cdots r_{i_{k}}(\mathcal{A})}{r_{i}^{k-1}(\mathcal{A})}$$
$$= \frac{\sum_{\{i,i_{2},...,i_{k}\} \in E_{i}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}},$$

which is called the average 2-degree of vertex i of \mathcal{G} (average of degrees of vertices in E_i) [12].

For a k-uniform hypergraph \mathcal{G} with maximum degree Δ , we know that $\rho(\mathcal{G}) \leq \Delta$ [2] and $\mu(\mathcal{G}) \leq 2\Delta$ [8] with either equality when \mathcal{G} is connected if

and only if \mathcal{G} is regular (see [9]). Recently, upper bounds for $\rho(\mathcal{G})$ and $\mu(\mathcal{G})$ are given in [12] using degree sequence. In this note, we present sharp upper bounds for $\rho(\mathcal{G})$ and $\mu(\mathcal{G})$ using average 2-degrees or degrees, and we also give a lower bound for $\rho(\mathcal{G})$. We compare these bounds with known bounds by examples.

2 Preliminaries

A nonnegative tensor \mathcal{T} of order $k \geq 2$ dimension n is called weakly irreducible if the associated directed graph $D_{\mathcal{T}}$ of \mathcal{T} is strongly connected, where $D_{\mathcal{T}}$ is the directed graph with vertex set $\{1, \ldots, n\}$ and arc set $\{(i, j) : a_{ii_2...i_k} \neq 0$ for some $i_s = j$ with $s = 2, \ldots, k\}$ [3, 8].

For an *n*-dimensional real vector x, let $||x||_k = (\sum_{i=1}^n |x_i|^k)^{\frac{1}{k}}$, and if $||x||_k = 1$, then we say that x is a unit vector. Let \mathbb{R}^n_+ be the set of *n*-dimensional nonnegative vectors.

Lemma 2.1. [3, 11] Let \mathcal{T} be a nonnegative tensor. Then $\rho(\mathcal{T})$ is an eigenvalue of \mathcal{T} and there is a unit nonnegative eigenvector corresponding to $\rho(\mathcal{T})$. If furthermore \mathcal{T} is weakly irreducible, then there is a unique unit positive eigenvector corresponding to $\rho(\mathcal{T})$.

Lemma 2.2. [8] Let \mathcal{G} be a k-uniform hypergraph with n vertices. Then $\rho(\mathcal{G}) = \max\{x^{\top}(\mathcal{A}(\mathcal{G})x) : x \in \mathbb{R}^n_+, ||x||_k = 1\}.$

Lemma 2.3. [6, 8] Let \mathcal{G} be a k-uniform hypergraph. Then $\mathcal{A}(\mathcal{G})$ ($\mathcal{Q}(G)$, respectively) is weakly irreducible if and only if \mathcal{G} is connected.

A hypergraph \mathcal{H} is a subhypergraph of \mathcal{G} if $V(\mathcal{H}) \subseteq V(\mathcal{G})$ and $E(\mathcal{H}) \subseteq E(\mathcal{G})$.

Lemma 2.4. [2, 4] Let \mathcal{G} be a connected k-uniform hypergraph and \mathcal{H} a subhypergraph of \mathcal{G} . Then $\rho(\mathcal{H}) \leq \rho(\mathcal{G})$ with equality if and only if $\mathcal{H} = \mathcal{G}$.

For two tensors \mathcal{M} and \mathcal{N} of order $k \geq 2$ and dimension n, if there is an $n \times n$ nonsingular diagonal matrix U such that $\mathcal{N} = U^{-(k-1)}\mathcal{M}U$, then we say that \mathcal{M} and \mathcal{N} are diagonal similar.

Lemma 2.5. [10] Let \mathcal{M} and \mathcal{N} be two diagonal similar tensors of order $k \geq 2$ and dimension n. Then \mathcal{M} and \mathcal{N} have the same real eigenvalues.

Lemma 2.6. [5, 11] Let \mathcal{T} be a nonnegative tensor of order $k \geq 2$ and dimension n. Then

$$\min_{1 \le i \le n} r_i(\mathcal{T}) \le \rho(\mathcal{T}) \le \max_{1 \le i \le n} r_i(\mathcal{T}).$$

Moreover, if \mathcal{T} is weakly irreducible, then either equality holds if and only if $r_1(\mathcal{T}) = \cdots = r_n(\mathcal{T})$.

Proposition 2.1. Let \mathcal{T} be a nonnegative tensor of order $k \geq 2$ and dimension n with all row sums positive. Then

$$\min_{1 \le i \le n} m_i(\mathcal{T}) \le \rho(\mathcal{T}) \le \max_{1 \le i \le n} m_i(\mathcal{T}).$$

Moreover, if \mathcal{T} is weakly irreducible, then either equality holds if and only if $m_1(\mathcal{T}) = \cdots = m_n(\mathcal{T}).$

Proof. Let $U = diag(r_1(\mathcal{T}), \ldots, r_n(\mathcal{T}))$ and $\mathcal{B} = U^{-(k-1)}\mathcal{T}U$. Then \mathcal{T} and \mathcal{B} are diagonal similar, and thus we have by Lemma 2.5 that $\rho(\mathcal{T}) = \rho(\mathcal{B})$. Obviously,

$$B_{i_1\dots i_k} = \frac{T_{i_1\dots i_k} r_{i_2}(\mathcal{T}) \cdots r_{i_k}(\mathcal{T})}{r_{i_1}^{k-1}(\mathcal{T})}$$

for $i_1, i_2, ..., i_k \in [n]$. Thus

$$r_i(\mathcal{B}) = \frac{\sum_{i_2,\dots,i_k \in [n]} T_{ii_2\dots i_k} r_{i_2}(\mathcal{T}) \cdots r_{i_k}(\mathcal{T})}{r_i^{k-1}(\mathcal{T})} = m_i(\mathcal{T})$$

for $i \in [n]$. By Lemma 2.6, we have

$$\min_{1 \le i \le n} m_i(\mathcal{T}) = \min_{1 \le i \le n} r_i(\mathcal{B}) \le \rho(\mathcal{T}) = \rho(\mathcal{B}) \le \max_{1 \le i \le n} r_i(\mathcal{B}) = \max_{1 \le i \le n} m_i(\mathcal{T}),$$

and if \mathcal{T} is weakly irreducible, then since $D_{\mathcal{T}} = D_{\mathcal{B}}$, \mathcal{B} is also weakly irreducible, and thus $\rho(\mathcal{T}) = \min_{1 \le i \le n} m_i(\mathcal{T})$ or $\rho(\mathcal{T}) = \max_{1 \le i \le n} m_i(\mathcal{T})$ if and only if $r_1(\mathcal{B}) = \cdots = r_n(\mathcal{B})$, i.e., $m_1(\mathcal{T}) = \cdots = m_n(\mathcal{T})$.

For a hypergraph \mathcal{G} , the blow-up of \mathcal{G} , denoted by \mathcal{G}^1 , is the hypergraph obtained from \mathcal{G} by adding a new common vertex v to each edge. If \mathcal{G} is a regular (k-1)-uniform hypergraph on n-1 vertices of degree d, then \mathcal{G}^1 is a k-uniform hypergraph on n vertices.

We use the techniques in [12].

3 Main results

Let \mathcal{G} be a k-uniform hypergraph on n vertices without isolated vertices with average 2-degrees $m_1 \geq \cdots \geq m_n$. By Proposition 2.1, $\rho(\mathcal{G}) \leq m_1$. In the following, we give a upper bound for $\rho(\mathcal{G})$ using m_1 and m_2 .

Theorem 3.1. Let \mathcal{G} be a k-uniform hypergraph on n vertices without isolated vertices with average 2-degrees $m_1 \geq \cdots \geq m_n$. Then

$$\rho(\mathcal{G}) \le m_1^{\frac{1}{k}} m_2^{1-\frac{1}{k}}.$$
(3.1)

Moreover, if \mathcal{G} is connected, then equality holds in (3.1) if and only if each vertex of \mathcal{G} has the same average 2-degree.

Proof. Let $\mathcal{A} = \mathcal{A}(\mathcal{G})$.

If $m_1 = m_2$, then by Proposition 2.1, we have

$$\rho(\mathcal{G}) \le m_1^{\frac{1}{k}} m_2^{1-\frac{1}{k}} = m_1,$$

and when \mathcal{G} is connected, A is weakly irreducible, and thus equality holds in (3.1) if and only if each vertex of \mathcal{G} has the same average 2-degree.

Suppose in the following that $m_1 > m_2$. Let d_1, d_2, \ldots, d_n be the degree sequence of \mathcal{G} . Let U be diagonal matrix $\operatorname{diag}(td_1, d_2, \ldots, d_n)$, where t > 1 is a variable to be determined later. Let $\mathcal{T} = U^{-(k-1)}\mathcal{A}U$. Then \mathcal{A} and \mathcal{T} are diagonal similar. By Lemma 2.5, \mathcal{A} and \mathcal{T} have the same real eigenvalues. By Lemma 2.1, $\rho(\mathcal{A})$ is an eigenvalue of \mathcal{A} and $\rho(\mathcal{T})$ is an eigenvalue of \mathcal{T} . Thus $\rho(\mathcal{G}) = \rho(\mathcal{A}) = \rho(\mathcal{T})$. Obviously,

$$T_{i_1\dots i_k} = U_{i_1i_1}^{-(k-1)} A_{i_1\dots i_k} U_{i_2i_2} \cdots U_{i_ki_k}$$

for $i_1, \ldots, i_k \in [n]$. Then

$$r_{1}(\mathcal{T}) = \sum_{i_{2},...,i_{k} \in [n]} T_{ii_{2}...i_{k}}$$

$$= \sum_{i_{2},...,i_{k} \in [n]} U_{11}^{-(k-1)} A_{1i_{2}...i_{k}} U_{i_{2}i_{2}} \cdots U_{i_{k}i_{k}}$$

$$= \sum_{i_{2},...,i_{k} \in [n] \setminus \{1\}} (td_{1})^{-(k-1)} A_{1i_{2}...i_{k}} d_{i_{2}} \cdots d_{i_{k}}$$

$$= \frac{\sum_{\{1,i_{2},...,i_{k}\} \in E_{1}} d_{i_{2}} \cdots d_{i_{k}}}{(td_{1})^{k-1}}$$

$$= \frac{m_{1}}{t^{k-1}}.$$

For $i = 2, \ldots, n$, let

$$m_{1,i} = m_i - \frac{\sum_{1 \notin \{i, i_2, \dots, i_k\} \in E_i} d_{i_2} \cdots d_{i_k}}{d_i^{k-1}}$$

and then

$$\begin{aligned} r_{i}(\mathcal{T}) &= \sum_{i_{2},\dots,i_{k}\in[n]} T_{ii_{2}\dots i_{k}} \\ &= \sum_{i_{2},\dots,i_{k}\in[n]} U_{ii}^{-(k-1)} A_{ii_{2}\dots i_{k}} U_{i_{2}i_{2}} \cdots U_{i_{k}i_{k}} \\ &= \sum_{\substack{i_{2},\dots,i_{k}\in[n]\\1\in\{i_{2},\dots,i_{k}\}}} d_{i}^{-(k-1)} A_{ii_{2}\dots i_{k}} d_{i_{2}} \cdots d_{i_{k}} + \sum_{\substack{i_{2},\dots,i_{k}\in[n]\\1\notin\{i_{2},\dots,i_{k}\}}} d_{i}^{-(k-1)} A_{ii_{2}\dots i_{k}} d_{i_{2}} \cdots d_{i_{k}} \\ &= \frac{\sum_{\substack{i_{2},\dots,i_{k}\in[n]\\1\in\{i_{2},\dots,i_{k}\}}} A_{ii_{2}\dots i_{k}} d_{i_{2}} d_{i_{3}} \cdots d_{i_{k}}}{d_{i}^{k-1}} + \frac{\sum_{\substack{i_{2},\dots,i_{k}\in[n]\\1\notin\{i_{2},\dots,i_{k}\}}} A_{ii_{2}\dots i_{k}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}} \\ &= \frac{\sum_{\{i,1,i_{3},\dots,i_{k}\}\in E_{i}}(td_{1})d_{i_{3}} \cdots d_{i_{k}}}{d_{i}^{k-1}} + \frac{\sum_{\substack{i_{2},\dots,i_{k}\in[n]\\1\notin\{i_{1},2,\dots,i_{k}\}\in E_{i}}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}} \end{aligned}$$

$$= \frac{t \sum_{\{i,1,i_3,\dots,i_k\} \in E_i} d_1 d_{i_3} \cdots d_{i_k}}{d_i^{k-1}} + \frac{\sum_{1 \notin \{i,i_2,\dots,i_k\} \in E_i} d_{i_2} \cdots d_{i_k}}{d_i^{k-1}}$$

= $tm_{1,i} + m_i - m_{1,i}$
 $\leq tm_i$
 $\leq tm_2,$

with equality if and only if $m_i = m_{1,i}$ and $m_i = m_2$. Take $t = \left(\frac{m_1}{m_2}\right)^{\frac{1}{k}}$. Obviously, t > 1. Then $r_1(\mathcal{T}) = m_1^{\frac{1}{k}} m_2^{1-\frac{1}{k}}$ and for $2 \le i \le n$, $r_i(\mathcal{T}) \le tm_2 = m_1^{\frac{1}{k}} m_2^{1-\frac{1}{k}}$. By Lemma 2.6,

$$\rho(\mathcal{G}) = \rho(\mathcal{T}) \le m_1^{\frac{1}{k}} m_2^{1-\frac{1}{k}}.$$

Now suppose that \mathcal{G} is connected. By Lemma 2.3, $\mathcal{A}(\mathcal{G})$ is weakly irreducible, and thus \mathcal{T} is weakly irreducible since $D_{\mathcal{A}(\mathcal{G})} = D_{\mathcal{T}}$.

Suppose that equality holds in (3.1). By Lemma 2.6, $r_1(\mathcal{T}) = \cdots = r_n(\mathcal{T})$. From the above argument, we have (i) $m_2 = \cdots = m_n$, and (ii) $m_i = m_{1,i}$ for $i = 2, \ldots, n$. From (ii) and the definition of $m_{1,i}$, each edge of \mathcal{G} contains vertex 1, which implies that $d_1(k-1) = \sum_{i=2}^n d_i$. From (i),

$$m_1 = \frac{\sum_{i=2}^n \frac{d_i^k m_i}{d_1}}{d_1^{k-1}(k-1)} = \frac{\sum_{i=2}^n d_i^k m_2}{d_1^k} = \frac{\sum_{i=2}^n d_i m_2 \left(\frac{d_i}{d_k}\right)^{k-1}}{d_1(k-1)} \le \frac{\sum_{i=2}^n m_2 d_i}{d_1(k-1)} = m_2,$$

a contradiction. Thus the inequality (3.1) is strict if $m_1 > m_2$.

Let \mathcal{G} be a connected k-uniform hypergraph on n vertices with degree sequence $d_1 \geq \cdots \geq d_n$. Then [12]

$$\rho(\mathcal{G}) \le d_1^{\frac{1}{k}} d_2^{1 - \frac{1}{k}} \tag{3.2}$$

with equality if and only if \mathcal{G} is a regular hypergraph or the blow-up hypergraph \mathcal{H}^1 of a regular (k-1)-uniform hypergraph \mathcal{H} on n-1 vertices.

Obviously, if \mathcal{G} is a regular hypergraph, then each vertex of \mathcal{G} has the same average 2-degree, and thus equality holds in (3.1) and (3.2). For the blow-up hypergraph \mathcal{H}^1 of a regular (k-1)-uniform hypergraph \mathcal{H} on n-1 vertices, the upper bound in (3.2) is attained, while the upper bound in (3.1) is not attained. However, in the following, we give two examples to show that there are irregular hypergraphs for which each vertex has the same average 2-degree. For such hypergraphs, the upper in (3.1) is attained, while the upper bound in (3.2) can not be attained.

Let \mathcal{H}_1 be a 3-uniform hypergraph with vertex set $V(\mathcal{H}_1) = [34]$ and

 $E(\mathcal{H}_1) = \{e_i : 1 \le i \le 51\}, \text{ where }$

$$\begin{array}{ll} e_1 = \{1,2,5\}, & e_2 = \{1,2,6\}, & e_3 = \{1,2,7\}, & e_4 = \{1,2,8\}, \\ e_5 = \{1,2,9\}, & e_6 = \{1,2,10\}, & e_7 = \{1,2,11\}, & e_8 = \{1,2,12\}, \\ e_9 = \{1,2,13\}, & e_{10} = \{3,4,5\}, & e_{11} = \{3,4,6\}, & e_{12} = \{3,4,7\}, \\ e_{13} = \{3,4,8\}, & e_{14} = \{3,4,9\}, & e_{15} = \{3,4,10\}, & e_{16} = \{3,4,11\}, \\ e_{17} = \{3,4,12\}, & e_{18} = \{3,4,13\}, & e_{19} = \{5,6,14\}, & e_{20} = \{6,7,15\}, \\ e_{21} = \{7,8,16\}, & e_{22} = \{8,9,17\}, & e_{23} = \{9,10,18\}, & e_{24} = \{10,11,19\}, \\ e_{25} = \{11,12,20\}, & e_{26} = \{12,13,21\}, & e_{27} = \{13,5,22\}, & e_{28} = \{5,23,24\}, \\ e_{29} = \{5,25,26\}, & e_{30} = \{6,27,28\}, & e_{31} = \{6,29,30\}, & e_{32} = \{7,31,32\}, \\ e_{33} = \{7,33,34\}, & e_{34} = \{8,23,24\}, & e_{35} = \{8,25,26\}, & e_{36} = \{9,27,28\}, \\ e_{41} = \{11,25,26\}, & e_{42} = \{12,27,28\}, & e_{43} = \{12,29,30\}, & e_{44} = \{13,31,32\}, \\ e_{45} = \{13,33,34\}, & e_{46} = \{14,15,16\}, & e_{47} = \{17,18,19\}, & e_{48} = \{20,21,22\}, \\ e_{49} = \{14,17,20\}, & e_{50} = \{15,18,21\}, & e_{51} = \{16,19,22\}. \end{array}$$

By direct calculation, we have

$$d_i = \begin{cases} 9 & \text{if } 1 \le i \le 4, \\ 6 & \text{if } 5 \le i \le 13, \\ 3 & \text{if } 14 \le i \le 34, \end{cases}$$

and

$$m_i = \begin{cases} \frac{(9 \times 6) \times 9}{9 \times 9} & \text{if } 1 \le i \le 4, \\ \frac{(9 \times 9) \times 2 + (6 \times 3) \times 2 + (3 \times 3) \times 2}{6 \times 6} & \text{if } 5 \le i \le 13, \\ \frac{6 \times 6 + (3 \times 3) \times 2}{3 \times 3} & \text{if } 14 \le i \le 22, \\ \frac{(6 \times 3) \times 3}{3 \times 3} & \text{if } 23 \le i \le 34 \\ = 6. \end{cases}$$

By Theorem 3.1, we have $\rho(\mathcal{H}_1) = 6$.

Let \mathcal{H}_2 be a 3-uniform hypergraph with vertex set $V(\mathcal{H}_2) = [54]$ and $E(\mathcal{H}_2) = \{e_i : 1 \le i \le 64\}$, where

$e_1 = \{1, 2, 3\},\$	$e_2 = \{3, 4, 6\},\$	$e_3 = \{3, 4, 5\},\$	$e_4 = \{4, 5, 6\},\$
$e_5 = \{1, 2, 6\},\$	$e_6 = \{1, 2, 5\},\$	$e_7 = \{1, 3, 4\},\$	$e_8 = \{2, 5, 6\},\$
$e_9 = \{1, 7, 8\},\$	$e_{10} = \{1, 8, 9\},\$	$e_{11} = \{1, 9, 10\},\$	$e_{12} = \{1, 10, 11\},\$
$e_{13} = \{2, 11, 12\},\$	$e_{14} = \{2, 12, 13\},\$	$e_{15} = \{2, 13, 14\},\$	$e_{16} = \{2, 14, 15\},\$
$e_{17} = \{3, 15, 16\},\$	$e_{18} = \{3, 16, 17\},\$	$e_{19} = \{3, 17, 18\},\$	$e_{20} = \{3, 18, 19\},\$
$e_{21} = \{4, 19, 20\},\$	$e_{22} = \{4, 20, 21\},\$	$e_{23} = \{4, 21, 22\},\$	$e_{24} = \{4, 22, 23\},\$
$e_{25} = \{5, 23, 24\},\$	$e_{26} = \{5, 24, 25\},\$	$e_{27} = \{5, 25, 26\},\$	$e_{28} = \{5, 26, 27\},\$
$e_{29} = \{6, 27, 28\},\$	$e_{30} = \{6, 28, 29\},\$	$e_{31} = \{6, 29, 30\},\$	$e_{32} = \{6, 30, 7\},\$
$e_{33} = \{7, 8, 31\},\$	$e_{34} = \{7, 8, 32\},\$	$e_{35} = \{9, 10, 33\},\$	$e_{36} = \{9, 10, 34\},\$
$e_{37} = \{11, 12, 35\},\$	$e_{38} = \{11, 12, 36\},\$	$e_{39} = \{13, 14, 37\},\$	$e_{40} = \{13, 14, 38\},\$
$e_{41} = \{15, 16, 39\},\$	$e_{42} = \{15, 16, 40\},\$	$e_{43} = \{17, 18, 41\},\$	$e_{44} = \{17, 18, 42\},\$
$e_{45} = \{19, 20, 43\},\$	$e_{46} = \{19, 20, 44\},\$	$e_{47} = \{21, 22, 45\},\$	$e_{48} = \{21, 22, 46\},\$
$e_{49} = \{23, 24, 47\},\$	$e_{50} = \{23, 24, 48\},\$	$e_{51} = \{25, 26, 49\},\$	$e_{52} = \{25, 26, 50\},\$
$e_{53} = \{27, 28, 51\},\$	$e_{54} = \{27, 28, 52\},\$	$e_{55} = \{29, 30, 53\},\$	$e_{56} = \{29, 30, 54\},\$
$e_{57} = \{31, 32, 33\},\$	$e_{58} = \{34, 35, 36\},\$	$e_{59} = \{37, 38, 39\},\$	$e_{60} = \{40, 41, 42\},\$
$e_{61} = \{43, 44, 45\},\$	$e_{62} = \{46, 47, 48\},\$	$e_{63} = \{49, 50, 51\},\$	$e_{64} = \{52, 53, 54\}.$

By direct calculation, we have

$$d_i = \begin{cases} 8 & \text{if } 1 \le i \le 6, \\ 4 & \text{if } 7 \le i \le 30, \\ 2 & \text{if } 31 \le i \le 54, \end{cases}$$

and

$$m_i = \begin{cases} \frac{(8\times8)\times4+(4\times4)\times4}{8\times8} & \text{if } 1 \le i \le 6, \\ \frac{(8\times4)\times2+(4\times3)\times2}{4\times4} & \text{if } 7 \le i \le 30, \\ \frac{4\times4+2\times2}{2\times2} & \text{if } 31 \le i \le 54 \\ = 5. \end{cases}$$

By Theorem 3.1, we have $\rho(\mathcal{H}_2) = 5$.

Let \mathcal{G} be a k-uniform hypergraph of order n without isolated vertices with maximum degree Δ and average 2-degrees $m_1 \geq \cdots \geq m_n$. Note that $\mu(\mathcal{G}) \leq \Delta + \rho(\mathcal{G})$. By Theorem 3.1, we have Then $\mu(\mathcal{G}) \leq m_1^{\frac{1}{k}} m_2^{1-\frac{1}{k}} + \Delta$.

If we take $U = \text{diag}(d_1, \ldots, d_{n-1}, yd_n)$ with $y = \left(\frac{m_n}{m_{n-1}}\right)^{\frac{1}{k}}$ in the proof of Theorem 3.1, then $\rho(\mathcal{G}) \ge m_n^{\frac{1}{k}} m_{n-1}^{1-\frac{1}{k}}$, and if \mathcal{G} is connected, then equality holds if and only if each vertex of \mathcal{G} has the same average 2-degree.

For a k-uniform hypergraph \mathcal{G} , if there is a disjoint partition of $V(\mathcal{G})$ as $V(\mathcal{G}) = V_0 \cup V_1 \cup \cdots \cup V_d$, where $|V_0| = 1, |V_1| = \cdots = |V_d| = k - 1$, and $E(\mathcal{G}) = \{V_0 \cup V_i : i \in [d]\}$, then \mathcal{G} is called a hyperstar, denoted by \mathcal{S}_d^k . The vertex (of degree d) in V_0 is called the heart. Obviously, it is an isolated vertex if d = 0.

For positive integers d_1, γ and nonnegative integer d_2 , let $\mathcal{G}_{d_1,d_2,\gamma}$ be the kuniform hypergraph obtained vertex-disjoint $\mathcal{S}_{d_1}^k$ and $\mathcal{S}_{d_2}^k$ by adding $\gamma(k-2)$ new vertices $v_{1,1}, \ldots, v_{1,k-2}, \ldots, v_{\gamma,1}, \ldots, v_{\gamma,k-2}$ and γ new edges e_1, \ldots, e_{γ} , where $e_i = \{u, v, v_{i,1}, \ldots, v_{i,k-2}\}$ for $i \in [\gamma]$, and u, v are the hearts of $\mathcal{S}_{d_1}^k$ and $\mathcal{S}_{d_2}^k$, respectively. Obviously, if $d_2 = 0$ and $\gamma = 1$, then $\mathcal{G}_{d_1,d_2,\gamma} \cong \mathcal{S}_{d_1+1}^k$.

Next we give a lower bound for $\rho(\mathcal{G})$ of a k-uniform hypergraph \mathcal{G} .

Theorem 3.2. Let \mathcal{G} be a k-uniform hypergraph with $u \in V(\mathcal{G})$ of maximum degree $\Delta \geq 1$. Let v be a neighbor of u with maximum degree. Then

$$\rho(\mathcal{G}) \ge \left(\frac{\Delta + \delta - 2\gamma + \gamma^2 + \sqrt{(\Delta - \delta)^2 + \gamma^4 + 2(\Delta + \delta - 2\gamma)\gamma^2}}{2}\right)^{\frac{1}{k}}, \quad (3.3)$$

where δ is the degree of v, and γ is the number of edges containing u and v. Moreover, if \mathcal{G} is connected, then equality holds in (3.3) if and only if $\mathcal{G} \cong \mathcal{G}_{\Delta,\delta,\gamma}$.

Proof. Let e_1, \ldots, e_{Δ} be the Δ edges of \mathcal{G} containing u. Among these edges, γ of them, say e_1, \ldots, e_{γ} , contain v. Let $e_{\Delta+1}, \ldots, e_{\Delta+\delta-\gamma}$ be the $\delta - \gamma$ edges of \mathcal{G} containing v different from e_1, \ldots, e_{γ} . Let \mathcal{G}_1 be the subhypergraph of \mathcal{G}

induced by $\{e_1, \ldots, e_{\Delta+\delta-\gamma}\}$. Then $V(\mathcal{G}_1) = \bigcup_{i=1}^{\Delta+\delta-\gamma} e_i$. For $1 \leq i \leq \Delta + \delta - \gamma$, let $e_i = \{v_{i,1}, \ldots, v_{i,k}\}$, where $v_{i,1} = u$ and $v_{i,2} = v$ if $1 \leq i \leq \gamma$, $v_{i,1} = u$ if $\gamma + 1 \leq i \leq \Delta$, and $v_{i,1} = v$ if $\Delta + 1 \leq i \leq \Delta + \delta - \gamma$. Note that maybe some of $v_{i,s}$ and $v_{j,t}$ for $1 \leq s, t \leq k$ and $1 \leq i < j \leq \Delta + \delta - \gamma$ with $v_{i,s}, v_{j,t} \neq u, v$ represent the same vertex.

Let \mathcal{G}'_1 be a new hypergraph such that $V(\mathcal{G}'_1) = \bigcup_{i=1}^{\Delta+\delta-\gamma} e'_i$ and $E(\mathcal{G}'_1) = \{e'_1, \ldots, e'_{\Delta+\delta-\gamma}\}$, where $e'_i = \{v'_{i,1}, \ldots, v'_{i,k}\}$ with $v'_{i,1} = u$ and $v'_{i,2} = v$ if $i = 1, \ldots, \gamma, v'_{i,1} = u$ if $\gamma + 1 \leq i \leq \Delta$, and $v'_{i,1} = v$ if $\Delta + 1 \leq i \leq \Delta + \delta - \gamma$. Note that $v \notin e_i$ for $\gamma + 1 \leq i \leq \Delta, u \notin e_i$ for $\Delta + 1 \leq i \leq \Delta + \delta - \gamma$, and $v'_{i,s}$ and $v'_{j,t}$ for $1 \leq s, t \leq k$ and $1 \leq i < j \leq \Delta + \delta - \gamma$ with $v'_{i,s}, v'_{j,t} \neq u, v$ are different vertices. Obviously, $\mathcal{G}'_1 \cong \mathcal{G}_{\Delta,\delta,\gamma}$

By Lemma 2.1, there is a unit positive eigenvector x of $\mathcal{A}(\mathcal{G}'_1)$ corresponding to $\rho(\mathcal{G}'_1)$, in which the entry at $v'_{i,s}$ is denoted by $x_{i,s}$, where $1 \leq i \leq \Delta + \delta - \gamma$ and $1 \leq s \leq k$. Then $\rho(\mathcal{G}'_1) = x^{\top}(\mathcal{A}(\mathcal{G}'_1)x)$. Let w be any vertex of $\bigcup_{i=\Delta-\gamma+1}^{\Delta}e_i \setminus \{u\}$. Since $\rho(\mathcal{G}'_1)x_w^{k-1} = x_u x_w^{k-2}$, we have $x_w = \frac{x_u}{\rho(\mathcal{G}'_1)}$. Thus the entry of x at each vertex of $\bigcup_{i=\Delta-\gamma+1}^{\Delta}e_i \setminus \{u\}$ is the same, denoted by a. Similarly, the entry of x'at each vertex of $\bigcup_{i=1}^{\gamma}e_i \setminus \{u,v\}$ is the same, denoted by b, and the entry of x'at each vertex of $\bigcup_{i=\Delta+1}^{\Delta+\delta-\gamma}e_i \setminus \{v\}$ is the same, denoted by c. Then

$$\begin{aligned} \rho(\mathcal{G}'_{1})a^{k-1} &= x_{u}a^{k-2}, \\ \rho(\mathcal{G}'_{1})x_{u}^{k-1} &= (\Delta - \gamma)a^{k-1} + \gamma b^{k-2}x_{v}, \\ \rho(\mathcal{G}'_{1})b^{k-1} &= x_{u}x_{v}b^{k-3}, \\ \rho(\mathcal{G}'_{1})x_{v}^{k-1} &= (\delta - \gamma)c^{k-1} + \gamma b^{k-2}x_{u}, \\ \rho(\mathcal{G}'_{1})c^{k-1} &= x_{v}c^{k-2}. \end{aligned}$$

Thus $\rho(\mathcal{G}'_1)$ is the largest root of the equation $f(\rho) = 0$, where $f(\rho) = (\rho^k - \Delta + \gamma) \left(\rho^{2k} - (\Delta + \delta - 2\gamma + \gamma^2)\rho^k + (\Delta - \gamma)(\delta - \gamma)\right)$. It follows that

$$\rho(\mathcal{G}_1') = \left(\frac{\Delta + \delta - 2\gamma + \gamma^2 + \sqrt{(\Delta - \delta)^2 + \gamma^4 + 2(\Delta + \delta - 2\gamma)\gamma^2}}{2}\right)^{\frac{1}{k}}.$$

Construct a surjection σ from $V(\mathcal{G}'_1)$ to $V(\mathcal{G}_1)$ such that $\sigma(v'_{i,s}) = v_{i,s}$ for $1 \leq i \leq \Delta + \delta - \gamma$ and $1 \leq s \leq k$. Let $y = (y_1, \ldots, y_{|V(\mathcal{G}_1)|})^\top$ such that $y_i = \max_{v'_{j,s} \in \sigma^{-1}(i)} \{x_{j,s}\}$ for $1 \leq i \leq |V(\mathcal{G}_1)|$. Obviously, $\|y\|_k \leq \|x\|_k = 1$. Let $z = \frac{y}{\|y\|_k}$. Then $\|z\|_k = 1$. By Lemma 2.2,

$$\rho(\mathcal{G}_1) \ge z^{\top}(\mathcal{A}(\mathcal{G}_1)z) = \frac{y^{\top}(\mathcal{A}(\mathcal{G}_1)y)}{\|y\|_k^k} \ge \frac{x^{\top}(\mathcal{A}(\mathcal{G}_1')x)}{\|x\|_k^k} = x^{\top}(\mathcal{A}(\mathcal{G}_1')x) = \rho(\mathcal{G}_1').$$
(3.4)

Since \mathcal{G}_1 is a subhypergraph of \mathcal{G} , we have by Lemma 2.4 that $\rho(\mathcal{G}) \geq \rho(\mathcal{G}_1)$. Thus

$$\rho(\mathcal{G}) \ge \rho(\mathcal{G}_1') = \left(\frac{\Delta + \delta - 2\gamma + \gamma^2 + \sqrt{(\Delta - \delta)^2 + \gamma^4 + 2(\Delta + \delta - 2\gamma)\gamma^2}}{2}\right)^{\frac{1}{k}}.$$

If $\mathcal{G} \cong \mathcal{G}_{\Delta,\delta,\gamma}$, then by the above proof, equality holds in (3.3).

Suppose that \mathcal{G} is connected and equality holds in (3.3). Then all equalities hold in (3.4) and $\rho(\mathcal{G}) = \rho(\mathcal{G}_1)$. Thus by the construction of \mathcal{G}'_1 , we have $\mathcal{G}_1 \cong$ \mathcal{G}'_1 . Otherwise, $|V(\mathcal{G}_1)| < |V(\mathcal{G}'_1)|$, and then $||y||_k < ||x||_k = 1$, a contradiction. By Lemma 2.4, we have $\mathcal{G} = \mathcal{G}_1$. Thus $\mathcal{G} \cong \mathcal{G}_{\Delta,\delta,\gamma}$.

Let \mathcal{G} be a k-uniform hypergraph with maximum degree $\Delta \geq 1$. Let $f(\Delta, \delta, \gamma)$ be the lower bound in (3.3). For $\gamma \leq \delta \leq \Delta$, $f(\Delta, \delta, \gamma)$ is a increasing function at δ . Note that $\gamma \geq 1$. By Theorem 3.2, $\rho(\mathcal{G}) \geq f(\Delta, 1, 1) = \Delta^{\frac{1}{k}}$. and if \mathcal{G} is connected, then equality holds if and only if \mathcal{G} is a hyperstar. Moreover, if \mathcal{G} is connected and is not a hyperstar, then $\rho(\mathcal{G}) \geq f(\Delta, 2, 1) =$ $\left(\frac{\Delta+1+\sqrt{\Delta^2-2\Delta+5}}{2}\right)^{\frac{1}{k}}$ with equality if and only if $\mathcal{G} \cong \mathcal{G}_{\Delta,2,1}$. In the following, we give upper bounds for $\mu(\mathcal{G})$ of a k-uniform hypergraph.

Theorem 3.3. Let \mathcal{G} be a k-uniform hypergraph on n vertices with degree sequence $d_1 \geq \cdots \geq d_n$. Let $d^* = 1$ if $d_1 = d_2$ and d^* be a root of h(t) = 0 in $\left(\left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}, \frac{d_1}{d_2}\right)$ if $d_1 > d_2$, where $h(t) = d_2 t^k + (d_2 - d_1)t^{k-1} - d_1$. Then

$$\mu(\mathcal{G}) \le d_1 + d_1 \left(\frac{1}{d^*}\right)^{k-1}.$$
(3.5)

Moreover, if \mathcal{G} is connected, then equality holds in (3.5) if and only if \mathcal{G} is a regular hypergraph or the blow-up hypergraph of a regular (k-1)-uniform hypergraph on n-1 vertices.

Proof. Let $\mathcal{Q} = \mathcal{Q}(\mathcal{G}), \ \mathcal{A} = \mathcal{A}(\mathcal{G}), \ \text{and} \ \mathcal{D} = \mathcal{D}(\mathcal{G}).$ If $d_1 = d_2$, then $d^* = 1$, and by Lemma 2.6, we have

$$\mu(\mathcal{G}) = \rho(\mathcal{Q}) \le \max_{1 \le i \le n} r_i(\mathcal{Q}) = \max_{1 \le i \le n} 2d_i = 2d_1 = d_1 + d_1 \left(\frac{1}{d^*}\right)^{k-1},$$

and when \mathcal{G} is connected, we have by Lemma 2.3 that \mathcal{Q} is weakly irreducible. and thus equality holds if and only if $r_1(\mathcal{Q}) = \cdots = r_n(\mathcal{Q})$, i.e., \mathcal{G} is a regular hypergraph.

Suppose in the following that $d_1 > d_2$. Let $U = \text{diag}(t, 1, \dots, 1)$ be an $n \times n$ diagonal matrix, where t > 1 is a variable to be determined later. Let $\mathcal{T} = U^{-(k-1)}\mathcal{Q}U$. By Lemma 2.5, \mathcal{Q} and \mathcal{T} have the same real eigenvalues. By Lemma 2.1, $\rho(\mathcal{Q})$ is an eigenvalue of \mathcal{Q} and $\rho(\mathcal{T})$ is an eigenvalue of \mathcal{T} . Thus $\mu(\mathcal{G}) = \rho(\mathcal{Q}) = \rho(\mathcal{T}).$ We have

$$r_{1}(\mathcal{T}) = \sum_{i_{2},...,i_{k} \in [n]} T_{1i_{2}...i_{k}}$$

$$= \sum_{i_{2},...,i_{k} \in [n]} U_{11}^{-(k-1)} A_{1i_{2}...i_{k}} U_{i_{2}i_{2}} \cdots U_{i_{k}i_{k}} + D_{1...1}$$

$$= \sum_{i_{2},...,i_{k} \in [n] \setminus \{1\}} \frac{1}{t^{k-1}} A_{1i_{2}...i_{k}} + d_{1}$$

$$= \frac{d_{1}}{t^{k-1}} + d_{1}.$$

For $i \in [n] \setminus \{1\}$, let $d_{1,i} = |\{e : 1, i \in e \in E(\mathcal{G})\}|$. Obviously, $d_{1,i} \leq d_i$. For $2 \leq i \leq n$, we have

$$\begin{split} r_{i}(\mathcal{T}) &= \sum_{\substack{i_{2}, \dots, i_{k} \in [n] \\ i_{2}, \dots, i_{k} \in [n] \\ }} T_{ii_{2} \dots i_{k}}}{U_{ii}^{-(k-1)} A_{ii_{2} \dots i_{k}} U_{i_{2}i_{2}} \cdots U_{i_{k}i_{k}} + D_{i\dots i}} \\ &= \sum_{\substack{i_{2}, \dots, i_{k} \in [n] \\ 1 \in \{i_{2}, \dots, i_{k}\} \\ + \sum_{\substack{i_{2}, \dots, i_{k} \in [n] \\ 1 \notin \{i_{2}, \dots, i_{k}\} \\ }} U_{ii}^{-(k-1)} A_{ii_{2} \dots i_{k}} U_{i_{2}i_{2}} \cdots U_{i_{k}i_{k}} + d_{i} \\ &= \sum_{\substack{i_{2}, \dots, i_{k} \in [n] \\ 1 \notin \{i_{2}, \dots, i_{k}\} \\ }} A_{ii_{2} \dots i_{k}} t + \sum_{\substack{i_{2}, \dots, i_{k} \in [n] \\ 1 \notin \{i_{2}, \dots, i_{k}\} \\ }} A_{ii_{2} \dots i_{k}} + d_{i} \\ &= td_{1,i} + d_{i} - d_{1,i} + d_{i} \\ &\leq (t+1)d_{i} \\ &\leq (t+1)d_{2} \end{split}$$

with equality if and only if $d_i = d_{1,i}$ and $d_i = d_2$.

Note that $h((\frac{d_1}{d_2})^{\frac{1}{k}}) = (d_2 - d_1) \left(\frac{d_1}{d_2}\right)^{1 - \frac{1}{k}} < 0 \text{ and } h((\frac{d_1}{d_2})) = d_1 \left(\left(\frac{d_1}{d_2}\right)^{k - 2} - 1\right) > 0$. Thus h(t) = 0 does have a root d^* in $((\frac{d_1}{d_2})^{\frac{1}{k}}, \frac{d_1}{d_2})$. Let $t = d^*$. Then t > 1. We have

$$r_1(\mathcal{T}) = d_1 + d_1 \left(\frac{1}{d^*}\right)^{k-1}$$

and for $2 \leq i \leq n$,

$$r_i(\mathcal{T}) \le d_1 + d_1 \left(\frac{1}{d^*}\right)^{k-1}$$

Thus by Lemma 2.6,

$$\mu(\mathcal{G}) = \rho(\mathcal{T}) \le \max_{1 \le i \le n} r_i(\mathcal{T}) = d_1 + d_1 \left(\frac{1}{d^*}\right)^{k-1}$$

This proves (3.5).

Now suppose that \mathcal{G} is connected. By Lemma 2.3, \mathcal{Q} is weakly irreducible, and so is \mathcal{T} .

If equality holds in (3.5), then by Lemma 2.6, $r_1(\mathcal{T}) = \cdots = r_n(\mathcal{T})$, and thus from the above arguments, we have $d_{1,i} = d_i$ for $i = 2, \ldots, n$ (implying that each edge of \mathcal{G} contains vertex 1), $d_2 = \cdots = d_n$, and thus \mathcal{G} is a blow-up hypergraph of a regular (k-1)-uniform hypergraph on n-1 vertices of degree d_2 .

Conversely, if $\mathcal{G} = \mathcal{H}^1$, where \mathcal{H} is a regular (k-1)-uniform hypergraph on n-1 vertices of degree d_2 , then by the above arguments, we have $r_i(\mathcal{T}) = d_1 + d_1(\frac{1}{d^*})^{k-1}$ for $1 \le i \le n$, and thus by Lemma 2.6, $\mu(\mathcal{G}) = \rho(\mathcal{Q}) = \rho(\mathcal{T}) = d_1 + d_1(\frac{1}{d^*})^{k-1}$. Let \mathcal{G} be a k-uniform hypergraph on n vertices with degree sequence $d_1 \geq \cdots \geq d_n$. If $d_1 > d_2$, then $d_1 + d_1(\frac{1}{d^*})^{k-1} < d_1 + d_1(\frac{d_2}{d_1})^{1-\frac{1}{k}} = d_1 + d_1^{\frac{1}{k}} d_2^{1-\frac{1}{k}}$. By Theorem 3.3, we have

$$\mu(\mathcal{G}) \le d_1 + d_1^{\frac{1}{k}} d_2^{1 - \frac{1}{k}},$$

and if \mathcal{G} is connected, then equality holds if and only if \mathcal{G} is a regular hypergraph, see [12].

Theorem 3.4. Let \mathcal{G} be a k-uniform hypergraph on n vertices without isolated vertices with average 2-degrees $m_1 \geq \cdots \geq m_n$, and degree sequence d_1, \ldots, d_n . Then

$$\mu(\mathcal{G}) \le \min_{1 \le j \le n} \max\left\{ m_1^{\frac{1}{k}} m_j^{1-\frac{1}{k}} + d_1, \theta_j \right\},$$
(3.6)

where

$$\theta_j = \max\left\{m_1^{\frac{1}{k}}m_im_j^{-\frac{1}{k}} + d_i : 2 \le i \le n\right\}.$$

Proof. Let $\mathcal{Q} = \mathcal{Q}(\mathcal{G})$. Let U be a diagonal matrix diag (td_1, d_2, \ldots, d_n) , where $t \geq 1$ is a variable to be determined later. Let $\mathcal{T} = U^{-(k-1)}\mathcal{Q}U$. Then \mathcal{Q} and \mathcal{T} are diagonal similar. By Lemma 2.5, $\mu(\mathcal{G}) = \rho(\mathcal{T})$. Obviously,

$$T_{i_1\dots i_k} = U_{i_1i_1}^{-(k-1)} Q_{i_1\dots i_k} U_{i_2i_2} \cdots U_{i_ki_k}$$

for $i_1, \ldots, i_k \in [n]$. Then

$$r_{1}(\mathcal{T}) = \sum_{i_{2},...,i_{k} \in [n]} U_{11}^{-(k-1)} Q_{1i_{2}...i_{k}} U_{i_{2}i_{2}} \cdots U_{i_{k}i_{k}}$$

$$= \frac{\sum_{i_{2},...,i_{k} \in [n] \setminus \{1\}} Q_{1i_{2}...i_{k}} d_{i_{2}} \cdots d_{i_{k}}}{(td_{1})^{k-1}}$$

$$= \frac{D_{1...1}(td_{1})^{k-1}}{(td_{1})^{k-1}} + \frac{\sum_{i_{2},...,i_{k} \in [n]} A_{1i_{2}...i_{k}} d_{i_{2}} \cdots d_{i_{k}}}{(td_{1})^{k-1}}$$

$$= d_{1} + \frac{\sum_{\{1,i_{2},...,i_{k}\} \in E_{1}} d_{i_{2}} \cdots d_{i_{k}}}{(td_{1})^{k-1}}$$

$$= d_{1} + \frac{m_{1}}{t^{k-1}}.$$

For $i = 2, \ldots, n$, let

$$m_{1,i} = m_i - \frac{\sum_{1 \notin \{i, i_2, \dots, i_k\} \in E_i} d_{i_2} \dots d_{i_k}}{d_i^{k-1}}$$

and then

$$r_{i}(\mathcal{T}) = \sum_{\substack{i_{2},...,i_{k} \in [n] \\ i \in \{i_{2},...,i_{k}\}}} U_{ii}^{-(k-1)}Q_{ii_{2}...i_{k}}U_{i_{2}i_{2}}\cdots U_{i_{k}i_{k}}}$$
$$= \sum_{\substack{i_{2},...,i_{k} \in [n] \\ i \in \{i_{2},...,i_{k}\}}} d_{i}^{-(k-1)}Q_{ii_{2}...i_{k}}d_{i_{2}}\cdots d_{i_{k}}}$$

$$\begin{split} &+ \sum_{\substack{i_2, \dots, i_k \in [n] \\ 1 \notin \{i_2, \dots, i_k\}}} d_i^{-(k-1)} Q_{ii_2 \dots i_k} d_{i_2} \cdots d_{i_k} \\ &= \frac{D_{i \dots i} d_i^{k-1}}{d_i^{k-1}} + \frac{\sum_{\substack{i_2, \dots, i_k \in [n] \\ 1 \in \{i_2, \dots, i_k\}}} A_{ii_2 \dots i_k} d_{i_2} \cdots d_{i_k}}{d_i^{k-1}} \\ &+ \frac{\sum_{\substack{i_2, \dots, i_k \in [n] \\ 1 \notin \{i_2, \dots, i_k\}}} A_{ii_2 \dots i_k} d_{i_2} d_{i_2} \cdots d_{i_k}}{d_i^{k-1}} \\ &= d_i + \frac{\sum_{\{i, 1, i_3, \dots, i_k\} \in E_i} (td_1) d_{i_3} \cdots d_{i_k}}{d_i^{k-1}} \\ &+ \frac{\sum_{\substack{1 \notin \{i, 1, \dots, i_k\} \in E_i} d_{i_2} \cdots d_{i_k}}}{d_i^{k-1}} \\ &= d_i + tm_{1,i} + m_i - m_{1,i} \\ &\leq d_i + tm_i. \end{split}$$

For an arbitrary fixed j with $1 \le j \le n$, let $t = \left(\frac{m_1}{m_j}\right)^{\frac{1}{k}}$. Obviously, $t \ge 1$. Then

$$r_1(\mathcal{T}) = m_1^{\frac{1}{k}} m_j^{1-\frac{1}{k}} + d_1,$$

for $2 \leq i \leq n$,

$$r_i(\mathcal{T}) \le tm_i + d_i = m_1^{\frac{1}{k}} m_i m_j^{-\frac{1}{k}} + d_i.$$

Let $\theta_j = \max\left\{m_1^{\frac{1}{k}} m_i m_j^{-\frac{1}{k}} + d_i : 2 \le i \le n\right\}$. Thus for $1 \le i \le n$, we have
 $r_i(\mathcal{T}) \le \max\left\{m_1^{\frac{1}{k}} m_j^{1-\frac{1}{k}} + d_1, \theta_j\right\}$

Thus

$$r_i(\mathcal{T}) \leq \min_{1 \leq j \leq n} \max\left\{m_1^{\frac{1}{k}} m_j^{1-\frac{1}{k}} + d_1, \theta_j\right\}.$$

Now the result follows from Lemma 2.6.

If we take $U = \text{diag}(d_1, \ldots, d_{n-1}, yd_n)$ with $y = \left(\frac{m_n}{m_j}\right)^{\frac{1}{k}}$ for an arbitrary fixed j in the above proof, then we have

$$\mu(\mathcal{G}) \ge \max_{1 \le j \le n} \min\left\{ m_n^{\frac{1}{k}} m_j^{1-\frac{1}{k}} + d_n, \gamma_j \right\},\,$$

where $\gamma_j = \min\left\{m_n^{\frac{1}{k}}m_im_j^{-\frac{1}{k}} + d_i : 2 \le i \le n\right\}$ for $1 \le j \le n$. Consider 4-uniform hypergraph \mathcal{G}_1 with vertex set [25] and edge set $E(\mathcal{G}_1) =$

 $\{e_1, \ldots, e_{14}\},$ where

$$\begin{array}{ll} e_1 = \{1,2,3,4\}, & e_2 = \{5,6,7,8\}, & e_3 = \{9,10,11,12\}, \\ e_4 = \{13,14,15,16\}, & e_5 = \{17,18,19,20\}, & e_6 = \{21,22,23,24\}, \\ e_7 = \{1,2,3,25\}, & e_8 = \{4,5,6,25\}, & e_9 = \{7,8,9,25\}, \\ e_{10} = \{10,11,12,25\}, & e_{11} = \{13,14,15,25\}, & e_{12} = \{16,17,18,25\} \\ e_{13} = \{19,20,21,25\}, & e_{14} = \{22,23,24,25\}. \end{array}$$

In notation of Theorem 3.4, we have

$$d_1 = \dots = d_{24} = 2, d_{25} = 8,$$

and

$$m_1 = \dots = m_{24} = 5, m_{25} = 0.125,$$

implying that $\theta_1 = \cdots = \theta_{24} = 14.5743$, $\theta_{25} \approx 8.31436$, and $m_1^{\frac{1}{4}} m_j^{\frac{3}{4}} + d_1 = 7$ for $1 \leq j \leq 24$ and $m_1^{\frac{1}{4}} m_{25}^{\frac{3}{4}} + d_1 \approx 2.31436$. Thus $\mu(\mathcal{G}_1) \leq 8.125$. Note that $8 = d_1 > d_2 = \cdots = d_{25} = 2$ in notation of Theorem 3.3 and that $h(d^*) = d_1 + d_1 \left(\frac{1}{d^*}\right)^{k-1}$ is a decreasing function for $d^* \in \left(\left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}, \frac{d_1}{d_2}\right)$. Then

$$d_1 + d_1 \left(\frac{1}{d^*}\right)^{k-1} > d_1 + d_1 \left(\frac{d_2}{d_1}\right)^{k-1} = d_1 + \frac{d_2^{k-1}}{d_1^{k-2}} = 8 + \frac{2^3}{8^2} = 8.125.$$

For \mathcal{G}_1 , the upper bound in (3.6) is smaller than the one in (3.5). Obviously, the blow-up hypergraph of a regular (k-1)-uniform hypergraph on n-1 vertices, the upper bound in (3.5) is smaller than the one in (3.6).

For a k-uniform hypergraph \mathcal{G} , let $d_1 \geq \cdots \geq d_n$ be the degree sequence of \mathcal{G} and m_1, \ldots, m_n be the average 2-degrees of \mathcal{G} . In [12], the following upper bounds for $\mu(\mathcal{G})$ are given.

$$\mu(\mathcal{G}) \le \max_{e \in E(\mathcal{G})} \max_{\{i,j\} \in e} (d_i + d_j), \tag{3.7}$$

$$\mu(\mathcal{G}) \le \max_{e \in E(\mathcal{G})} \max_{\{i,j\} \in e} \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2}.$$
(3.8)

Consider 3-uniform hypergraph \mathcal{G}_2 with vertex set [9] and edge set $E(\mathcal{G}_2) = \{e_1, \ldots, e_4\}$, where

$$e_1 = \{1, 2, 9\}, e_2 = \{3, 4, 8\}, e_3 = \{5, 6, 7\}, e_4 = \{7, 8, 9\}.$$

In notation of Theorem 3.4, we have

$$d_1 = \dots = d_6 = 1, d_7 = d_8 = d_9 = 2,$$

and

$$m_1 = \dots = m_6 = 2, m_7 = m_8 = m_9 = \frac{5}{4},$$

implying that $\theta_1 = \cdots = \theta_6 = 3.25$, $\theta_7 = \theta_8 = \theta_9 \approx 3.462$, and $m_1^{\frac{1}{3}} m_j^{\frac{2}{3}} + d_1 = 3$ when $1 \leq j \leq 6$ and $m_1^{\frac{1}{3}} m_j^{\frac{2}{3}} + d_1 \approx 2.462$ when $7 \leq j \leq 9$, and thus $\mu(\mathcal{G}_2) \leq 3.25$. By direct calculation, the bounds in (3.7) and (3.8) are 4 and 3.25, respectively. For \mathcal{G}_2 , the upper bound in (3.6) is smaller than the upper bound in (3.7).

Consider 4-uniform hypergraph \mathcal{G}_3 with vertex set [7] and edge set $E(\mathcal{G}_3) = \{e_1, \ldots, e_8\}$, where

$$e_{1} = \{1, 2, 3, 4\}, \quad e_{2} = \{1, 5, 6, 7\}, \quad e_{3} = \{2, 3, 4, 5\}, \\ e_{4} = \{3, 4, 5, 6\}, \quad e_{5} = \{4, 5, 6, 7\}, \quad e_{6} = \{5, 6, 7, 2\}, \\ e_{7} = \{6, 7, 2, 3\}, \quad e_{8} = \{7, 2, 3, 4\}.$$

In notation of Theorem 3.4, we have

$$d_1 = 2, d_2 = \dots = d_7 = 5,$$

and

$$m_1 = 31.25, m_2 = \dots = m_7 = 4.4,$$

implying that $\theta_1 = 9.4$, and $\theta_2 = \cdots = \theta_7 \approx 12.18294$, and $m_1^{\frac{1}{4}} m_j^{\frac{3}{4}} + d_1 = 33.25$ for j = 1 and $m_1^{\frac{1}{4}} m_j^{\frac{3}{4}} + d_1 \approx 9.18294$ for $2 \le j \le 7$. Thus $\mu(\mathcal{G}_3) \le 12.18294$. It is easily seen that the bounds in (3.7) and (3.8) are 10 and 15.32159, respectively. For \mathcal{G}_3 , the upper bound in Theorem 3.4 is smaller than the one in (3.8) but larger than the one in (3.7).

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