# Sharp bounds for ordinary and signless Laplacian spectral radii of uniform hypergraphs 

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#### Abstract

We give sharp upper bounds for the ordinary spectral radius and signless Laplacian spectral radius of a uniform hypergraph in terms of the average 2-degrees or degrees of vertices, respectively, and we also give a lower bound for the ordinary spectral radius. We also compare these bounds with known ones.


Key words: tensor, eigenvalues of tensors, uniform hypergraph, average 2-degree, adjacency tensor, signless Laplacian tensor

## 1 Introduction

For positive integers $k$ and $n$ with $k \leq n$, a tensor $\mathcal{T}=\left(T_{i_{1} \ldots i_{k}}\right)$ of order $k$ and dimension $n$ refers to a multidimensional array with complex entries $T_{i_{1} \ldots i_{k}}$ for $i_{j} \in[n]:=\{1, \ldots, n\}$ and $j \in[k]$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2 .

Let $\mathcal{M}$ be a tensor of order $s \geq 2$ and dimension $n$, and $\mathcal{N}$ a tensor of order $k \geq 1$ and dimension $n$. The product $\mathcal{M} \mathcal{N}$ is the tensor of order $(s-1)(k-1)+1$ and dimension $n$ with entries 10

$$
(\mathcal{M N})_{i j_{1} \ldots j_{s-1}}=\sum_{i_{2}, \ldots, i_{s} \in[n]} M_{i i_{2} \ldots i_{s}} N_{i_{2} j_{1}} \cdots N_{i_{s} j_{s-1}}
$$

with $i \in[n]$ and $j_{1}, \ldots, j_{s-1} \in[n]^{k-1}$.
For a tensor $\mathcal{T}$ of order $k \geq 2$ and dimension $n$ and a vector $x=\left(x_{1}, \ldots, x_{n}\right)^{\top}$, $\mathcal{T} x$ is an $n$-dimensional vector whose $i$-th entry is

$$
(\mathcal{T} x)_{i}=\sum_{i_{2}, \ldots, i_{k} \in[n]} T_{i i_{2} \ldots i_{k}} x_{i_{2}} \cdots x_{i_{k}}
$$

[^0]where $i \in[n]$. Let $x^{[r]}=\left(x_{1}^{r}, \ldots, x_{n}^{r}\right)^{\top}$. For some complex $\rho$, if there is a nonzero $n$-dimensional vector $x$ such that
$$
\mathcal{T} x=\rho x^{[k-1]},
$$
then $\rho$ is called an eigenvalue of $\mathcal{T}$, and $x$ an eigenvector of $\mathcal{T}$ corresponding to $\rho$, see [7, 8]. Let $\rho(\mathcal{T})$ be the largest modulus of the eigenvalues of $\mathcal{T}$.

Let $\mathcal{G}$ be a hypergraph with vertex set $V(\mathcal{G})=[n]$ and edge set $E(\mathcal{G})$, see [1]. If every edge of $\mathcal{G}$ has cardinality $k$, then we say that $\mathcal{G}$ is a $k$-uniform hypergraph. Throughout this paper, we consider $k$-uniform hypergraphs on $n$ vertices with $2 \leq k \leq n$. A uniform hypergraph is a hypergraph that is $k$-uniform for some $k$. For $i \in[n], E_{i}$ denotes the set of edges of $\mathcal{G}$ containing $i$. The degree of a vertex $i$ in $\mathcal{G}$ is defined as $d_{i}=\left|E_{i}\right|$. If $d_{i}=d$ for $i \in V(\mathcal{G})$, then $\mathcal{G}$ is called a regular hypergraph (of degree $d$ ). For $i, j \in V(\mathcal{G})$, if there is a sequence of edges $e_{1}, \ldots, e_{r}$ such that $i \in e_{1}, j \in e_{r}$ and $e_{s} \cap e_{s+1} \neq \emptyset$ for all $s \in[r-1]$, then we say that $i$ and $j$ are connected. A hypergraph is connected if every pair of different vertices of $\mathcal{G}$ is connected.

The adjacency tensor of a $k$-uniform hypergraph $\mathcal{G}$ on $n$ vertices is defined as the tensor $\mathcal{A}(\mathcal{G})$ of order $k$ and dimension $n$ whose $\left(i_{1} \ldots i_{k}\right)$-entry is

$$
A_{i_{1} \ldots i_{k}}= \begin{cases}\frac{1}{(k-1)!} & \text { if }\left\{i_{1}, \ldots, i_{k}\right\} \in E(\mathcal{G}) \\ 0 & \text { otherwise }\end{cases}
$$

Let $\mathcal{D}(\mathcal{G})$ be the diagonal tensor of order $k$ and dimension $n$ with its diagonal entry $D_{i \ldots i}$ the degree of vertex $i$ for $i \in[n]$. Then $\mathcal{Q}(\mathcal{G})=\mathcal{D}(\mathcal{G})+\mathcal{A}(\mathcal{G})$ is the signless Laplacian tensor of $\mathcal{G}$. We call $\rho(\mathcal{A}(\mathcal{G}))$ the (ordinary) spectral radius of $\mathcal{G}$, which is denoted by $\rho(\mathcal{G})$, and $\rho(\mathcal{Q}(\mathcal{G}))$ the signless Laplacian spectral radius of $\mathcal{G}$, which is denoted by $\mu(\mathcal{G})$.

For a nonnegative tensor $\mathcal{T}$ of order $k \geq 2$ and dimension $n$, the $i$-th row sum of $\mathcal{T}$ is $r_{i}(\mathcal{T})=\sum_{i_{2}, \ldots, i_{k} \in[n]} T_{i i_{2} \ldots i_{k}}$. If $r_{i}(\mathcal{T})>0$, then the $i$-th average 2-row sum of $\mathcal{T}$ is defined as

$$
m_{i}(\mathcal{T})=\frac{\sum_{i_{2}, \ldots, i_{k} \in[n]} T_{i i_{2} \ldots i_{k}} r_{i_{2}}(\mathcal{T}) \cdots r_{i_{k}}(\mathcal{T})}{r_{i}^{k-1}(\mathcal{T})}
$$

Let $\mathcal{G}$ be a $k$-uniform hypergraph on $n$ vertices. Let $\mathcal{A}=\mathcal{A}(\mathcal{G})$. For $i \in V(\mathcal{G})$ with $d_{i}>0$,

$$
\begin{aligned}
m_{i}(\mathcal{A}) & =\frac{\sum_{i_{2}, \ldots, i_{k} \in[n]} A_{i i_{2} \ldots i_{k}} r_{i_{2}}(\mathcal{A}) \cdots r_{i_{k}}(\mathcal{A})}{r_{i}^{k-1}(\mathcal{A})} \\
& =\frac{\sum_{\left\{i, i_{2}, \ldots, i_{k}\right\} \in E_{i}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}}
\end{aligned}
$$

which is called the average 2-degree of vertex $i$ of $\mathcal{G}$ (average of degrees of vertices in $E_{i}$ ) [12].

For a $k$-uniform hypergraph $\mathcal{G}$ with maximum degree $\Delta$, we know that $\rho(\mathcal{G}) \leq \Delta[2]$ and $\mu(\mathcal{G}) \leq 2 \Delta$ [8] with either equality when $\mathcal{G}$ is connected if
and only if $\mathcal{G}$ is regular (see [9]). Recently, upper bounds for $\rho(\mathcal{G})$ and $\mu(\mathcal{G})$ are given in [12] using degree sequence. In this note, we present sharp upper bounds for $\rho(\mathcal{G})$ and $\mu(\mathcal{G})$ using average 2-degrees or degrees, and we also give a lower bound for $\rho(\mathcal{G})$. We compare these bounds with known bounds by examples.

## 2 Preliminaries

A nonnegative tensor $\mathcal{T}$ of order $k \geq 2$ dimension $n$ is called weakly irreducible if the associated directed graph $D_{\mathcal{T}}$ of $\mathcal{T}$ is strongly connected, where $D_{\mathcal{T}}$ is the directed graph with vertex set $\{1, \ldots, n\}$ and arc set $\left\{(i, j): a_{i i_{2} \ldots i_{k}} \neq\right.$ 0 for some $i_{s}=j$ with $\left.s=2, \ldots, k\right\}$ [3, 8].

For an $n$-dimensional real vector $x$, let $\|x\|_{k}=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{k}\right)^{\frac{1}{k}}$, and if $\|x\|_{k}=$ 1 , then we say that $x$ is a unit vector. Let $\mathbb{R}_{+}^{n}$ be the set of $n$-dimensional nonnegative vectors.

Lemma 2.1. [3, 11] Let $\mathcal{T}$ be a nonnegative tensor. Then $\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$ and there is a unit nonnegative eigenvector corresponding to $\rho(\mathcal{T})$. If furthermore $\mathcal{T}$ is weakly irreducible, then there is a unique unit positive eigenvector corresponding to $\rho(\mathcal{T})$.

Lemma 2.2. [8] Let $\mathcal{G}$ be a $k$-uniform hypergraph with $n$ vertices. Then $\rho(\mathcal{G})=$ $\max \left\{x^{\top}(\mathcal{A}(\mathcal{G}) x): x \in \mathbb{R}_{+}^{n},\|x\|_{k}=1\right\}$.

Lemma 2.3. [6, 8] Let $\mathcal{G}$ be a k-uniform hypergraph. Then $\mathcal{A}(\mathcal{G})(\mathcal{Q}(G)$, respectively) is weakly irreducible if and only if $\mathcal{G}$ is connected.

A hypergraph $\mathcal{H}$ is a subhypergraph of $\mathcal{G}$ if $V(\mathcal{H}) \subseteq V(\mathcal{G})$ and $E(\mathcal{H}) \subseteq$ $E(\mathcal{G})$.

Lemma 2.4. [2, 4] Let $\mathcal{G}$ be a connected $k$-uniform hypergraph and $\mathcal{H}$ a subhypergraph of $\mathcal{G}$. Then $\rho(\mathcal{H}) \leq \rho(\mathcal{G})$ with equality if and only if $\mathcal{H}=\mathcal{G}$.

For two tensors $\mathcal{M}$ and $\mathcal{N}$ of order $k \geq 2$ and dimension $n$, if there is an $n \times n$ nonsingular diagonal matrix $U$ such that $\mathcal{N}=U^{-(k-1)} \mathcal{M} U$, then we say that $\mathcal{M}$ and $\mathcal{N}$ are diagonal similar.

Lemma 2.5. [10] Let $\mathcal{M}$ and $\mathcal{N}$ be two diagonal similar tensors of order $k \geq 2$ and dimension $n$. Then $\mathcal{M}$ and $\mathcal{N}$ have the same real eigenvalues.

Lemma 2.6. [5, 11] Let $\mathcal{T}$ be a nonnegative tensor of order $k \geq 2$ and dimension $n$. Then

$$
\min _{1 \leq i \leq n} r_{i}(\mathcal{T}) \leq \rho(\mathcal{T}) \leq \max _{1 \leq i \leq n} r_{i}(\mathcal{T})
$$

Moreover, if $\mathcal{T}$ is weakly irreducible, then either equality holds if and only if $r_{1}(\mathcal{T})=\cdots=r_{n}(\mathcal{T})$.

Proposition 2.1. Let $\mathcal{T}$ be a nonnegative tensor of order $k \geq 2$ and dimension $n$ with all row sums positive. Then

$$
\min _{1 \leq i \leq n} m_{i}(\mathcal{T}) \leq \rho(\mathcal{T}) \leq \max _{1 \leq i \leq n} m_{i}(\mathcal{T})
$$

Moreover, if $\mathcal{T}$ is weakly irreducible, then either equality holds if and only if $m_{1}(\mathcal{T})=\cdots=m_{n}(\mathcal{T})$.

Proof. Let $U=\operatorname{diag}\left(r_{1}(\mathcal{T}), \ldots, r_{n}(\mathcal{T})\right)$ and $\mathcal{B}=U^{-(k-1)} \mathcal{T} U$. Then $\mathcal{T}$ and $\mathcal{B}$ are diagonal similar, and thus we have by Lemma 2.5 that $\rho(\mathcal{T})=\rho(\mathcal{B})$. Obviously,

$$
B_{i_{1} \ldots i_{k}}=\frac{T_{i_{1} \ldots i_{k}} r_{i_{2}}(\mathcal{T}) \cdots r_{i_{k}}(\mathcal{T})}{r_{i_{1}}^{k-1}(\mathcal{T})}
$$

for $i_{1}, i_{2}, \ldots, i_{k} \in[n]$. Thus

$$
r_{i}(\mathcal{B})=\frac{\sum_{i_{2}, \ldots, i_{k} \in[n]} T_{i i_{2} \ldots i_{k}} r_{i_{2}}(\mathcal{T}) \cdots r_{i_{k}}(\mathcal{T})}{r_{i}^{k-1}(\mathcal{T})}=m_{i}(\mathcal{T})
$$

for $i \in[n]$. By Lemma [2.6, we have

$$
\min _{1 \leq i \leq n} m_{i}(\mathcal{T})=\min _{1 \leq i \leq n} r_{i}(\mathcal{B}) \leq \rho(\mathcal{T})=\rho(\mathcal{B}) \leq \max _{1 \leq i \leq n} r_{i}(\mathcal{B})=\max _{1 \leq i \leq n} m_{i}(\mathcal{T})
$$

and if $\mathcal{T}$ is weakly irreducible, then since $D_{\mathcal{T}}=D_{\mathcal{B}}, \mathcal{B}$ is also weakly irreducible, and thus $\rho(\mathcal{T})=\min _{1 \leq i \leq n} m_{i}(\mathcal{T})$ or $\rho(\mathcal{T})=\max _{1 \leq i \leq n} m_{i}(\mathcal{T})$ if and only if $r_{1}(\mathcal{B})=\cdots=r_{n}(\mathcal{B})$, i.e., $m_{1}(\mathcal{T})=\cdots=m_{n}(\mathcal{T})$.

For a hypergraph $\mathcal{G}$, the blow-up of $\mathcal{G}$, denoted by $\mathcal{G}^{1}$, is the hypergraph obtained from $\mathcal{G}$ by adding a new common vertex $v$ to each edge. If $\mathcal{G}$ is a regular ( $k-1$ )-uniform hypergraph on $n-1$ vertices of degree $d$, then $\mathcal{G}^{1}$ is a $k$-uniform hypergraph on $n$ vertices.

We use the techniques in [12].

## 3 Main results

Let $\mathcal{G}$ be a $k$-uniform hypergraph on $n$ vertices without isolated vertices with average 2-degrees $m_{1} \geq \cdots \geq m_{n}$. By Proposition [2.1, $\rho(\mathcal{G}) \leq m_{1}$. In the following, we give a upper bound for $\rho(\mathcal{G})$ using $m_{1}$ and $m_{2}$.

Theorem 3.1. Let $\mathcal{G}$ be a $k$-uniform hypergraph on $n$ vertices without isolated vertices with average 2 -degrees $m_{1} \geq \cdots \geq m_{n}$. Then

$$
\begin{equation*}
\rho(\mathcal{G}) \leq m_{1}^{\frac{1}{k}} m_{2}^{1-\frac{1}{k}} \tag{3.1}
\end{equation*}
$$

Moreover, if $\mathcal{G}$ is connected, then equality holds in (3.1) if and only if each vertex of $\mathcal{G}$ has the same average 2 -degree.

Proof. Let $\mathcal{A}=\mathcal{A}(\mathcal{G})$.
If $m_{1}=m_{2}$, then by Proposition 2.1, we have

$$
\rho(\mathcal{G}) \leq m_{1}^{\frac{1}{k}} m_{2}^{1-\frac{1}{k}}=m_{1},
$$

and when $\mathcal{G}$ is connected, $A$ is weakly irreducible, and thus equality holds in (3.1) if and only if each vertex of $\mathcal{G}$ has the same average 2-degree.

Suppose in the following that $m_{1}>m_{2}$. Let $d_{1}, d_{2}, \ldots, d_{n}$ be the degree sequence of $\mathcal{G}$. Let $U$ be diagonal matrix $\operatorname{diag}\left(t d_{1}, d_{2}, \ldots, d_{n}\right)$, where $t>1$ is a variable to be determined later. Let $\mathcal{T}=U^{-(k-1)} \mathcal{A} U$. Then $\mathcal{A}$ and $\mathcal{T}$ are diagonal similar. By Lemma 2.5, $\mathcal{A}$ and $\mathcal{T}$ have the same real eigenvalues. By Lemma 2.1, $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ and $\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$. Thus $\rho(\mathcal{G})=\rho(\mathcal{A})=\rho(\mathcal{T})$. Obviously,

$$
T_{i_{1} \ldots i_{k}}=U_{i_{1} i_{1}}^{-(k-1)} A_{i_{1} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}}
$$

for $i_{1}, \ldots, i_{k} \in[n]$. Then

$$
\begin{aligned}
r_{1}(\mathcal{T}) & =\sum_{i_{2}, \ldots, i_{k} \in[n]} T_{i i_{2} \ldots i_{k}} \\
& =\sum_{i_{2}, \ldots, i_{k} \in[n]} U_{11}^{-(k-1)} A_{1 i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}} \\
& =\sum_{i_{2}, \ldots, i_{k} \in[n] \backslash\{1\}}\left(t d_{1}\right)^{-(k-1)} A_{1 i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}} \\
& =\frac{\sum_{\left\{1, i_{2}, \ldots, i_{k}\right\} \in E_{1}} d_{i_{2}} \cdots d_{i_{k}}}{\left(t d_{1}\right)^{k-1}} \\
& =\frac{m_{1}}{t^{k-1}}
\end{aligned}
$$

For $i=2, \ldots, n$, let

$$
m_{1, i}=m_{i}-\frac{\sum_{1 \notin\left\{i, i_{2}, \ldots, i_{k}\right\} \in E_{i}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}}
$$

and then

$$
\begin{aligned}
& r_{i}(\mathcal{T})=\sum_{i_{2}, \ldots, i_{k} \in[n]} T_{i i_{2} \ldots i_{k}} \\
&=\sum_{i_{2}, \ldots, i_{k} \in[n]} U_{i i}^{-(k-1)} A_{i i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}} \\
&=\sum_{\substack{i_{2}, \ldots, i_{k} \in[n] \\
1 \in\left\{i_{2}, \ldots, i_{k}\right\}}} d_{i}^{-(k-1)} A_{i i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}}+\sum_{\substack{i_{2}, \ldots, i_{k} \in[n] \\
1 \notin\left\{i_{2}, \ldots, i_{k}\right\}}} d_{i}^{-(k-1)} A_{i i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}} \\
&=\frac{\sum_{\substack{i_{2}, \ldots, i_{k} \in[n] \\
1 \in\left\{i_{2}, \ldots, i_{k}\right\}}} A_{i i_{2} \ldots i_{k}} d_{i_{2}} d_{i_{3}} \cdots d_{i_{k}}}{d_{i}}+\frac{d_{i}^{k-1}, \ldots, i_{k} \in[n]}{1 \notin\left\{i_{2}, \ldots, i_{k}\right\}}< \\
& d_{i i_{2} \ldots i_{k}}^{k-1} d_{i_{2}} \cdots d_{i_{k}} \\
& d_{i}^{k-1} \\
&=\frac{\sum_{\left\{i, 1, i_{3}, \ldots, i_{k}\right\} \in E_{i}}\left(t d_{1}\right) d_{i_{3}} \cdots d_{i_{k}}}{d_{i}^{k-1}}+\frac{\sum_{1 \notin\left\{i, i_{2}, \ldots, i_{k}\right\} \in E_{i}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{t \sum_{\left\{i, 1, i_{3}, \ldots, i_{k}\right\} \in E_{i}} d_{1} d_{i_{3}} \cdots d_{i_{k}}}{d_{i}^{k-1}}+\frac{\sum_{1 \notin\left\{i, i_{2}, \ldots, i_{k}\right\} \in E_{i}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}} \\
& =\operatorname{tm}_{1, i}+m_{i}-m_{1, i} \\
& \leq t m_{i} \\
& \leq \operatorname{tm}_{2},
\end{aligned}
$$

with equality if and only if $m_{i}=m_{1, i}$ and $m_{i}=m_{2}$. Take $t=\left(\frac{m_{1}}{m_{2}}\right)^{\frac{1}{k}}$. Obviously, $t>1$. Then $r_{1}(\mathcal{T})=m_{1}^{\frac{1}{k}} m_{2}^{1-\frac{1}{k}}$ and for $2 \leq i \leq n, r_{i}(\mathcal{T}) \leq t m_{2}=$ $m_{1}^{\frac{1}{k}} m_{2}^{1-\frac{1}{k}}$. By Lemma 2.6,

$$
\rho(\mathcal{G})=\rho(\mathcal{T}) \leq m_{1}^{\frac{1}{k}} m_{2}^{1-\frac{1}{k}}
$$

Now suppose that $\mathcal{G}$ is connected. By Lemma 2.3, $\mathcal{A}(\mathcal{G})$ is weakly irreducible, and thus $\mathcal{T}$ is weakly irreducible since $D_{\mathcal{A}(\mathcal{G})}=D_{\mathcal{T}}$.

Suppose that equality holds in (3.1). By Lemma [2.6, $r_{1}(\mathcal{T})=\cdots=r_{n}(\mathcal{T})$. From the above argument, we have (i) $m_{2}=\cdots=m_{n}$, and (ii) $m_{i}=m_{1, i}$ for $i=2, \ldots, n$. From (ii) and the definition of $m_{1, i}$, each edge of $\mathcal{G}$ contains vertex 1 , which implies that $d_{1}(k-1)=\sum_{i=2}^{n} d_{i}$. From (i),

$$
m_{1}=\frac{\sum_{i=2}^{n} \frac{d_{i}^{k} m_{i}}{d_{1}}}{d_{1}^{k-1}(k-1)}=\frac{\sum_{i=2}^{n} d_{i}^{k} m_{2}}{d_{1}^{k}}=\frac{\sum_{i=2}^{n} d_{i} m_{2}\left(\frac{d_{i}}{d_{k}}\right)^{k-1}}{d_{1}(k-1)} \leq \frac{\sum_{i=2}^{n} m_{2} d_{i}}{d_{1}(k-1)}=m_{2}
$$

a contradiction. Thus the inequality (3.1) is strict if $m_{1}>m_{2}$.
Let $\mathcal{G}$ be a connected $k$-uniform hypergraph on $n$ vertices with degree sequence $d_{1} \geq \cdots \geq d_{n}$. Then [12]

$$
\begin{equation*}
\rho(\mathcal{G}) \leq d_{1}^{\frac{1}{k}} d_{2}^{1-\frac{1}{k}} \tag{3.2}
\end{equation*}
$$

with equality if and only if $\mathcal{G}$ is a regular hypergraph or the blow-up hypergraph $\mathcal{H}^{1}$ of a regular $(k-1)$-uniform hypergraph $\mathcal{H}$ on $n-1$ vertices.

Obviously, if $\mathcal{G}$ is a regular hypergraph, then each vertex of $\mathcal{G}$ has the same average 2-degree, and thus equality holds in (3.1) and (3.2). For the blow-up hypergraph $\mathcal{H}^{1}$ of a regular $(k-1)$-uniform hypergraph $\mathcal{H}$ on $n-1$ vertices, the upper bound in (3.2) is attained, while the upper bound in (3.1) is not attained. However, in the following, we give two examples to show that there are irregular hypergraphs for which each vertex has the same average 2-degree. For such hypergraphs, the upper in (3.1) is attained, while the upper bound in (3.2) can not be attained.

Let $\mathcal{H}_{1}$ be a 3 -uniform hypergraph with vertex set $V\left(\mathcal{H}_{1}\right)=[34]$ and
$E\left(\mathcal{H}_{1}\right)=\left\{e_{i}: 1 \leq i \leq 51\right\}$, where

$$
\begin{array}{rrrr}
e_{1}=\{1,2,5\}, & e_{2}=\{1,2,6\}, & e_{3}=\{1,2,7\}, & e_{4}=\{1,2,8\}, \\
e_{5}=\{1,2,9\}, & e_{6}=\{1,2,10\}, & e_{7}=\{1,2,11\}, & e_{8}=\{1,2,12\}, \\
e_{9}=\{1,2,13\}, & e_{10}=\{3,4,5\}, & e_{11}=\{3,4,6\}, & e_{12}=\{3,4,7\}, \\
e_{13}=\{3,4,8\}, & e_{14}=\{3,4,9\}, & e_{15}=\{3,4,10\}, & e_{16}=\{3,4,11\}, \\
e_{17}=\{3,4,12\}, & e_{18}=\{3,4,13\}, & e_{19}=\{5,6,14\}, & e_{20}=\{6,7,15\}, \\
e_{21}=\{7,8,16\}, & e_{22}=\{8,9,17\}, & e_{23}=\{9,10,18\}, & e_{24}=\{10,11,19\}, \\
e_{25}=\{11,12,20\}, & e_{26}=\{12,13,21\}, & e_{27}=\{13,5,22\}, & e_{28}=\{5,23,24\}, \\
e_{29}=\{5,25,26\}, & e_{30}=\{6,27,28\}, & e_{31}=\{6,29,30\}, & e_{32}=\{7,31,32\}, \\
e_{33}=\{7,33,34\}, & e_{34}=\{8,23,24\}, & e_{35}=\{8,25,26\}, & e_{36}=\{9,27,28\}, \\
e_{37}=\{9,29,30\}, & e_{38}=\{10,31,32\}, & e_{39}=\{10,33,34\}, & e_{40}=\{11,23,24\}, \\
e_{41}=\{11,25,26\}, & e_{42}=\{12,27,28\}, & e_{43}=\{12,29,30\}, & e_{44}=\{13,31,32\}, \\
e_{45}=\{13,33,34\}, & e_{46}=\{14,15,16\}, & e_{47}=\{17,18,19\}, & e_{48}=\{20,21,22\}, \\
e_{49}=\{14,17,20\}, & e_{50}=\{15,18,21\}, & e_{51}=\{16,19,22\}, &
\end{array}
$$

By direct calculation, we have

$$
d_{i}= \begin{cases}9 & \text { if } 1 \leq i \leq 4 \\ 6 & \text { if } 5 \leq i \leq 13 \\ 3 & \text { if } 14 \leq i \leq 34\end{cases}
$$

and

$$
\begin{aligned}
m_{i} & = \begin{cases}\frac{(9 \times 6) \times 9}{9 \times 9} & \text { if } 1 \leq i \leq 4 \\
\frac{(9 \times 9) \times 2+(6 \times 3) \times 2+(3 \times 3) \times 2}{6 \times 6} & \text { if } 5 \leq i \leq 13 \\
\frac{6 \times 6+(3 \times 3) \times 2}{3 \times 3} & \text { if } 14 \leq i \leq 22 \\
\frac{(6 \times 3) \times 3}{3 \times 3} & \text { if } 23 \leq i \leq 34\end{cases} \\
& =6 .
\end{aligned}
$$

By Theorem 3.1, we have $\rho\left(\mathcal{H}_{1}\right)=6$.
Let $\mathcal{H}_{2}$ be a 3-uniform hypergraph with vertex set $V\left(\mathcal{H}_{2}\right)=[54]$ and $E\left(\mathcal{H}_{2}\right)=\left\{e_{i}: 1 \leq i \leq 64\right\}$, where

$$
\begin{array}{rlrl}
e_{1}=\{1,2,3\}, & e_{2}=\{3,4,6\}, & & e_{3}=\{3,4,5\}, \\
e_{5}=\{1,2,6\}, & e_{6}=\{1,2,5\}, & e_{4}=\{4,5,6\}, \\
e_{9}=\{1,7,8\}, & e_{10}=\{1,8,9\}, & e_{11}=\{1,9,4\}, & e_{8}=\{2,5,6\}, \\
e_{13}=\{2,11,12\}, & e_{14}=\{2,12,13\}, & e_{15}=\{2,13,14\}, & e_{12}=\{1,10,11\}, \\
e_{17}=\{3,15,16\}, & e_{18}=\{3,16,17\}, & e_{19}=\{3,17,18\}, & e_{20}=\{3,14,15\}, \\
e_{21}=\{4,19,20\}, & e_{22}=\{4,20,21\}, & e_{23}=\{4,21,22\}, & e_{24}=\{4,22,23\}, \\
e_{25}=\{5,23,24\}, & e_{26}=\{5,24,25\}, & e_{27}=\{5,25,26\}, & e_{28}=\{5,26,27\}, \\
e_{29}=\{6,27,28\}, & e_{30}=\{6,28,29\}, & e_{31}=\{6,29,30\}, & e_{32}=\{6,30,7\}, \\
e_{33}=\{7,8,31\}, & e_{34}=\{7,8,32\}, & e_{35}=\{9,10,33\}, & e_{36}=\{9,10,34\}, \\
e_{37}=\{11,12,35\}, & e_{38}=\{11,12,36\}, & e_{39}=\{13,14,37\}, & e_{40}=\{13,14,38\}, \\
e_{41}=\{15,16,39\}, & e_{42}=\{15,16,40\}, & e_{43}=\{17,18,41\}, & e_{44}=\{17,18,42\}, \\
e_{45}=\{19,20,43\}, & e_{46}=\{19,20,44\}, & e_{47}=\{21,22,45\}, & e_{48}=\{21,22,46\}, \\
e_{49}=\{23,24,47\}, & e_{50}=\{23,24,48\}, & e_{51}=\{25,26,49\}, & e_{52}=\{25,26,50\}, \\
e_{53}=\{27,28,51\}, & e_{54}=\{27,28,52\}, & e_{55}=\{29,30,53\}, & e_{56}=\{29,30,54\}, \\
e_{57}=\{31,32,33\}, & e_{58}=\{34,35,36\}, & e_{59}=\{37,38,39\}, & e_{60}=\{40,41,42\}, \\
e_{61}=\{43,44,45\}, & e_{62}=\{46,47,48\}, & e_{63}=\{49,50,51\}, & e_{64}=\{52,53,54\} .
\end{array}
$$

By direct calculation, we have

$$
d_{i}= \begin{cases}8 & \text { if } 1 \leq i \leq 6 \\ 4 & \text { if } 7 \leq i \leq 30 \\ 2 & \text { if } 31 \leq i \leq 54\end{cases}
$$

and

$$
\begin{aligned}
m_{i} & = \begin{cases}\frac{(8 \times 8) \times 4+(4 \times 4) \times 4}{8 \times 8} & \text { if } 1 \leq i \leq 6, \\
\frac{(8 \times 4) \times 2+(4 \times 3) \times 2}{4 \times 4} & \text { if } 7 \leq i \leq 30, \\
\frac{4 \times 4+2 \times 2}{2 \times 2} & \text { if } 31 \leq i \leq 54\end{cases} \\
& =5 .
\end{aligned}
$$

By Theorem 3.1, we have $\rho\left(\mathcal{H}_{2}\right)=5$.
Let $\mathcal{G}$ be a $k$-uniform hypergraph of order $n$ without isolated vertices with maximum degree $\Delta$ and average 2-degrees $m_{1} \geq \cdots \geq m_{n}$. Note that $\mu(\mathcal{G}) \leq$ $\Delta+\rho(\mathcal{G})$. By Theorem 3.1, we have Then $\mu(\mathcal{G}) \leq m_{1}^{\frac{1}{k}} m_{2}^{1-\frac{1}{k}}+\Delta$.

If we take $U=\operatorname{diag}\left(d_{1}, \ldots, d_{n-1}, y d_{n}\right)$ with $y=\left(\frac{m_{n}}{m_{n-1}}\right)^{\frac{1}{k}}$ in the proof of Theorem 3.1, then $\rho(\mathcal{G}) \geq m_{n}^{\frac{1}{k}} m_{n-1}^{1-\frac{1}{k}}$, and if $\mathcal{G}$ is connected, then equality holds if and only if each vertex of $\mathcal{G}$ has the same average 2-degree.

For a $k$-uniform hypergraph $\mathcal{G}$, if there is a disjoint partition of $V(\mathcal{G})$ as $V(\mathcal{G})=V_{0} \cup V_{1} \cup \cdots \cup V_{d}$, where $\left|V_{0}\right|=1,\left|V_{1}\right|=\cdots=\left|V_{d}\right|=k-1$, and $E(\mathcal{G})=\left\{V_{0} \cup V_{i}: i \in[d]\right\}$, then $\mathcal{G}$ is called a hyperstar, denoted by $\mathcal{S}_{d}^{k}$. The vertex (of degree $d$ ) in $V_{0}$ is called the heart. Obviously, it is an isolated vertex if $d=0$.

For positive integers $d_{1}, \gamma$ and nonnegative integer $d_{2}$, let $\mathcal{G}_{d_{1}, d_{2}, \gamma}$ be the $k$ uniform hypergraph obtained vertex-disjoint $\mathcal{S}_{d_{1}}^{k}$ and $\mathcal{S}_{d_{2}}^{k}$ by adding $\gamma(k-2)$ new vertices $v_{1,1}, \ldots, v_{1, k-2}, \ldots, v_{\gamma, 1}, \ldots, v_{\gamma, k-2}$ and $\gamma$ new edges $e_{1}, \ldots, e_{\gamma}$, where $e_{i}=\left\{u, v, v_{i, 1}, \ldots, v_{i, k-2}\right\}$ for $i \in[\gamma]$, and $u, v$ are the hearts of $\mathcal{S}_{d_{1}}^{k}$ and $\mathcal{S}_{d_{2}}^{k}$, respectively. Obviously, if $d_{2}=0$ and $\gamma=1$, then $\mathcal{G}_{d_{1}, d_{2}, \gamma} \cong \mathcal{S}_{d_{1}+1}^{k}$.

Next we give a lower bound for $\rho(\mathcal{G})$ of a $k$-uniform hypergraph $\mathcal{G}$.
Theorem 3.2. Let $\mathcal{G}$ be a $k$-uniform hypergraph with $u \in V(\mathcal{G})$ of maximum degree $\Delta \geq 1$. Let $v$ be a neighbor of $u$ with maximum degree. Then

$$
\begin{equation*}
\rho(\mathcal{G}) \geq\left(\frac{\Delta+\delta-2 \gamma+\gamma^{2}+\sqrt{(\Delta-\delta)^{2}+\gamma^{4}+2(\Delta+\delta-2 \gamma) \gamma^{2}}}{2}\right)^{\frac{1}{k}} \tag{3.3}
\end{equation*}
$$

where $\delta$ is the degree of $v$, and $\gamma$ is the number of edges containing $u$ and v. Moreover, if $\mathcal{G}$ is connected, then equality holds in (3.3) if and only if $\mathcal{G} \cong \mathcal{G}_{\Delta, \delta, \gamma}$.

Proof. Let $e_{1}, \ldots, e_{\Delta}$ be the $\Delta$ edges of $\mathcal{G}$ containing $u$. Among these edges, $\gamma$ of them, say $e_{1}, \ldots, e_{\gamma}$, contain $v$. Let $e_{\Delta+1}, \ldots, e_{\Delta+\delta-\gamma}$ be the $\delta-\gamma$ edges of $\mathcal{G}$ containing $v$ different from $e_{1}, \ldots, e_{\gamma}$. Let $\mathcal{G}_{1}$ be the subhypergraph of $\mathcal{G}$
induced by $\left\{e_{1}, \ldots, e_{\Delta+\delta-\gamma}\right\}$. Then $V\left(\mathcal{G}_{1}\right)=\cup_{i=1}^{\Delta+\delta-\gamma} e_{i}$. For $1 \leq i \leq \Delta+\delta-\gamma$, let $e_{i}=\left\{v_{i, 1}, \ldots, v_{i, k}\right\}$, where $v_{i, 1}=u$ and $v_{i, 2}=v$ if $1 \leq i \leq \gamma, v_{i, 1}=u$ if $\gamma+1 \leq i \leq \Delta$, and $v_{i, 1}=v$ if $\Delta+1 \leq i \leq \Delta+\delta-\gamma$. Note that maybe some of $v_{i, s}$ and $v_{j, t}$ for $1 \leq s, t \leq k$ and $1 \leq i<j \leq \Delta+\delta-\gamma$ with $v_{i, s}, v_{j, t} \neq u, v$ represent the same vertex.

Let $\mathcal{G}_{1}^{\prime}$ be a new hypergraph such that $V\left(\mathcal{G}_{1}^{\prime}\right)=\cup_{i=1}^{\Delta+\delta-\gamma} e_{i}^{\prime}$ and $E\left(\mathcal{G}_{1}^{\prime}\right)=$ $\left\{e_{1}^{\prime}, \ldots, e_{\Delta+\delta-\gamma}^{\prime}\right\}$, where $e_{i}^{\prime}=\left\{v_{i, 1}^{\prime}, \ldots, v_{i, k}^{\prime}\right\}$ with $v_{i, 1}^{\prime}=u$ and $v_{i, 2}^{\prime}=v$ if $i=$ $1, \ldots, \gamma, v_{i, 1}^{\prime}=u$ if $\gamma+1 \leq i \leq \Delta$, and $v_{i, 1}^{\prime}=v$ if $\Delta+1 \leq i \leq \Delta+\delta-\gamma$. Note that $v \notin e_{i}$ for $\gamma+1 \leq i \leq \Delta, u \notin e_{i}$ for $\Delta+1 \leq i \leq \Delta+\delta-\gamma$, and $v_{i, s}^{\prime}$ and $v_{j, t}^{\prime}$ for $1 \leq s, t \leq k$ and $1 \leq i<j \leq \Delta+\delta-\gamma$ with $v_{i, s}^{\prime}, v_{j, t}^{\prime} \neq u, v$ are different vertices. Obviously, $\mathcal{G}_{1}^{\prime} \cong \mathcal{G}_{\Delta, \delta, \gamma}$

By Lemma 2.1, there is a unit positive eigenvector $x$ of $\mathcal{A}\left(\mathcal{G}_{1}^{\prime}\right)$ corresponding to $\rho\left(\mathcal{G}_{1}^{\prime}\right)$, in which the entry at $v_{i, s}^{\prime}$ is denoted by $x_{i, s}$, where $1 \leq i \leq \Delta+\delta-\gamma$ and $1 \leq s \leq k$. Then $\rho\left(\mathcal{G}_{1}^{\prime}\right)=x^{\top}\left(\mathcal{A}\left(\mathcal{G}_{1}^{\prime}\right) x\right)$. Let $w$ be any vertex of $\cup_{i=\Delta-\gamma+1}^{\Delta} e_{i} \backslash\{u\}$. Since $\rho\left(\mathcal{G}_{1}^{\prime}\right) x_{w}^{k-1}=x_{u} x_{w}^{k-2}$, we have $x_{w}=\frac{x_{u}}{\rho\left(\mathcal{G}_{1}^{\prime}\right)}$. Thus the entry of $x$ at each vertex of $\cup_{i=\Delta-\gamma+1}^{\Delta} e_{i} \backslash\{u\}$ is the same, denoted by $a$. Similarly, the entry of $x^{\prime}$ at each vertex of $\cup_{i=1}^{\gamma} e_{i} \backslash\{u, v\}$ is the same, denoted by $b$, and the entry of $x^{\prime}$ at each vertex of $\cup_{i=\Delta+1}^{\Delta+\delta-\gamma} e_{i} \backslash\{v\}$ is the same, denoted by $c$. Then

$$
\begin{aligned}
\rho\left(\mathcal{G}_{1}^{\prime}\right) a^{k-1} & =x_{u} a^{k-2} \\
\rho\left(\mathcal{G}_{1}^{\prime}\right) x_{u}^{k-1} & =(\Delta-\gamma) a^{k-1}+\gamma b^{k-2} x_{v} \\
\rho\left(\mathcal{G}_{1}^{\prime}\right) b^{k-1} & =x_{u} x_{v} b^{k-3} \\
\rho\left(\mathcal{G}_{1}^{\prime}\right) x_{v}^{k-1} & =(\delta-\gamma) c^{k-1}+\gamma b^{k-2} x_{u} \\
\rho\left(\mathcal{G}_{1}^{\prime}\right) c^{k-1} & =x_{v} c^{k-2}
\end{aligned}
$$

Thus $\rho\left(\mathcal{G}_{1}^{\prime}\right)$ is the largest root of the equation $f(\rho)=0$, where $f(\rho)=\left(\rho^{k}-\right.$ $\Delta+\gamma)\left(\rho^{2 k}-\left(\Delta+\delta-2 \gamma+\gamma^{2}\right) \rho^{k}+(\Delta-\gamma)(\delta-\gamma)\right)$. It follows that

$$
\rho\left(\mathcal{G}_{1}^{\prime}\right)=\left(\frac{\Delta+\delta-2 \gamma+\gamma^{2}+\sqrt{(\Delta-\delta)^{2}+\gamma^{4}+2(\Delta+\delta-2 \gamma) \gamma^{2}}}{2}\right)^{\frac{1}{k}}
$$

Construct a surjection $\sigma$ from $V\left(\mathcal{G}_{1}^{\prime}\right)$ to $V\left(\mathcal{G}_{1}\right)$ such that $\sigma\left(v_{i, s}^{\prime}\right)=v_{i, s}$ for $1 \leq i \leq \Delta+\delta-\gamma$ and $1 \leq s \leq k$. Let $y=\left(y_{1}, \ldots, y_{\left|V\left(\mathcal{G}_{1}\right)\right|}\right)^{\top}$ such that $y_{i}=\max _{v_{j, s}^{\prime} \in \sigma^{-1}(i)}\left\{x_{j, s}\right\}$ for $1 \leq i \leq\left|V\left(\mathcal{G}_{1}\right)\right|$. Obviously, $\|y\|_{k} \leq\|x\|_{k}=1$. Let $z=\frac{y}{\|y\|_{k}}$. Then $\|z\|_{k}=1$. By Lemma [2.2,

$$
\begin{equation*}
\rho\left(\mathcal{G}_{1}\right) \geq z^{\top}\left(\mathcal{A}\left(\mathcal{G}_{1}\right) z\right)=\frac{y^{\top}\left(\mathcal{A}\left(\mathcal{G}_{1}\right) y\right)}{\|y\|_{k}^{k}} \geq \frac{x^{\top}\left(\mathcal{A}\left(\mathcal{G}_{1}^{\prime}\right) x\right)}{\|x\|_{k}^{k}}=x^{\top}\left(\mathcal{A}\left(\mathcal{G}_{1}^{\prime}\right) x\right)=\rho\left(\mathcal{G}_{1}^{\prime}\right) \tag{3.4}
\end{equation*}
$$

Since $\mathcal{G}_{1}$ is a subhypergraph of $\mathcal{G}$, we have by Lemma 2.4 that $\rho(\mathcal{G}) \geq \rho\left(\mathcal{G}_{1}\right)$. Thus
$\rho(\mathcal{G}) \geq \rho\left(\mathcal{G}_{1}^{\prime}\right)=\left(\frac{\Delta+\delta-2 \gamma+\gamma^{2}+\sqrt{(\Delta-\delta)^{2}+\gamma^{4}+2(\Delta+\delta-2 \gamma) \gamma^{2}}}{2}\right)^{\frac{1}{k}}$.
If $\mathcal{G} \cong \mathcal{G}_{\Delta, \delta, \gamma}$, then by the above proof, equality holds in (3.3).

Suppose that $\mathcal{G}$ is connected and equality holds in (3.3). Then all equalities hold in (3.4) and $\rho(\mathcal{G})=\rho\left(\mathcal{G}_{1}\right)$. Thus by the construction of $\mathcal{G}_{1}^{\prime}$, we have $\mathcal{G}_{1} \cong$ $\mathcal{G}_{1}^{\prime}$. Otherwise, $\left|V\left(\mathcal{G}_{1}\right)\right|<\left|V\left(\mathcal{G}_{1}^{\prime}\right)\right|$, and then $\|y\|_{k}<\|x\|_{k}=1$, a contradiction. By Lemma 2.4, we have $\mathcal{G}=\mathcal{G}_{1}$. Thus $\mathcal{G} \cong \mathcal{G}_{\Delta, \delta, \gamma}$.

Let $\mathcal{G}$ be a $k$-uniform hypergraph with maximum degree $\Delta \geq 1$. Let $f(\Delta, \delta, \gamma)$ be the lower bound in (3.3). For $\gamma \leq \delta \leq \Delta, f(\Delta, \delta, \gamma)$ is a increasing function at $\delta$. Note that $\gamma \geq 1$. By Theorem 3.2, $\rho(\mathcal{G}) \geq f(\Delta, 1,1)=\Delta^{\frac{1}{k}}$, and if $\mathcal{G}$ is connected, then equality holds if and only if $\mathcal{G}$ is a hyperstar. Moreover, if $\mathcal{G}$ is connected and is not a hyperstar, then $\rho(\mathcal{G}) \geq f(\Delta, 2,1)=$ $\left(\frac{\Delta+1+\sqrt{\Delta^{2}-2 \Delta+5}}{2}\right)^{\frac{1}{k}}$ with equality if and only if $\mathcal{G} \cong \mathcal{G}_{\Delta, 2,1}$.

In the following, we give upper bounds for $\mu(\mathcal{G})$ of a $k$-uniform hypergraph.
Theorem 3.3. Let $\mathcal{G}$ be a $k$-uniform hypergraph on $n$ vertices with degree sequence $d_{1} \geq \cdots \geq d_{n}$. Let $d^{*}=1$ if $d_{1}=d_{2}$ and $d^{*}$ be a root of $h(t)=0$ in $\left(\left(\frac{d_{1}}{d_{2}}\right)^{\frac{1}{k}}, \frac{d_{1}}{d_{2}}\right)$ if $d_{1}>d_{2}$, where $h(t)=d_{2} t^{k}+\left(d_{2}-d_{1}\right) t^{k-1}-d_{1}$. Then

$$
\begin{equation*}
\mu(\mathcal{G}) \leq d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1} \tag{3.5}
\end{equation*}
$$

Moreover, if $\mathcal{G}$ is connected, then equality holds in (3.5) if and only if $\mathcal{G}$ is a regular hypergraph or the blow-up hypergraph of a regular $(k-1)$-uniform hypergraph on $n-1$ vertices.

Proof. Let $\mathcal{Q}=\mathcal{Q}(\mathcal{G}), \mathcal{A}=\mathcal{A}(\mathcal{G})$, and $\mathcal{D}=\mathcal{D}(\mathcal{G})$.
If $d_{1}=d_{2}$, then $d^{*}=1$, and by Lemma 2.6, we have

$$
\mu(\mathcal{G})=\rho(\mathcal{Q}) \leq \max _{1 \leq i \leq n} r_{i}(\mathcal{Q})=\max _{1 \leq i \leq n} 2 d_{i}=2 d_{1}=d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}
$$

and when $\mathcal{G}$ is connected, we have by Lemma 2.3 that $\mathcal{Q}$ is weakly irreducible, and thus equality holds if and only if $r_{1}(\mathcal{Q})=\cdots=r_{n}(\mathcal{Q})$, i.e., $\mathcal{G}$ is a regular hypergraph.

Suppose in the following that $d_{1}>d_{2}$. Let $U=\operatorname{diag}(t, 1, \ldots, 1)$ be an $n \times n$ diagonal matrix, where $t>1$ is a variable to be determined later. Let $\mathcal{T}=U^{-(k-1)} \mathcal{Q U}$. By Lemma 2.5, $\mathcal{Q}$ and $\mathcal{T}$ have the same real eigenvalues. By Lemma 2.1, $\rho(\mathcal{Q})$ is an eigenvalue of $\mathcal{Q}$ and $\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$. Thus $\mu(\mathcal{G})=\rho(\mathcal{Q})=\rho(\mathcal{T})$. We have

$$
\begin{aligned}
r_{1}(\mathcal{T}) & =\sum_{i_{2}, \ldots, i_{k} \in[n]} T_{1 i_{2} \ldots i_{k}} \\
& =\sum_{i_{2}, \ldots, i_{k} \in[n]} U_{11}^{-(k-1)} A_{1 i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}}+D_{1 \ldots 1} \\
& =\sum_{i_{2}, \ldots, i_{k} \in[n] \backslash\{1\}} \frac{1}{t^{k-1}} A_{1 i_{2} \ldots i_{k}}+d_{1} \\
& =\frac{d_{1}}{t^{k-1}}+d_{1} .
\end{aligned}
$$

For $i \in[n] \backslash\{1\}$, let $d_{1, i}=|\{e: 1, i \in e \in E(\mathcal{G})\}|$. Obviously, $d_{1, i} \leq d_{i}$. For $2 \leq i \leq n$, we have

$$
\begin{aligned}
r_{i}(\mathcal{T})= & \sum_{\substack{i_{2}, \ldots, i_{k} \in[n]}} T_{i i_{2} \ldots i_{k}} \\
= & \sum_{i_{2}, \ldots, i_{k} \in[n]} U_{i i}^{-(k-1)} A_{i i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}}+D_{i \ldots i} \\
= & \sum_{\substack{i_{2}, \ldots, i_{i} \in[n] \\
1 \in\left\{i_{2}, \ldots, i_{k}\right\}}} U_{i i}^{-(k-1)} A_{i i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}} \\
& +\sum_{\substack{\left.i_{2}, \ldots, i_{k} \in[n] \\
1 \notin i_{2}, \ldots, i_{k}\right\}}} U_{i i}^{-(k-1)} A_{i i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}}+d_{i} \\
= & \sum_{\substack{i_{2}, \ldots, i_{k} \in[n] \\
1 \in\left\{i_{2}, \ldots, i_{k}\right\}}} A_{i i_{2} \ldots i_{k}} t+\sum_{\substack{i_{2}, \ldots, i_{k} \in[n] \\
1 \notin\left\{i_{2}, \ldots, i_{k}\right\}}} A_{i i_{2} \ldots i_{k}}+d_{i} \\
= & t d_{1, i}+d_{i}-d_{1, i}+d_{i} \\
\leq & (t+1) d_{i} \\
\leq & (t+1) d_{2}
\end{aligned}
$$

with equality if and only if $d_{i}=d_{1, i}$ and $d_{i}=d_{2}$.
Note that $h\left(\left(\frac{d_{1}}{d_{2}}\right)^{\frac{1}{k}}\right)=\left(d_{2}-d_{1}\right)\left(\frac{d_{1}}{d_{2}}\right)^{1-\frac{1}{k}}<0$ and $h\left(\left(\frac{d_{1}}{d_{2}}\right)\right)=d_{1}\left(\left(\frac{d_{1}}{d_{2}}\right)^{k-2}-1\right)>$
0 . Thus $h(t)=0$ does have a root $d^{*}$ in $\left(\left(\frac{d_{1}}{d_{2}}\right)^{\frac{1}{k}}, \frac{d_{1}}{d_{2}}\right)$. Let $t=d^{*}$. Then $t>1$. We have

$$
r_{1}(\mathcal{T})=d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}
$$

and for $2 \leq i \leq n$,

$$
r_{i}(\mathcal{T}) \leq d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}
$$

Thus by Lemma 2.6,

$$
\mu(\mathcal{G})=\rho(\mathcal{T}) \leq \max _{1 \leq i \leq n} r_{i}(\mathcal{T})=d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}
$$

This proves (3.5).
Now suppose that $\mathcal{G}$ is connected. By Lemma 2.3, $\mathcal{Q}$ is weakly irreducible, and so is $\mathcal{T}$.

If equality holds in (3.5), then by Lemma 2.6, $r_{1}(\mathcal{T})=\cdots=r_{n}(\mathcal{T})$, and thus from the above arguments, we have $d_{1, i}=d_{i}$ for $i=2, \ldots, n$ (implying that each edge of $\mathcal{G}$ contains vertex 1 ), $d_{2}=\cdots=d_{n}$, and thus $\mathcal{G}$ is a blow-up hypergraph of a regular $(k-1)$-uniform hypergraph on $n-1$ vertices of degree $d_{2}$.

Conversely, if $\mathcal{G}=\mathcal{H}^{1}$, where $\mathcal{H}$ is a regular $(k-1)$-uniform hypergraph on $n-1$ vertices of degree $d_{2}$, then by the above arguments, we have $r_{i}(\mathcal{T})=$ $d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}$ for $1 \leq i \leq n$, and thus by Lemma 2.6, $\mu(\mathcal{G})=\rho(\mathcal{Q})=\rho(\mathcal{T})=$ $d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}$.

Let $\mathcal{G}$ be a $k$-uniform hypergraph on $n$ vertices with degree sequence $d_{1} \geq$ $\cdots \geq d_{n}$. If $d_{1}>d_{2}$, then $d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}<d_{1}+d_{1}\left(\frac{d_{2}}{d_{1}}\right)^{1-\frac{1}{k}}=d_{1}+d_{1}^{\frac{1}{k}} d_{2}^{1-\frac{1}{k}}$. By Theorem [3.3, we have

$$
\mu(\mathcal{G}) \leq d_{1}+d_{1}^{\frac{1}{k}} d_{2}^{1-\frac{1}{k}},
$$

and if $\mathcal{G}$ is connected, then equality holds if and only if $\mathcal{G}$ is a regular hypergraph, see [12].

Theorem 3.4. Let $\mathcal{G}$ be a $k$-uniform hypergraph on $n$ vertices without isolated vertices with average 2 -degrees $m_{1} \geq \cdots \geq m_{n}$, and degree sequence $d_{1}, \ldots, d_{n}$. Then

$$
\begin{equation*}
\mu(\mathcal{G}) \leq \min _{1 \leq j \leq n} \max \left\{m_{1}^{\frac{1}{k}} m_{j}^{1-\frac{1}{k}}+d_{1}, \theta_{j}\right\} \tag{3.6}
\end{equation*}
$$

where

$$
\theta_{j}=\max \left\{m_{1}^{\frac{1}{k}} m_{i} m_{j}^{-\frac{1}{k}}+d_{i}: 2 \leq i \leq n\right\}
$$

Proof. Let $\mathcal{Q}=\mathcal{Q}(\mathcal{G})$. Let $U$ be a diagonal matrix $\operatorname{diag}\left(t d_{1}, d_{2}, \ldots, d_{n}\right)$, where $t \geq 1$ is a variable to be determined later. Let $\mathcal{T}=U^{-(k-1)} \mathcal{Q} U$. Then $\mathcal{Q}$ and $\mathcal{T}$ are diagonal similar. By Lemma 2.5, $\mu(\mathcal{G})=\rho(\mathcal{T})$. Obviously,

$$
T_{i_{1} \ldots i_{k}}=U_{i_{1} i_{1}}^{-(k-1)} Q_{i_{1} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}} .
$$

for $i_{1}, \ldots, i_{k} \in[n]$. Then

$$
\begin{aligned}
r_{1}(\mathcal{T}) & =\sum_{i_{2}, \ldots, i_{k} \in[n]} U_{11}^{-(k-1)} Q_{1 i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}} \\
& =\frac{\sum_{i_{2}, \ldots, i_{k} \in[n] \backslash\{1\}} Q_{1 i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}}}{\left(t d_{1}\right)^{k-1}} \\
& =\frac{D_{1 \ldots 1}\left(t d_{1}\right)^{k-1}}{\left(t d_{1}\right)^{k-1}}+\frac{\sum_{i_{2}, \ldots, i_{k} \in[n]} A_{1 i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}}}{\left(t d_{1}\right)^{k-1}} \\
& =d_{1}+\frac{\sum_{\left\{1, i_{2}, \ldots, i_{k}\right\} \in E_{1}} d_{i_{2}} \cdots d_{i_{k}}}{\left(t d_{1}\right)^{k-1}} \\
& =d_{1}+\frac{m_{1}}{t^{k-1}} .
\end{aligned}
$$

For $i=2, \ldots, n$, let

$$
m_{1, i}=m_{i}-\frac{\sum_{1 \notin\left\{i, i_{2}, \ldots, i_{k}\right\} \in E_{i}} d_{i_{2}} \ldots d_{i_{k}}}{d_{i}^{k-1}}
$$

and then

$$
\begin{aligned}
r_{i}(\mathcal{T}) & =\sum_{\substack{i_{2}, \ldots, i_{k} \in[n]}} U_{i i}^{-(k-1)} Q_{i i_{2} \ldots i_{k}} U_{i_{2} i_{2}} \cdots U_{i_{k} i_{k}} \\
& =\sum_{\substack{i_{2}, \ldots, i_{i} \in[n] \\
1 \in\left\{i_{2}, \ldots, i_{k}\right\}}} d_{i}^{-(k-1)} Q_{i i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{\substack{\left.i_{2}, \ldots, i_{k} \in[n] \\
1 \notin i_{2}, \ldots, i_{k}\right\}}} d_{i}^{-(k-1)} Q_{i i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}} \\
= & \frac{D_{i \ldots . .} d_{i}^{k-1}}{d_{i}^{k-1}}+\frac{\sum_{\substack{i_{2}, \ldots, i_{k} \in[n] \\
1 \in\left\{i_{2}, \ldots, i_{k}\right\}}} A_{i i_{2} \ldots i_{k}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}} \\
& +\frac{\sum_{\substack{i_{2}, \ldots, i_{k} \in[n] \\
1 \notin\left\{i_{2}, \ldots, i_{k}\right\}}} A_{i i_{2} \ldots i_{k}} d_{i_{2}} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}} \\
= & d_{i}+\frac{\sum_{\left\{i, 1, i_{3}, \ldots, i_{k}\right\} \in E_{i}}\left(t d_{1}\right) d_{i_{3}} \cdots d_{i_{k}}}{d_{i}^{k-1}} \\
& +\frac{\sum_{1 \notin\left\{i, i_{2}, \ldots, i_{k}\right\} \in E_{i}}^{k-1} d_{i_{2}} \cdots d_{i_{k}}}{d_{i}^{k-1}} \\
= & d_{i}+t m_{1, i}+m_{i}-m_{1, i} \\
\leq & d_{i}+t m_{i} .
\end{aligned}
$$

For an arbitrary fixed $j$ with $1 \leq j \leq n$, let $t=\left(\frac{m_{1}}{m_{j}}\right)^{\frac{1}{k}}$. Obviously, $t \geq 1$. Then

$$
r_{1}(\mathcal{T})=m_{1}^{\frac{1}{k}} m_{j}^{1-\frac{1}{k}}+d_{1}
$$

for $2 \leq i \leq n$,

$$
r_{i}(\mathcal{T}) \leq t m_{i}+d_{i}=m_{1}^{\frac{1}{k}} m_{i} m_{j}^{-\frac{1}{k}}+d_{i}
$$

Let $\theta_{j}=\max \left\{m_{1}^{\frac{1}{k}} m_{i} m_{j}^{-\frac{1}{k}}+d_{i}: 2 \leq i \leq n\right\}$. Thus for $1 \leq i \leq n$, we have

$$
r_{i}(\mathcal{T}) \leq \max \left\{m_{1}^{\frac{1}{k}} m_{j}^{1-\frac{1}{k}}+d_{1}, \theta_{j}\right\}
$$

Thus

$$
r_{i}(\mathcal{T}) \leq \min _{1 \leq j \leq n} \max \left\{m_{1}^{\frac{1}{k}} m_{j}^{1-\frac{1}{k}}+d_{1}, \theta_{j}\right\}
$$

Now the result follows from Lemma 2.6.
If we take $U=\operatorname{diag}\left(d_{1}, \ldots, d_{n-1}, y d_{n}\right)$ with $y=\left(\frac{m_{n}}{m_{j}}\right)^{\frac{1}{k}}$ for an arbitrary fixed $j$ in the above proof, then we have

$$
\mu(\mathcal{G}) \geq \max _{1 \leq j \leq n} \min \left\{m_{n}^{\frac{1}{k}} m_{j}^{1-\frac{1}{k}}+d_{n}, \gamma_{j}\right\}
$$

where $\gamma_{j}=\min \left\{m_{n}^{\frac{1}{k}} m_{i} m_{j}^{-\frac{1}{k}}+d_{i}: 2 \leq i \leq n\right\}$ for $1 \leq j \leq n$.
Consider 4-uniform hypergraph $\mathcal{G}_{1}$ with vertex set [25] and edge set $E\left(\mathcal{G}_{1}\right)=$ $\left\{e_{1}, \ldots, e_{14}\right\}$, where

$$
\begin{array}{ccc}
e_{1}=\{1,2,3,4\}, & e_{2}=\{5,6,7,8\}, & e_{3}=\{9,10,11,12\}, \\
e_{4}=\{13,14,15,16\}, & e_{5}=\{17,18,19,20\}, & e_{6}=\{21,22,23,24\}, \\
e_{7}=\{1,2,3,25\}, & e_{8}=\{4,5,6,25\}, & e_{9}=\{7,8,9,25\}, \\
e_{10}=\{10,11,12,25\}, & e_{11}=\{13,14,15,25\}, & e_{12}=\{16,17,18,25\} \\
e_{13}=\{19,20,21,25\}, & e_{14}=\{22,23,24,25\} . &
\end{array}
$$

In notation of Theorem 3.4, we have

$$
d_{1}=\cdots=d_{24}=2, d_{25}=8
$$

and

$$
m_{1}=\cdots=m_{24}=5, m_{25}=0.125
$$

implying that $\theta_{1}=\cdots=\theta_{24}=14.5743, \theta_{25} \approx 8.31436$, and $m_{1}^{\frac{1}{4}} m_{j}^{\frac{3}{4}}+d_{1}=7$ for $1 \leq j \leq 24$ and $m_{1}^{\frac{1}{4}} m_{25}^{\frac{3}{4}}+d_{1} \approx 2.31436$. Thus $\mu\left(\mathcal{G}_{1}\right) \leq 8.125$. Note that $8=d_{1}>d_{2}=\cdots=d_{25}=2$ in notation of Theorem 3.3 and that $h\left(d^{*}\right)=d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}$ is a decreasing function for $d^{*} \in\left(\left(\frac{d_{1}}{d_{2}}\right)^{\frac{1}{k}}, \frac{d_{1}}{d_{2}}\right)$. Then

$$
d_{1}+d_{1}\left(\frac{1}{d^{*}}\right)^{k-1}>d_{1}+d_{1}\left(\frac{d_{2}}{d_{1}}\right)^{k-1}=d_{1}+\frac{d_{2}^{k-1}}{d_{1}^{k-2}}=8+\frac{2^{3}}{8^{2}}=8.125 .
$$

For $\mathcal{G}_{1}$, the upper bound in (3.6) is smaller than the one in (3.5). Obviously, the blow-up hypergraph of a regular $(k-1)$-uniform hypergraph on $n-1$ vertices, the upper bound in (3.5) is smaller than the one in (3.6).

For a $k$-uniform hypergraph $\mathcal{G}$, let $d_{1} \geq \cdots \geq d_{n}$ be the degree sequence of $\mathcal{G}$ and $m_{1}, \ldots, m_{n}$ be the average 2-degrees of $\mathcal{G}$. In [12], the following upper bounds for $\mu(\mathcal{G})$ are given.

$$
\begin{gather*}
\mu(\mathcal{G}) \leq \max _{e \in E(\mathcal{G})} \max _{\{i, j\} \in e}\left(d_{i}+d_{j}\right)  \tag{3.7}\\
\mu(\mathcal{G}) \leq \max _{e \in E(\mathcal{G})} \max _{\{i, j\} \in e} \frac{d_{i}+d_{j}+\sqrt{\left(d_{i}-d_{j}\right)^{2}+4 m_{i} m_{j}}}{2} \tag{3.8}
\end{gather*}
$$

Consider 3-uniform hypergraph $\mathcal{G}_{2}$ with vertex set [9] and edge set $E\left(\mathcal{G}_{2}\right)=$ $\left\{e_{1}, \ldots, e_{4}\right\}$, where

$$
e_{1}=\{1,2,9\}, e_{2}=\{3,4,8\}, e_{3}=\{5,6,7\}, e_{4}=\{7,8,9\}
$$

In notation of Theorem 3.4, we have

$$
d_{1}=\cdots=d_{6}=1, d_{7}=d_{8}=d_{9}=2
$$

and

$$
m_{1}=\cdots=m_{6}=2, m_{7}=m_{8}=m_{9}=\frac{5}{4}
$$

implying that $\theta_{1}=\cdots=\theta_{6}=3.25, \theta_{7}=\theta_{8}=\theta_{9} \approx 3.462$, and $m_{1}^{\frac{1}{3}} m_{j}^{\frac{2}{3}}+d_{1}=3$ when $1 \leq j \leq 6$ and $m_{1}^{\frac{1}{3}} m_{j}^{\frac{2}{3}}+d_{1} \approx 2.462$ when $7 \leq j \leq 9$, and thus $\mu\left(\mathcal{G}_{2}\right) \leq$ 3.25. By direct calculation, the bounds in (3.7) and (3.8) are 4 and 3.25, respectively. For $\mathcal{G}_{2}$, the upper bound in (3.6) is smaller than the upper bound in (3.7).

Consider 4-uniform hypergraph $\mathcal{G}_{3}$ with vertex set [7] and edge set $E\left(\mathcal{G}_{3}\right)=$ $\left\{e_{1}, \ldots, e_{8}\right\}$, where

$$
\begin{array}{lll}
e_{1}=\{1,2,3,4\}, & e_{2}=\{1,5,6,7\}, & e_{3}=\{2,3,4,5\}, \\
e_{4}=\{3,4,5,6\}, & e_{5}=\{4,5,6,7\}, & e_{6}=\{5,6,7,2\}, \\
e_{7}=\{6,7,2,3\}, & e_{8}=\{7,2,3,4\} . &
\end{array}
$$

In notation of Theorem 3.4, we have

$$
d_{1}=2, d_{2}=\cdots=d_{7}=5,
$$

and

$$
m_{1}=31.25, m_{2}=\cdots=m_{7}=4.4
$$

implying that $\theta_{1}=9.4$, and $\theta_{2}=\cdots=\theta_{7} \approx 12.18294$, and $m_{1}^{\frac{1}{4}} m_{j}^{\frac{3}{4}}+d_{1}=33.25$ for $j=1$ and $m_{1}^{\frac{1}{4}} m_{j}^{\frac{3}{4}}+d_{1} \approx 9.18294$ for $2 \leq j \leq 7$. Thus $\mu\left(\mathcal{G}_{3}\right) \leq 12.18294$. It is easily seen that the bounds in (3.7) and (3.8) are 10 and 15.32159, respectively. For $\mathcal{G}_{3}$, the upper bound in Theorem 3.4 is smaller than the one in (3.8) but larger than the one in (3.7).

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## References

[1] C. Berge, Hypergraphs: Combinatorics of Finite Sets, North-Holland, Amsterdam, 1989.
[2] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl. 436 (2012) 3268-3292.
[3] S. Friedland, S. Gaubert, L. Han, Perron-Frobenius theorems for nonnegative multilinear forms and extension, Linear Algebra Appl. 438 (2013) 738-749.
[4] M. Khan, Y.Z. Fan, On the spectral radius of a class of non-odd-bipartite even uniform hypergraphs, Linear Algebra Appl. 480 (2015) 93-106.
[5] C. Li, Z. Chen, Y. Li, A new eigenvalue inclusion set for tensors and its applications, Linear Algebra Appl. 481 (2015) 36-53.
[6] K. Pearson, T. Zhang, On spectral hypergraph theory of the adjacency tensor, Graphs Combin. 30 (2014) 1233-1248.
[7] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbol Comput. 40 (2005) 1302-1324.
[8] L. Qi, $H^{+}$-eigenvalues of Laplacian and signless Laplacian tensors, Commun. Math. Sci. 12 (2014) 1045-1064.
[9] L. Qi, J.-Y. Shao, Q. Wang, Regular uniform hypergraphs, s-cycles, spaths and their largest Laplacian eigenvalues, Linear Algebra Appl. 443 (2014) 215-227.
[10] J.-Y. Shao, A general product of tensors with applications, Linear Algebra Appl. 439 (2012) 2350-2366.
[11] Y. Yang, Q. Yang, Further results for Perron-Frobenius theorem for nonegative tensors, SIAM J. Matrix Anal. Appl. 31 (2010) 2517-2530.
[12] X. Yuan, M. Zhang, M. Lu, Some upper bounds on the eigenvalues of uniform hypergraphs, Linear Algebra Appl. 484 (2015) 540-549.


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