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# Bivariate orthogonal polynomials, 2D Toda lattices and Lax-type pairs[☆]

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ABSTRACT

We explore the connection between an infinite system of particles in $\mathbb{R}^2$ described by a bi-dimensional version of the Toda equations with the theory of orthogonal polynomials in two variables. We define a 2D Toda lattice in the sense that we consider only one time variable and two space variables describing a mesh of interacting particles over the plane. We show that this 2D Toda lattice is related with the matrix coefficients of the three term relations of bivariate orthogonal polynomials associated with an exponential modification of a positive measure. Moreover, *block Lax pairs* for 2D Toda lattices are deduced.



## 1. Introduction

Oscillations of an infinite system of points joined by spring masses, where the interaction is an exponential function of the distance between two consecutive particles, are described by the so-called *Toda equations* [14,15]. An explicit solution of the Toda lattice equations in one time variable can be deduced by using orthogonal polynomials associated to an exponential modification of a positive measure (see, for instance, [7,10]). *Lax pairs* [8] of matrices associated with the coefficients of the three term recurrence relation for orthonormal and non-orthonormal polynomials can be deduced. Moreover, orthogonal polynomials associated with symmetric measures are related with the so-called *Langmuir lattice*, *Volterra lattice* or *finite difference Korteweg–de Vries equation*.

Several modifications (or perturbations) of Toda equations have been studied during years. In particular, extensions of Toda equations with two-dimensional discrete variables and one temporal variable were considered in [11], and later the relations with 2D Lotka–Volterra equation were considered in [5]. In [2] the authors consider discrete and continuous deformations of a measure, also the multivariate orthogonal polynomials and study the resulting integrable systems. In fact, they consider continuous Toda deformations of the measure by using $d$-dimensional vectors and covectors of several time variables, and relate them with Christoffel perturbations of the measure. A similar work of these authors about multispectral Toda hierarchy can be found in [3], where they deal with two continuous time parameter sequences.

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Recently, Aptekarev et al. [1] have presented multidimensional analogues of continuous and discrete-time Toda lattices, where they have considered integrable systems with two or more space coordinates, and relate them with multiple orthogonal polynomials.

For more information and references about Toda lattice we refer to [1–3]. These references contain complete introduction about this topic, and a complete set of references related with Toda equations and Lax pairs can be found therein.

In this paper, we consider a continuous Toda lattice in only one time variable $t$ and two space variables describing a mesh of interacting points over the plane. We prove that this kind of extension for Toda equations can be related with bivariate orthogonal polynomials associated with an exponential modification of a positive measure by using their matrix three term relations. In fact, using the vector representation for bivariate polynomials, the so-called *orthogonal polynomial systems*, we deduce matrix Toda-type equations for the matrix coefficients of the three term relations for bivariate polynomials. In addition, we deduce a Lax-type pair related with the obtained 2D Toda equations. Moreover, the case when the bivariate measure is centrally symmetric is considered, and Langmuir equations are deduced.

Since the coefficients of the three term relations are matrices of increasing size, our Toda-type equations are also given in terms of matrices of increasing size. Moreover, the product of matrices is non-commutative, then it is necessary to take into account the size of the involving matrices. Therefore, the matrix manipulation is not a trivial extension of the univariate case.

This paper is structured as follows. Section 2 is devoted to recall the basic facts about the continuous classical Toda equations and their relations with the coefficients of the three term recurrence relation for standard orthogonal polynomials associated with an exponential modification of a positive measure. Moreover, we review Lax and Lax–Nakamura pairs for continuous Toda equations.

Some basic theory about bivariate orthogonal polynomials is described in Section 3. Given a positive measure defined over a domain in $\mathbb{R}^2$, we define an *Orthogonal Polynomial System* as an orthogonal polynomial sequence organized as vectors of increasing size $n+1$ such that their entries are independent bivariate polynomials of total degree $n$. This section ends with the description of the matrix three term relations for bivariate orthogonal polynomials, the key feature of the rest of the paper.

In the next section we establish the continuous bivariate Toda equations depending on a single time-variable, and we study their relations with standard bivariate orthogonal polynomials. In particular, we deduce the matrix 2D Toda equations from an exponential modification of the original measure. In the centrally symmetric case, by using similar tools, we obtain an analogue of the Langmuir lattice. Toda equations for orthonormal polynomials are deduced as well.

We devote Section 5 to the matrix analogue of the Lax pair associated with the bivariate Toda lattice. Using a block matrix formulation, a Lax–Nakamura-type pair, that is, a Lax pair for non-orthonormal polynomials is deduced for the bivariate polynomials and for the centrally symmetric case. In the orthonormal case, due of the non-commutativity of the product of matrices, we deduce a perturbed matrix Lax-type pair.

In the last section we describe in detail the particular case when the bivariate positive measure is given by the tensor product of two univariate positive measures. We also show how the bivariate Toda equations recover classical Toda equations for the univariate measures.

## 2. Toda lattices and orthogonal polynomials in one variable

As it is well known, continuous Toda lattices and orthogonal polynomials can be related. Toda lattice equations describe the oscillations of an infinite system of particles $x_n$, $n \geq 0$, joined by spring masses, where the interaction is described by the exponential of the distance between two masses as follows:

$$\ddot{x}_n = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \quad n \geq 1,$$

where we use the standard notation $\dot{y}(t) = dy(t)/dt$.

Following for instance Suris [13], the above equation can be transformed into a system of differential equations by taking

$$d_n(t) = \dot{x}_n, \qquad c_n(t) = e^{x_{n-1}-x_n}, \qquad n \geq 1,$$

and then, setting $c_0(t) = d_{-1}(t) = 0$, we obtain, for $n \geq 0$,

$$\dot{d}_n(t) = c_n(t) - c_{n+1}(t), \tag{1}$$

$$\dot{c}_n(t) = c_n(t)\,[d_{n-1}(t) - d_n(t)]. \tag{2}$$

The coefficients of the three term recurrence relations for a special family of orthogonal polynomials constitute an explicit solution for the system (1) and (2) (see, for instance, [7,10]). In fact,

**Theorem 1** [7,10]. *Let $d\mu(x)$ be a real positive measure with finite moments, and let $\{P_n(x)\}_{n\geq 0}$ be the monic orthogonal polynomial sequence (MOPS, in short) associated with $d\mu(x)$. Let $c_n$, $d_n$ be the coefficients of the three term recurrence relation, that is,*

$$P_{n+1}(x) = (x - d_n)\,P_n(x) - c_n\,P_{n-1}(x), \quad n \geq 0, \tag{3}$$

*where* $P_{-1}(x) = 0,\ P_0(x) = 1,$ *and*

$$d_n = \frac{\int_{\mathbb{R}} x\, P_n(x)^2\, d\mu(x)}{\int_{\mathbb{R}} P_n(x)^2\, d\mu(x)}, \quad n \ge 0, \quad c_n = \frac{\int_{\mathbb{R}} P_n(x)^2\, d\mu(x)}{\int_{\mathbb{R}} P_{n-1}(x)^2\, d\mu(x)}, \quad n \ge 1.$$

*Suppose that the moments*

$$\int_{\mathbb{R}} x^n\, e^{-xt}\, d\mu(x)$$

*exist for* $n \ge 0$, *and let* $\{P_n(x, t)\}_{n \ge 0}$ *be the MOPS associated with* $d\tilde{\mu}(x,t) = e^{-xt}\, d\mu(x)$. *Let* $c_n(t)$, $d_n(t)$ *be the coefficients of the three term recurrence relation*

$$P_{n+1}(x,t) = (x - d_n(t))\, P_n(x,t) - c_n(t)\, P_{n-1}(x,t), \quad n \ge 0, \tag{4}$$

*where* $P_{-1}(x,t) = 0,\ P_0(x,t) = 1.$

*Then, the coefficients* $c_n(t)$, $d_n(t)$ *satisfy the system (1) and (2), with initial conditions* $c_n(0) = c_n,\ d_n(0) = d_n$. *Moreover, the correspondence is unique.*

If we suppose that the positive measure $d\mu(x)$ is symmetric, then $d_n = 0$, for $n \ge 0$, and the modified measure is given by $d\tilde{\mu}(x,t) = e^{-x^2 t}\, d\mu(x)$. Then, three term recurrence relation (4) becomes

$$P_{n+1}(x,t) = x\, P_n(x,t) - c_n(t)\, P_{n-1}(x,t), \quad n \ge 0.$$

In this case, the coefficients $c_n(t)$ satisfy the *Langmuir lattice* (see [10])

$$\dot{c}_n(t) = c_n(t)[c_{n-1}(t) - c_{n+1}(t)]. \tag{5}$$

### 2.1. Toda lattices and orthonormal polynomials

Let $\{Q_n(x, t)\}_{n \ge 0}$ denote the *sequence of orthonormal polynomials* with respect to the measure $d\tilde{\mu}(x,t) = e^{-xt} d\mu(x)$. It is well known that

$$Q_n(x,t) = P_n(x,t)\, h_n(t)^{-1/2}, \quad \text{with} \quad h_n(t) = \int_{\mathbb{R}} P_n(x,t)^2\, e^{-xt}\, d\mu(x).$$

From (3), three term recurrence relation for orthonormal polynomials is given by

$$x\, Q_n(x,t) = a_n(t)\, Q_{n+1}(x,t) + d_n(t)\, Q_n(x,t) + a_{n-1}(t)\, Q_{n-1}(x,t), \quad n \ge 0,$$

with $Q_{-1}(x,t) = 0,\ Q_0(x,t) = h_0(t)^{-1/2}$, and $a_n(t) = c_{n+1}(t)^{-1/2}$.

Setting $a_{-1}(t) = 0$, Toda equations (1) and (2) become

$$\dot{d}_n(t) = a_{n-1}^2(t) - a_n^2(t), \tag{6}$$

$$\dot{a}_n(t) = \frac{a_n(t)}{2}\, [d_n(t) - d_{n+1}(t)], \tag{7}$$

for $n \ge 0$. We refer to the lattice given by (6) and (7) as *Toda lattice for univariate orthonormal polynomials*.

When the positive measure is symmetric, then Langmuir lattice (5) transforms into

$$\dot{a}_n(t) = \frac{a_n(t)}{2}[a_{n-1}^2(t) - a_{n+1}^2(t)], \quad n \ge 0. \tag{8}$$

### 2.2. Lax pairs for orthonormal polynomials

Usually, Toda equations for orthonormal polynomials are represented by means of a *Lax pair*. Here, we describe this construction.

Let us define the infinite tridiagonal matrices

$$\mathcal{L} = \begin{pmatrix} d_0 & a_0 & & & & \\ a_0 & d_1 & a_1 & & & \\ & a_1 & d_2 & \ddots & & \\ & & \ddots & \ddots & a_{n-1} & \\ & & & a_{n-1} & d_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{B} = \frac{1}{2}\begin{pmatrix} 0 & a_0 & & & & \\ -a_0 & 0 & a_1 & & & \\ & -a_1 & 0 & \ddots & & \\ & & \ddots & \ddots & a_{n-1} & \\ & & & -a_{n-1} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

where we have omitted the time variable $t$ for simplification. Then, Toda equations (6) and (7) can be written in a matrix form as

$$\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}] = \mathcal{L}\,\mathcal{B} - \mathcal{B}\,\mathcal{L}. \tag{9}$$

The pair of matrices $\{\mathcal{L}, \mathcal{B}\}$ is called *Lax pair* and the representation (9) is called *Lax representation* for the Toda lattice (6) and (7) (see [10]).

For the symmetric case, we consider the infinite pentadiagonal matrix

$$\mathcal{B}_s = \frac{1}{2}\begin{pmatrix} 0 & 0 & a_0a_1 & & & & \\ 0 & 0 & 0 & a_1a_2 & & & \\ -a_0a_1 & 0 & 0 & 0 & \ddots & & \\ & -a_1a_2 & 0 & 0 & \ddots & a_{n-1}a_n & \\ & & \ddots & \ddots & \ddots & 0 & \ddots \\ & & & -a_{n-1}a_n & 0 & 0 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

and using that $d_n(t) = 0$ for $n \geq 0$, one can obtain a Lax representation associated with (8) in the form

$$\dot{\mathcal{L}} = \mathcal{L}\,\mathcal{B}_s - \mathcal{B}_s\,\mathcal{L} = [\mathcal{L}, \mathcal{B}_s].$$

### *2.3. Lax–Nakamura pairs for monic orthogonal polynomials*

Usually Lax pairs are deduced for the sequence of orthonormal polynomials, as above. However, following Nakamura [9], a Lax pair for monic orthogonal polynomials can be established using Eqs. (1) and (2). Define the infinite matrices

$$\mathcal{L} = \begin{pmatrix} d_0 & 1 & & & & \\ c_1 & d_1 & 1 & & & \\ & c_2 & d_2 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & c_n & d_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & & & & \\ -c_1 & 0 & & & \\ & -c_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & -c_n & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}.$$

Observe that $\mathcal{B}$ is the opposite of the lower triangular part of $\mathcal{L}$. Then, we can see that Eqs. (1) and (2) can be represented as the Lax pair $\{\mathcal{L}, \mathcal{B}\}$, i.e.,

$$\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}] = \mathcal{L}\,\mathcal{B} - \mathcal{B}\,\mathcal{L}.$$

For the symmetric case, since $d_n(t) = 0, n \geq 0,$ therefore (5) can be represented again as a Lax–Nakamura pair defining

$$\mathcal{B}_s = \begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ -c_1c_2 & 0 & 0 & & & \\ & -c_2c_3 & 0 & \ddots & & \\ & & \ddots & \ddots & 0 & \\ & & & -c_nc_{n+1} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

and, as before, we obtain $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}_s] = \mathcal{L}\,\mathcal{B}_s - \mathcal{B}_s\,\mathcal{L}$.

## 3. Orthogonal polynomials in two variables

Along this paper, we need some definitions and general properties about bivariate orthogonal polynomials. For an exhaustive description of this and other related subjects see, for instance, [4,12].

Let $\Pi$ denote the linear space of real polynomials in two variables, and let $\Pi_n$ denote the subspace of polynomials of total degree not greater than $n$.

Let $\mathcal{M}_{h\times k}(\mathbb{R})$ and $\mathcal{M}_{h\times k}(\Pi)$ denote the linear spaces of $h \times k$ matrices with real or polynomial entries, respectively. When $h = k$, the second index will be omitted.

Given a matrix $A$, we denote by $A^T$ its transpose, and by $\det(A)$ its determinant if $A$ is a square matrix. As usual, we say that $A$ is non-singular if $\det(A) \neq 0$. Furthermore, we introduce $I_h$ as the identity matrix of dimension $h$.

**Definition 1.** A *polynomial system (PS)* is a vector sequence $\{\mathbb{P}_n\}_{n\geq 0}$ such that

$$\mathbb{P}_n = \mathbb{P}_n(x,y) = (P_{n,0}(x,y), P_{n-1,1}(x,y), \ldots, P_{0,n}(x,y))^T \in \mathcal{M}_{(n+1)\times 1}(\Pi_n),$$

where $\{P_{n,0}(x,y), P_{n-1,1}(x,y), \ldots, P_{0,n}(x,y)\}$ are polynomials of total degree $n$ independent modulo $\Pi_{n-1}$.

The simplest example for PS is the so-called *canonical basis*, defined as

$$\{\mathbb{X}_n = (x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n)^T,\ n \geq 0\}.$$

Observe that

$$x\,\mathbb{X}_n = L_{n,1}\,\mathbb{X}_{n+1}, \qquad y\,\mathbb{X}_n = L_{n,2}\,\mathbb{X}_{n+1},$$

where $L_{n,\,i}$, $i = 1, 2$, are $(n+1)\times(n+2)$ matrices defined as (see [4], p. 76)

$$L_{n,1} = \left(\begin{array}{ccc|c} 1 & & \bigcirc & 0 \\ & \ddots & & \vdots \\ \bigcirc & & 1 & 0 \end{array}\right) \quad \text{and} \quad L_{n,2} = \left(\begin{array}{c|ccc} 0 & 1 & & \bigcirc \\ \vdots & & \ddots & \\ 0 & \bigcirc & & 1 \end{array}\right). \tag{10}$$

Observe that $L_{n,\,i}$, $i = 1, 2$ are full rank matrices, and $L_{n,i}L_{n,i}^T = I_{n+1}$.

Moreover, we can express a vector polynomial $\mathbb{P}_n$ in terms of the *canonical basis* as follows:

$$\mathbb{P}_n = G_n^n\,\mathbb{X}_n + G_{n-1}^n\,\mathbb{X}_{n-1} + \cdots + G_0^n\,\mathbb{X}_0,$$

where $G_i^n$, for $i = 0, 1, \ldots, n$, are $(n+1)\times(i+1)$ constant matrices.

The square matrix $G_n^n$ is called the *leading coefficient* of $\mathbb{P}_n$, and it is a non-singular matrix since the entries of $\mathbb{P}_n$ are independent polynomials. In this way, we say that $\mathbb{P}_n$ is monic if $G_n^n = I_{n+1}$, that is, every polynomial entry in $\mathbb{P}_n$ has a unique term of higher degree

$$P_{n-k,k}(x,y) = x^{n-k}y^k + \text{lower degree terms}, \qquad 0 \leq k \leq n.$$

A *monic PS* $\{\mathbb{P}_n\}_{n\geq 0}$ is a PS such that $\mathbb{P}_n$ is monic, for $n \geq 0$.

Let $\Omega \subset \mathbb{R}^2$ be a domain having a nonempty interior, and let $d\mu(x, y)$ be a measure defined on the domain $\Omega$ such that all moments

$$\int_\Omega x^h y^k\, d\mu(x,y) < +\infty \tag{11}$$

exist for $h, k \geq 0$. Let $\langle \cdot, \cdot \rangle$ denote the inner product defined on $\Pi$ by means of

$$\langle p, q\rangle = \int_\Omega p(x,y)\,q(x,y)d\mu(x,y).$$

Let $A = (a_{i,j}(x,y))_{i,j=1}^{h,k} \in \mathcal{M}_{h\times k}(\Pi)$ and $B = (b_{i,j}(x,y))_{i,j=1}^{k,l} \in \mathcal{M}_{k\times l}(\Pi)$ be two polynomial matrices, that is, $a_{i,\,j}(x, y)$, $b_{i,\,j}(x, y) \in \Pi$. The action of the above inner product over polynomial matrices is defined as the $h \times l$ matrix [4],

$$\langle A, B\rangle = \int_\Omega A(x,y)\,B(x,y)\,d\mu(x,y) = \left(\int_\Omega c_{i,j}(x,y)d\mu(x,y)\right)_{i,j=1}^{h,l},$$

where $C = A\cdot B = (c_{i,j}(x,y))_{i,j=1}^{h,l} \in \mathcal{M}_{h\times l}(\Pi)$.

We say that a polynomial $p \in \Pi_n$ is *orthogonal* with respect to $\langle \cdot, \cdot \rangle$ if

$$\langle p, q\rangle = 0, \quad \forall q \in \Pi, \quad \deg q < \deg p.$$

Then, we can define

$$\mathcal{V}_n = \{p \in \Pi_n : \langle p, q\rangle = 0, \forall q \in \Pi_{n-1}\}.$$

Observe that $\mathcal{V}_n$ is a linear space of dimension $n + 1$.

**Definition 2** [4]**.** We say that a PS $\{\mathbb{P}_n\}_{n\geq 0}$ is an *orthogonal polynomial system (OPS)* with respect to the inner product $\langle \cdot, \cdot \rangle$ if

$$\langle \mathbb{P}_n, \mathbb{P}_m^T\rangle = 0, \quad n \neq m,$$
$$\langle \mathbb{P}_n, \mathbb{P}_n^T\rangle = H_n, \quad n = 0, 1, 2, \ldots,$$

where $H_n \in \mathcal{M}_{n+1}(\mathbb{R})$ is a symmetric and positive-definite matrix.

In the particular case where $H_n$ is a diagonal matrix, we say that the OPS $\{\mathbb{P}_n\}_{n\geq 0}$ is a *mutually orthogonal polynomial system*. Moreover, if $H_n = I_{n+1}$, we call $\{\mathbb{P}_n\}_{n\geq 0}$ an *orthonormal polynomial system*. In addition, there exists a unique monic orthogonal polynomial system associated with $d\mu(x, y)$.

Now, we must describe the three term relations for monic orthogonal polynomials in two variables [4, p. 75].

**Theorem 2.** *Let $\{\mathbb{P}_n\}_{n\geq 0}$ be the monic orthogonal polynomial system associated with a measure $d\mu(x, y)$. Then, for $n \geq 0$, there exist full rank matrices $D_{n,\,i}$, $C_{n,\,i}$ of respective sizes $(n+1)\times(n+1)$ and $(n+1)\times n$, $i = 1, 2$, such that*

$$x\,\mathbb{P}_n = L_{n,1}\,\mathbb{P}_{n+1} + D_{n,1}\,\mathbb{P}_n + C_{n,1}\,\mathbb{P}_{n-1}, \tag{12}$$

$$y\,\mathbb{P}_n = L_{n,2}\,\mathbb{P}_{n+1} + D_{n,2}\,\mathbb{P}_n + C_{n,2}\,\mathbb{P}_{n-1}, \tag{13}$$

*where $\mathbb{P}_{-1} = 0$ and $C_{-1,i} = 0$. Moreover,*

$$\begin{array}{ll} D_{n,1}\,H_n = \langle x\,\mathbb{P}_n, \mathbb{P}_n^T\rangle, & C_{n,1}\,H_{n-1} = H_n\,L_{n-1,1}^T, \\ D_{n,2}\,H_n = \langle y\,\mathbb{P}_n, \mathbb{P}_n^T\rangle, & C_{n,2}\,H_{n-1} = H_n\,L_{n-1,2}^T. \end{array}$$

Observe that we can add relations (12) and (13), and we get

$$(x+y)\,\mathbb{P}_n = L_n\,\mathbb{P}_{n+1} + D_n\,\mathbb{P}_n + C_n\,\mathbb{P}_{n-1},$$

where

$$\begin{aligned} L_n &= L_{n,1} + L_{n,2} \in \mathcal{M}_{(n+1)\times(n+2)}(\mathbb{R}), \\ D_n &= D_{n,1} + D_{n,2} \in \mathcal{M}_{(n+1)\times(n+1)}(\mathbb{R}), \\ C_n &= C_{n,1} + C_{n,2} \in \mathcal{M}_{(n+1)\times n}(\mathbb{R}). \end{aligned}$$

A bivariate measure $d\mu(x, y)$ defined on $\Omega \subset \mathbb{R}^2$ is *centrally symmetric* ([4, p. 76]) if

$$(x, y) \in \Omega \quad \Longrightarrow \quad (-x, -y) \in \Omega,$$

and all moments of odd order vanish, i.e.,

$$\int_\Omega x^h\,y^k d\mu(x, y) = 0, \quad h, k \geq 0, \quad h + k = \text{odd integer}. \tag{14}$$

As in the univariate case, the properties of symmetry from the inner product can be related with the coefficient matrices of the three term relations. In [4, p. 77], it is shown that a measure $d\mu(x, y)$ is centrally symmetric if and only if the matrices $D_{n,\,i} \equiv 0$, for $n \geq 0$ and $i = 1, 2$.

## 4. 2D Toda lattices and bivariate orthogonal polynomials

Consider the bivariate Toda lattice given by the oscillations of a mesh of particles on $\mathbb{R}^2$ given by the coordinates

$$\{(x_h, y_k) : h, k \geq 0\}.$$

For $n \geq 0$, we define the $(n+1)\times(n+1)$ matrices

$$\mathcal{X}_{n,1}(t) = \begin{pmatrix} x_0 & x_0 & \cdots & x_0 \\ x_1 & x_1 & \cdots & x_1 \\ \vdots & \vdots & & \vdots \\ x_n & x_n & \cdots & x_n \end{pmatrix}, \qquad \mathcal{X}_{n,2}(t) = \begin{pmatrix} y_0 & y_1 & \cdots & y_n \\ y_0 & y_1 & \cdots & y_n \\ \vdots & \vdots & & \vdots \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$$

and

$$\mathcal{X}_n(t) = \mathcal{X}_{n,1}(t) + \mathcal{X}_{n,2}(t) = \begin{pmatrix} \chi_{0,0} & \chi_{0,1} & \cdots & \chi_{0,n} \\ \chi_{1,0} & \chi_{1,1} & \cdots & \chi_{1,n} \\ \vdots & \vdots & & \vdots \\ \chi_{n,0} & \chi_{n,1} & \cdots & \chi_{n,n} \end{pmatrix},$$

where $\chi_{h,k} = x_h + y_k$, $0 \leq h, k \leq n$.

Suppose that the interaction between the masses is given by the exponential matrix of $\mathcal{X}_{n,i}$, that is, equation

$$\ddot{\mathcal{X}}_{n,1} = e^{-\mathcal{X}_n}\,L_{n-1}^T\,e^{\mathcal{X}_{n-1}}\,L_{n-1,1} - L_{n,1}\,e^{-\mathcal{X}_{n+1}}\,L_n^T\,e^{\mathcal{X}_n},$$

describes the oscillations in the $X$-axis, and equation

$$\ddot{\mathcal{X}}_{n,2} = e^{-\mathcal{X}_n}\,L_{n-1}^T\,e^{\mathcal{X}_{n-1}}\,L_{n-1,2} - L_{n,2}\,e^{-\mathcal{X}_{n+1}}\,L_n^T\,e^{\mathcal{X}_n},$$

describes the oscillations in the $Y$-axis. Summing both equations, we can describe the total oscillations as

$$\ddot{\mathcal{X}}_n = \ddot{\mathcal{X}}_{n,1} + \ddot{\mathcal{X}}_{n,2} = e^{-\mathcal{X}_n}\,L_{n-1}^T\,e^{\mathcal{X}_{n-1}}\,L_{n-1} - L_n\,e^{-\mathcal{X}_{n+1}}\,L_n^T\,e^{\mathcal{X}_n}.$$

For $n \geq 0$ and $i = 1, 2,$ we define the matrices depending on $t$ given by the expressions

$$\begin{aligned} D_{n,i}(t) &= \dot{\mathcal{X}}_{n,i} \\ C_{n,i}(t) &= e^{-\mathcal{X}_n} L_{n-1,i}^T e^{\mathcal{X}_{n-1}}, \end{aligned}$$

of dimensions $(n+1) \times (n+1)$ and $(n+1) \times n,$ respectively. If $\mathcal{X}_n$ and $\dot{\mathcal{X}}_n$ commute, then

$$\dot{D}_{n,i}(t) = C_n(t)\, L_{n-1,i} - L_{n,i}\, C_{n+1}(t), \tag{15}$$

$$\dot{C}_{n,i}(t) = C_{n,i}(t)\, D_{n-1}(t) - D_n(t)\, C_{n,i}(t), \tag{16}$$

for $i = 1, 2,$ where $D_n(t) = D_{n,1}(t) + D_{n,2}(t)$ and $C_n(t) = C_{n,1}(t) + C_{n,2}(t)$. Moreover,

$$\dot{D}_n(t) = C_n(t)\, L_{n-1} - L_n\, C_{n+1}(t), \tag{17}$$

$$\dot{C}_n(t) = C_n(t)\, D_{n-1}(t) - D_n(t)\, C_n(t). \tag{18}$$

We say that (15) and (16), for $i = 1, 2,$ or (17) and (18), is a *2D Toda lattice*.

Now, we want to relate the above 2D Toda lattice with bivariate orthogonal polynomials. Let $d\mu(x, y)$ be a positive measure defined on a domain $\Omega \subset \mathbb{R}^2$, and suppose that all moments (11) exist. Let $\{\mathbb{P}_n\}_{n\geq 0}$ be the monic OPS associated with $d\mu(x, y)$.

Define the modified measure

$$d\tilde{\mu}(t) \equiv d\tilde{\mu}(x, y, t) = e^{-(x+y)t} d\mu(x, y),$$

and suppose that all moments

$$\int_\Omega x^h\, y^k\, e^{-(x+y)t}\, d\mu(x, y) < +\infty$$

exist for $h,\ k \geq 0$, that is, we suppose that the new measure depends on the time variable $t$ as well as the two variables $x$ and $y$. Then, there exists a monic orthogonal polynomial system $\{\mathbb{P}_n(t)\}_{n\geq 0} \equiv \{\mathbb{P}_n(x, y, t)\}_{n\geq 0}$ associated with $d\tilde{\mu}$. Clearly, every polynomial in such a system is a polynomial in two variables $x$ and $y$ whose coefficients depend on $t$. Obviously, $\mathbb{P}_n(0) = \mathbb{P}_n,$ and

$$\dot{\mathbb{P}}_n(t) = \frac{d}{dt}\mathbb{P}_n(t) = (\dot{P}_{n,0}(x, y, t), \dot{P}_{n-1,1}(x, y, t), \dots, \dot{P}_{0,n}(x, y, t))^T \in \mathcal{M}_{(n+1)\times 1}(\Pi_{n-1}).$$

Since $P_{n-k,k}(x, y, t)$ is a monic polynomial, then $\dot{P}_{n-k,k}(x, y, t) \in \Pi_{n-1}$.

In addition, the symmetric positive-definite matrix

$$H_n(t) = \langle \mathbb{P}_n(t), \mathbb{P}_n^T(t) \rangle,$$

also depends on $t$, and $H_n(0) = H_n,$ for $n \geq 0$.

The system $\{\mathbb{P}_n(t)\}_{n\geq 0}$ satisfies three term relations as (12) and (13), but now, the matrix coefficients depend on $t$. In this way, for $n \geq 0$, there exist matrices $D_{n,\,i}(t)$, $C_{n,\,i}(t)$ of respective sizes $(n+1) \times (n+1)$ and $(n+1) \times n,$ $i = 1, 2,$ such that

$$x\,\mathbb{P}_n(t) = L_{n,1}\,\mathbb{P}_{n+1}(t) + D_{n,1}(t)\,\mathbb{P}_n(t) + C_{n,1}(t)\,\mathbb{P}_{n-1}(t), \tag{19}$$

$$y\,\mathbb{P}_n(t) = L_{n,2}\,\mathbb{P}_{n+1}(t) + D_{n,2}(t)\,\mathbb{P}_n(t) + C_{n,2}(t)\,\mathbb{P}_{n-1}(t), \tag{20}$$

where $\mathbb{P}_{-1}(t) = 0$ and $C_{-1,i}(t) = 0$. Moreover,

$$D_{n,1}(t)\, H_n(t) = \langle x\,\mathbb{P}_n(t), \mathbb{P}_n^T(t)\rangle, \quad D_{n,2}(t)\, H_n(t) = \langle y\,\mathbb{P}_n(t), \mathbb{P}_n^T(t)\rangle, \tag{21}$$

$$C_{n,1}(t)\, H_{n-1}(t) = H_n(t)\, L_{n-1,1}^T, \quad C_{n,2}(t)\, H_{n-1}(t) = H_n(t)\, L_{n-1,2}^T, \tag{22}$$

and $D_{n,i}(0) = D_{n,i},$ $C_{n,i}(0) = C_{n,i},$ $i = 1, 2$. We remark that $L_{n,\,i}$ are given in (10) and they are independent of $t$. We also define

$$\begin{aligned} D_n(t) &= D_{n,1}(t) + D_{n,2}(t), \\ C_n(t) &= C_{n,1}(t) + C_{n,2}(t). \end{aligned}$$

Now we present our first result.

**Lemma 3.** *For* $n \geq 0$,

$$\dot{H}_n(t) = -D_n(t)\,H_n(t). \tag{23}$$

**Proof.** Since

$$H_n(t) = \int_\Omega \mathbb{P}_n(t)\,\mathbb{P}_n^T(t) e^{-(x+y)t} d\mu(x,y),$$

we get

$$\begin{aligned}\dot{H}_n(t) &= \int_\Omega \dot{\mathbb{P}}_n(t)\,\mathbb{P}_n^T(t) e^{-(x+y)t} d\mu(x,y) + \int_\Omega \mathbb{P}_n(t)\,\dot{\mathbb{P}}_n^T(t) e^{-(x+y)t} d\mu(x,y) \\ &\quad - \int_\Omega (x+y)\mathbb{P}_n(t)\,\mathbb{P}_n^T(t) e^{-(x+y)t} d\mu(x,y) \\ &= -D_{n,1}(t)\,H_n(t) - D_{n,2}(t)\,H_n(t),\end{aligned}$$

using (21), and the fact that $\deg \dot{\mathbb{P}}_n(t) < n$. □

On the coefficients of the three term relations (19) and (20) we have the following.

**Theorem 4.** *For* $n \geq 1$ *and* $i = 1, 2,$ *it is satisfied*

$$\dot{C}_{n,i}(t) = C_{n,i}(t)\,D_{n-1}(t) - D_n(t)\,C_{n,i}(t). \tag{24}$$

*Moreover,*

$$\dot{C}_n(t) = C_n(t)\,D_{n-1}(t) - D_n(t)\,C_n(t). \tag{25}$$

**Proof.** Taking derivatives in (22), we get

$$\dot{H}_n(t)\,L_{n-1,i}^T = \dot{C}_{n,i}(t)\,H_{n-1}(t) + C_{n,i}(t)\,\dot{H}_{n-1}(t),$$

and using (23), we obtain

$$-D_n(t)\,H_n(t)\,L_{n-1,i}^T = \dot{C}_{n,i}(t)\,H_{n-1}(t) - C_{n,i}(t)\,D_{n-1}(t)\,H_{n-1}(t).$$

Since $H_{n-1}(t)$ is a non-singular matrix, therefore

$$\dot{C}_{n,i}(t) = C_{n,i}(t)\,D_{n-1}(t) - D_n(t)\,H_n(t)\,L_{n-1,i}^T H_{n-1}^{-1}(t),$$

and using again (22), we obtain (24). Summing above equation for $i = 1, 2,$ we deduce (25). □

**Theorem 5.** *For* $n \geq 1$, *and* $i = 1, 2,$ *the following holds:*

$$\dot{D}_{n,i}(t) = C_n(t)\,L_{n-1,i} - L_{n,i}\,C_{n+1}(t). \tag{26}$$

*In addition,*

$$\dot{D}_n(t) = C_n(t)\,L_{n-1} - L_n\,C_{n+1}(t). \tag{27}$$

**Proof.** Eq. (21) can be written as follows:

$$D_{n,1}(t)\,H_n(t) = \int_\Omega x\,\mathbb{P}_n(t)\,\mathbb{P}_n^T(t) e^{-(x+y)t}\,d\mu(x,y).$$

By taking derivative with respect to $t$ we get

$$\begin{aligned}\dot{D}_{n,1}\,H_n + D_{n,1}\,\dot{H}_n &= \int_\Omega x\,\dot{\mathbb{P}}_n\,\mathbb{P}_n^T e^{-(x+y)t}\,d\mu(x,y) + \int_\Omega x\,\mathbb{P}_n\,\dot{\mathbb{P}}_n^T e^{-(x+y)t}\,d\mu(x,y) \\ &\quad - \int_\Omega x\,(x+y)\,\mathbb{P}_n\,\mathbb{P}_n^T e^{-(x+y)t}\,d\mu(x,y),\end{aligned} \tag{28}$$

where we have omitted the time variable $t$ for simplicity.

We study the first and third integral in (28), since the second integral is the transpose of the first one. Using (19), we get

$$\int_\Omega x\,\dot{\mathbb{P}}_n\,\mathbb{P}_n^T e^{-(x+y)t}\,d\mu(x,y) = \int_\Omega \dot{\mathbb{P}}_n\,\mathbb{P}_{n-1}^T e^{-(x+y)t}\,d\mu(x,y)\;\cdot C_{n,1}^T.$$

On the other hand, since $\int_\Omega \mathbb{P}_n\,\mathbb{P}_{n-1}^T e^{-(x+y)t}\,d\mu(x,y) = 0$ by orthogonality, then

$$\begin{aligned}0 &= \int_\Omega \dot{\mathbb{P}}_n\,\mathbb{P}_{n-1}^T e^{-(x+y)t}\,d\mu(x,y) + \int_\Omega \mathbb{P}_n\,\dot{\mathbb{P}}_{n-1}^T e^{-(x+y)t}\,d\mu(x,y) \\ &\quad - \int_\Omega (x+y)\,\mathbb{P}_n\,\mathbb{P}_{n-1}^T e^{-(x+y)t}\,d\mu(x,y),\end{aligned}$$

and therefore

$$\int_\Omega \dot{\mathbb{P}}_n \, \mathbb{P}_{n-1}^T e^{-(x+y)t} \, d\mu(x,y) = C_{n,1} \, H_{n-1} + C_{n,2} \, H_{n-1}.$$

For the third integral in (28), using again (19), we obtain

$$\begin{aligned}\int_\Omega x^2 \, \mathbb{P}_n \, \mathbb{P}_n^T e^{-(x+y)t} d\mu(x,y) &= \int_\Omega (x \, \mathbb{P}_n) \, (x \, \mathbb{P}_n)^T e^{-(x+y)t} d\mu(x,y) \\ &= L_{n,1} \, H_{n+1} \, L_{n,1}^T + D_{n,1} \, H_n \, D_{n,1}^T + C_{n,1} \, H_{n-1} \, C_{n,1}^T,\end{aligned}$$

and

$$\int_\Omega x \, y \, \mathbb{P}_n \, \mathbb{P}_n^T e^{-(x+y)t} d\mu(x,y) = L_{n,1} \, H_{n+1} \, L_{n,2}^T + D_{n,1} \, H_n \, D_{n,2}^T + C_{n,1} \, H_{n-1} \, C_{n,2}^T.$$

In this way, (28) becomes

$$\dot{D}_{n,1} \, H_n + D_{n,1} \, \dot{H}_n = C_{n,2} \, H_{n-1} \, C_{n,1}^T + C_{n,1} \, H_{n-1} \, C_{n,1}^T - L_{n,1} \, H_{n+1} \, L_{n,1}^T - D_{n,1} \, H_n \, D_{n,1}^T - L_{n,1} \, H_{n+1} \, L_{n,2}^T - D_{n,1} \, H_n \, D_{n,2}^T.$$

On the other hand, exploiting (21), and using (19), we get

$$\begin{aligned}D_{n,1} \, H_n &= \langle x \, \mathbb{P}_n, \, \mathbb{P}_n^T \rangle = \int_\Omega x \, \mathbb{P}_n \, \mathbb{P}_n^T e^{-(x+y)t} d\mu(x,y) \\ &= \int_\Omega \mathbb{P}_n \, (x \, \mathbb{P}_n)^T e^{-(x+y)t} d\mu(x,y) = H_n \, D_{n,1}^T,\end{aligned}$$

and then, by (23),

$$D_{n,1} \, \dot{H}_n = -D_{n,1} \, D_{n,1} \, H_n - D_{n,1} \, D_{n,2} \, H_n = -D_{n,1} \, H_n \, D_{n,1}^T - D_{n,1} \, H_n \, D_{n,2}^T.$$

Therefore, using again (22), we obtain

$$\begin{aligned}\dot{D}_{n,1} \, H_n &= C_{n,2} \, H_{n-1} \, C_{n,1}^T + C_{n,1} \, H_{n-1} \, C_{n,1}^T - L_{n,1} \, H_{n+1} \, L_{n,1}^T - L_{n,1} \, H_{n+1} \, L_{n,2}^T \\ &= C_{n,2} \, L_{n-1,1} \, H_n + C_{n,1} \, L_{n-1,1} \, H_n - L_{n,1} \, C_{n+1,1} \, H_n - L_{n,1} \, C_{n+1,2} \, H_n.\end{aligned}$$

Multiplying by $H_n^{-1}$, the result (26) follows for $i = 1$. The case $i = 2$ can be shown similarly. Summing Eq. (26) for $i = 1, 2$, we obtain (27). □

Theorems 4 and 5 show that the matrix coefficients of the three term relation of a bivariate monic orthogonal polynomials satisfy the 2D Toda lattice (15) and (16) for $i = 1, 2$, and the 2D Toda lattice (17) and (18).

### *4.1. Centrally symmetric measures*

Let us consider a bivariate centrally symmetric positive measure $d\mu(x, y)$ defined by (14). Since $D_{n,\,i}(t) \equiv 0, \quad$ for $n \geq 0$, and $i = 1, 2$, relations (19) and (20) become

$$\begin{aligned}x \, \mathbb{P}_n(t) &= L_{n,1} \, \mathbb{P}_{n+1}(t) + C_{n,1}(t) \, \mathbb{P}_{n-1}(t), \\ y \, \mathbb{P}_n(t) &= L_{n,2} \, \mathbb{P}_{n+1}(t) + C_{n,2}(t) \, \mathbb{P}_{n-1}(t),\end{aligned}$$

where $\mathbb{P}_{-1}(t) = 0$ and $C_{-1,i}(t) = 0$. Similarly as before we define the modified measure

$$d\tilde{\mu}(t) \equiv d\tilde{\mu}(x,y,t) = e^{-(x+y)^2 t} d\mu(x,y),$$

and we suppose that all of the moments

$$\int_\Omega x^h \, y^k \, e^{-(x+y)^2 t} \, d\mu(x,y) < +\infty$$

exist for $h, k \geq 0$. We can prove the following.

**Lemma 6.** *For a centrally symmetric bivariate measure,*

$$\dot{H}_n(t) = -E_n(t) \, H_n(t), \quad n \geq 0, \tag{29}$$

*where* $E_n(t) = L_n \, C_{n+1}(t) + C_n(t) \, L_{n-1}$.

**Proof.** Since $H_n(t) = \int_\Omega \mathbb{P}_n(t) \, \mathbb{P}_n^T(t) e^{-(x+y)^2 t} d\mu(x,y)$, and $\deg \dot{\mathbb{P}}_n(t) < n$, we get

$$\dot{H}_n(t) = -\int_\Omega (x^2 + 2xy + y^2) \mathbb{P}_n(t) \, \mathbb{P}_n^T(t) e^{-(x+y)^2 t} d\mu(x,y),$$

using (19) and (20), we get

$$\begin{aligned}\dot{H}_n(t) &= -[L_{n,1}C_{n+1,1}(t) + C_{n,1}(t)L_{n-1,1} + L_{n,1}C_{n+1,2}(t) + C_{n,1}(t)L_{n-1,2} \\ &\quad + L_{n,2}C_{n+1,1}(t) + C_{n,2}(t)L_{n-1,1} + L_{n,2}C_{n+1,2}(t) + C_{n,2}(t)L_{n-1,2}]\, H_n(t) \\ &= -[L_nC_{n+1}(t) + C_n(t)L_{n-1}]\, H_n(t).\end{aligned}$$

□

The next theorem relates centrally symmetric positive measures with 2D Toda systems.

**Theorem 7.** *For a bivariate centrally symmetric positive measure, it follows that:*

$$\dot{C}_n(t) = C_n(t)\, C_{n-1}(t)\, L_{n-2} - L_n\, C_{n+1}(t)\, C_n(t), \quad n \geq 1, \tag{30}$$

*taking* $C_0(t) = 0$.

**Proof.** Taking derivatives in (22), we get

$$\dot{H}_n(t)\, L^T_{n-1,i} = \dot{C}_{n,i}(t)\, H_{n-1}(t) + C_{n,i}(t)\, \dot{H}_{n-1}(t),$$

and using (29), we obtain

$$-E_n(t)\, H_n(t)\, L^T_{n-1,i} = \dot{C}_{n,i}(t)\, H_{n-1}(t) - C_{n,i}(t)\, E_{n-1}(t)\, H_{n-1}(t).$$

$H_{n-1}(t)$ is a non-singular matrix, therefore

$$\dot{C}_{n,i}(t) = C_{n,i}(t)\, E_{n-1}(t) - E_n(t)\, H_n(t)\, L^T_{n-1,i} H^{-1}_{n-1}(t),$$

using again (22) and the definition of $E_n(t)$, we get

$$\begin{aligned}\dot{C}_{n,i}(t) &= C_{n,i}(t)\, E_{n-1}(t) - E_n(t)\, C_{n,i}(t) \\ &= C_{n,i}(t)[L_{n-1}C_n(t) + C_{n-1}(t)L_{n-2}] - [L_nC_{n+1}(t) + C_n(t)L_{n-1}]C_{n,i}(t).\end{aligned}$$

Hence,

$$\begin{aligned}\dot{C}_n(t) &= C_n(t)[L_{n-1}C_n(t) + C_{n-1}(t)L_{n-2}] - [L_nC_{n+1}(t) + C_n(t)L_{n-1}]C_n(t) \\ &= C_n(t)C_{n-1}(t)L_{n-2} - L_nC_{n+1}(t)C_n(t),\end{aligned}$$

and the result is proved. □

The lattice given by Eq. (30) is referred as a *2D Langmuir lattice*.

### *4.2. 2D Toda lattices and bivariate orthonormal polynomials*

In this section, we want to write the 2D Toda equations (15) and (16) for orthonormal bivariate polynomials. Since

$$H_n(t) = \langle \mathbb{P}_n(t), \mathbb{P}_n^T(t) \rangle,$$

is a $(n+1)$ symmetric and positive definite matrix, there exists another symmetric and positive definite matrix $H_n(t)^{\frac{1}{2}}$, the so-called *square root* of the matrix $H_n(t)$[6, p. 405], such that

$$H_n(t) = H_n(t)^{\frac{1}{2}}\, H_n(t)^{\frac{1}{2}}.$$

Moreover, we define $H_n(t)^{-\frac{1}{2}} = (H_n(t)^{\frac{1}{2}})^{-1}$, and it can be checked that $(H_n(t)^{\frac{1}{2}})^{-1} = (H_n(t)^{-1})^{\frac{1}{2}}$.

Let $\{\mathbb{Q}_n(t)\}_{n\geq 0} = \{\mathbb{Q}_n(x, y, t)\}_{n\geq 0}$ be the OPS defined by

$$\mathbb{Q}_n(t) = H_n(t)^{-\frac{1}{2}}\, \mathbb{P}_n(t). \tag{31}$$

Then, $\langle \mathbb{Q}_n(t), \mathbb{Q}_n^T(t) \rangle = H_n(t)^{-\frac{1}{2}}\, \langle \mathbb{P}_n(t), \mathbb{P}_n^T(t) \rangle H_n(t)^{-\frac{1}{2}} = I_{n+1}$, and therefore $\{\mathbb{Q}_n\}_{n\geq 0}$ is an orthonormal polynomial system. Multiplying (19) and (20) by $H_n(t)^{-\frac{1}{2}}$, it can be proved that $\{\mathbb{Q}_n\}_{n\geq 0}$ satisfy the three term relations

$$\begin{aligned}x\, \mathbb{Q}_n(t) &= A_{n,1}(t)\, \mathbb{Q}_{n+1}(t) + B_{n,1}(t)\, \mathbb{Q}_n(t) + A^T_{n-1,1}(t)\, \mathbb{Q}_{n-1}(t), \\ y\, \mathbb{Q}_n(t) &= A_{n,2}(t)\, \mathbb{Q}_{n+1}(t) + B_{n,2}(t)\, \mathbb{Q}_n(t) + A^T_{n-1,2}(t)\, \mathbb{Q}_{n-1}(t),\end{aligned}$$

where

$$A_{n,i}(t) = H_n(t)^{-\frac{1}{2}}\, L_{n,i}\, H_{n+1}(t)^{\frac{1}{2}} = H_n(t)^{\frac{1}{2}}\, C^T_{n+1,i}(t)\, H_{n+1}(t)^{-\frac{1}{2}}, \tag{32}$$

$$B_{n,i}(t) = H_n(t)^{-\frac{1}{2}}\, D_{n,i}(t)\, H_n(t)^{\frac{1}{2}}, \tag{33}$$

are matrices of respective sizes $(n+1)\times(n+2)$ and $(n+1)\times(n+1)$, $i=1,2$. Adding the above matrices for $i=1,2$, also we get

$$A_n(t) = H_n(t)^{-\frac{1}{2}}\,L_n\,H_{n+1}(t)^{\frac{1}{2}} = H_n(t)^{\frac{1}{2}}\,C_{n+1}^T(t)\,H_{n+1}(t)^{-\frac{1}{2}}, \tag{34}$$

$$B_n(t) = H_n(t)^{-\frac{1}{2}}\,D_n(t)\,H_n(t)^{\frac{1}{2}}, \tag{35}$$

Since the product of matrices is not commutative, then, in general $B_{n,\,i}(t)\neq D_{n,\,i}(t)$. On the other hand,

$$\begin{aligned}
\dot{A}_{n,i} &= \dot{H}_n^{\frac{1}{2}}\,C_{n+1,i}^T\,H_{n+1}^{-\frac{1}{2}} + H_n^{\frac{1}{2}}\,\dot{C}_{n+1,i}^T\,H_{n+1}^{-\frac{1}{2}} + H_n^{\frac{1}{2}}\,C_{n+1,i}^T\,\dot{H}_{n+1}^{-\frac{1}{2}},\\
\dot{B}_{n,i} &= \dot{H}_n^{-\frac{1}{2}}\,D_{n,i}\,H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\,\dot{D}_{n,i}\,H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\,D_{n,i}\,\dot{H}_n^{\frac{1}{2}},
\end{aligned}$$

and

$$\dot{A}_n = \dot{H}_n^{\frac{1}{2}}\,C_{n+1}^T\,H_{n+1}^{-\frac{1}{2}} + H_n^{\frac{1}{2}}\,\dot{C}_{n+1}^T\,H_{n+1}^{-\frac{1}{2}} + H_n^{\frac{1}{2}}\,C_{n+1}^T\,\dot{H}_{n+1}^{-\frac{1}{2}}, \tag{36}$$

$$\dot{B}_n = \dot{H}_n^{-\frac{1}{2}}\,D_n\,H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\,\dot{D}_n\,H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\,D_n\,\dot{H}_n^{\frac{1}{2}}, \tag{37}$$

where we omitted the time variable for brevity. On the other hand,

$$\begin{aligned}
B_{n,i}^T(t) &= H_n(t)^{\frac{1}{2}}\,D_{n,i}^T(t)\,H_n(t)^{-\frac{1}{2}} = H_n(t)^{\frac{1}{2}}\,H_n(t)^{-1}\,\langle x\mathbb{P}_n(t),\mathbb{P}_n^T(t)\rangle^T\,H_n(t)^{-\frac{1}{2}}\\
&= H_n(t)^{-\frac{1}{2}}\,\langle \mathbb{P}_n(t),(x\mathbb{P}_n(t))^T\rangle\,H_n(t)^{-\frac{1}{2}} = \langle \mathbb{Q}_n(t),(x\mathbb{Q}_n(t))^T\rangle = B_{n,i}(t),
\end{aligned}$$

showing that $B_{n,\,i}(t)$ is a symmetric matrix, and also $B_n(t)$.

Now we deal with Eq. (26). Using (32) and (34), we deduce

$$\begin{aligned}
H_n(t)^{-\frac{1}{2}}\,\dot{D}_{n,i}(t)\,H_n(t)^{\frac{1}{2}} &= H_n(t)^{-\frac{1}{2}}\,C_n(t)\,L_{n-1,i}\,H_n(t)^{\frac{1}{2}} - H_n(t)^{-\frac{1}{2}}\,L_{n,i}\,C_{n+1}(t)\,H_n(t)^{\frac{1}{2}}\\
&= H_n(t)^{-\frac{1}{2}}\,C_n(t)\,H_{n-1}(t)^{\frac{1}{2}}\,H_{n-1}(t)^{-\frac{1}{2}}\,L_{n-1,i}\,H_n(t)^{\frac{1}{2}}\\
&\quad - H_n(t)^{-\frac{1}{2}}\,L_{n,i}\,H_{n+1}(t)^{\frac{1}{2}}\,H_{n+1}(t)^{-\frac{1}{2}}\,C_{n+1}(t)\,H_n(t)^{\frac{1}{2}}\\
&= A_{n-1}^T(t)\,A_{n-1,i}(t) - A_{n,i}(t)\,A_n^T(t).
\end{aligned}$$

In the same way, we can use (33) and (35) into (24), and we find that

$$H_n(t)^{-\frac{1}{2}}\,\dot{C}_{n,i}(t)\,H_{n-1}(t)^{\frac{1}{2}} = A_{n-1,i}^T(t)\,B_{n-1}(t) - B_n(t)\,A_{n-1,i}^T(t).$$

We resume our results in the following propositions.

**Proposition 1.** *For $n\geq 0$, and $i=1,2$, we get*

$$H_n^{-\frac{1}{2}}\,\dot{C}_{n,i}\,H_{n-1}^{\frac{1}{2}} = A_{n-1,i}^T\,B_{n-1,i} - B_{n,i}\,A_{n-1,i}^T. \tag{38}$$

$$H_n(t)^{-\frac{1}{2}}\,\dot{D}_{n,i}\,H_n^{\frac{1}{2}} = A_{n-1}^T\,A_{n-1,i} - A_{n,i}\,A_n^T, \tag{39}$$

*Summing above equations for $i=1,2$, we get*

$$H_n^{-\frac{1}{2}}\,\dot{C}_n\,H_{n-1}^{\frac{1}{2}} = A_{n-1}^T\,B_{n-1} - B_n\,A_{n-1}^T. \tag{40}$$

$$H_n^{-\frac{1}{2}}\,\dot{D}_n\,H_n^{\frac{1}{2}} = A_{n-1}^T\,A_{n-1} - A_n\,A_n^T, \tag{41}$$

*where we have omitted the variable t.*

Using (36) and (37) in (40) and (41) we get

**Proposition 2.** *For $n\geq 1$,*

$$\dot{A}_{n-1}^T = A_{n-1}^T\,B_{n-1} - B_n\,A_{n-1}^T + \dot{H}_n^{-\frac{1}{2}}\,C_n\,H_{n-1}^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\,C_n\,\dot{H}_{n-1}^{\frac{1}{2}}, \tag{42}$$

$$\dot{B}_n = A_{n-1}^T\,A_{n-1} - A_n\,A_n^T + \dot{H}_n^{-\frac{1}{2}}\,D_n\,H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\,D_n\,\dot{H}_n^{\frac{1}{2}}, \tag{43}$$

*where we have omitted the variable t.*

*4.3. 2D Langmuir lattices and centrally symmetric bivariate orthonormal polynomials*

We study now the orthonormal polynomials (31) for centrally symmetric measures, that is, $D_{n,i}(t) \equiv 0$, $n \geq 0$, $i = 1, 2$. If we consider the 2D Langmuir lattice given by (30), and we use definition (34), it follows:

$$H_{n+1}^{-\frac{1}{2}} \dot{C}_{n+1} H_n^{\frac{1}{2}} = A_n^T A_{n-1}^T A_{n-1} - A_{n+1} A_{n+1}^T A_n^T, \quad n \geq 0, \tag{44}$$

and substituting (36), we get

$$\dot{A}_n^T = A_n^T A_{n-1}^T A_{n-1} - A_{n+1} A_{n+1}^T A_n^T + \dot{H}_{n+1}^{-\frac{1}{2}} C_{n+1} H_n^{\frac{1}{2}} + H_{n+1}^{-\frac{1}{2}} C_{n+1} \dot{H}_n^{\frac{1}{2}}, \quad n \geq 0, \tag{45}$$

where, again we have omitted $t$.

## 5. Block Lax representation for bivariate orthogonal polynomials

In this section we want to give a block matrix perspective of the 2D Toda equations (17) and (18) and the 2D Langmuir equation (30). In particular, we deduce a *block Lax–Nakamura pair* for bivariate monic orthogonal polynomials.

Furthermore, we give a *block Lax-type* pair for bivariate orthonormal polynomials using Eqs. (42) and (43) and for centrally symmetric case using Eq. (45).

We define the following infinite block matrices:

$$\mathcal{L} = \begin{pmatrix} D_0 & L_0 & & & & \\ C_1 & D_1 & L_1 & & & \\ & C_2 & D_2 & \ddots & & \\ & & \ddots & \ddots & L_{n-1} & \\ & & & C_n & D_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & & & & & \\ -C_1 & 0 & & & & \\ & -C_2 & 0 & & & \\ & & -C_3 & \ddots & & \\ & & & \ddots & 0 & \\ & & & & -C_n & \ddots \\ & & & & & \ddots \end{pmatrix},$$

where we have omitted the time variable $t$ for brevity.

**Proposition 3.** *The 2D Toda equations (17) and (18) can be given as the block Lax–Nakamura pair representation*

$$\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}] = \mathcal{L}\mathcal{B} - \mathcal{B}\mathcal{L}. \tag{46}$$

For the centrally symmetric case, since $D_n(t) \equiv 0$, we define the infinite block matrix

$$\mathcal{B}_s = \begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ -C_1C_2 & 0 & 0 & & & \\ & -C_2C_3 & 0 & \ddots & & \\ & & \ddots & \ddots & 0 & \\ & & & -C_nC_{n+1} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}.$$

In this case, the 2D Langmuir lattice (30) can be express as a block Lax–Nakamura representation as follows:

$$\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}_s] = \mathcal{L}\mathcal{B}_s - \mathcal{B}_s\mathcal{L}.$$

Now we want to deduce a *block Lax-type* pair for bivariate orthonormal polynomials, that is, we want to express relations (42) and (43) as a Lax-type representation.

Define the following infinite diagonal block matrices:

$$\mathcal{H}^{\frac{1}{2}} = \operatorname{diag}\{H_n^{\frac{1}{2}}, \ n = 0, 1, \ldots\},$$
$$\mathcal{H}^{-\frac{1}{2}} = \operatorname{diag}\{H_n^{-\frac{1}{2}}, \ n = 0, 1, \ldots\},$$

$$\mathcal{L} = \begin{pmatrix} B_0 & A_0 & & & & \\ A_0^T & B_1 & A_1 & & & \\ & A_1^T & B_2 & \ddots & & \\ & & \ddots & \ddots & A_n & \\ & & & A_n^T & B_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & A_0 & & & & \\ -A_0^T & 0 & A_1 & & & \\ & -A_1^T & 0 & \ddots & & \\ & & \ddots & \ddots & A_n & \\ & & & -A_n^T & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

$$\mathcal{K} = \frac{1}{2}\begin{pmatrix} 0 & A_0 & & & & \\ A_0^T & 0 & A_1 & & & \\ & A_1^T & 0 & \ddots & & \\ & & \ddots & \ddots & A_n & \\ & & & A_n^T & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} D_0 & C_1^T & & & & \\ C_1 & D_1 & C_2^T & & & \\ & C_2 & D_2 & \ddots & & \\ & & \ddots & \ddots & C_n^T & \\ & & & C_n & D_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

then, we can represent Eqs. (42) and (43) by the following block Lax-type representation:

$$\dot{\mathcal{L}} = [\mathcal{L} - \mathcal{K}, \mathcal{B}] + \dot{\mathcal{H}}^{-\frac{1}{2}} \mathcal{J} \mathcal{H}^{\frac{1}{2}} + \mathcal{H}^{-\frac{1}{2}} \mathcal{J} \dot{\mathcal{H}}^{\frac{1}{2}}.$$

Now we can deduce a *block Lax-type* pair for centrally symmetric bivariate orthonormal polynomials. In this case, $B_n(t) \equiv 0$ and $D_n(t) \equiv 0$, and we define

$$\mathcal{B}_s = \begin{pmatrix} 0 & 0 & A_0A_1 & & & & \\ 0 & 0 & 0 & A_1A_2 & & & \\ -A_1^TA_0^T & 0 & 0 & 0 & \ddots & & \\ & -A_2^TA_1^T & 0 & 0 & \ddots & A_{n-1}A_n & \\ & & \ddots & \ddots & \ddots & 0 & \ddots \\ & & & -A_{n+1}^TA_n^T & 0 & 0 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then, Eq. (45) can be represented by the following Lax-type representation:

$$\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}_s] + \dot{\mathcal{H}}^{-\frac{1}{2}} \mathcal{J} \mathcal{H}^{\frac{1}{2}} + \mathcal{H}^{-\frac{1}{2}} \mathcal{J} \dot{\mathcal{H}}^{\frac{1}{2}}.$$

## 6. A particular case: tensor product of univariate polynomials

It is well known that we can define orthogonal polynomials in two variables as the product of orthogonal polynomials in one variable, the so-called *tensor product*.

Let $d\mu_i(x)$, for $i = 1, 2$, be two real measures with finite moments, and let $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ be the monic orthogonal polynomial sequences associated with $d\mu_i(x)$, respectively. Then, for $n \geq 0$, both sequences satisfy the following three term recurrence relations:

$$\begin{aligned} x\,p_n(x) &= p_{n+1}(x) + d_n^{(1)}\,p_n(x) + c_n^{(1)}\,p_{n-1}(x), \quad p_{-1}(x) = 0, \quad p_0(x) = 1, \\ x\,q_n(x) &= q_{n+1}(x) + d_n^{(2)}\,q_n(x) + c_n^{(2)}\,q_{n-1}(x), \quad q_{-1}(x) = 0, \quad q_0(x) = 1. \end{aligned}$$

As above, for $i = 1, 2$, we suppose that all the moments

$$\int_{\mathbb{R}} x^n e^{-xt}\,d\mu_i(x),$$

exist for $n \geq 0$, and let $\{p_n(x, t)\}_{n\geq 0}$ and $\{q_n(x, t)\}_{n\geq 0}$ be the MOPSs associated with $e^{-xt}\,d\mu_i(x)$, respectively. Now, let $c_n^{(i)}(t)$, $d_n^{(i)}(t)$ be the coefficients of the three term recurrence relations

$$x\,p_n(x,t) = p_{n+1}(x,t) + d_n^{(1)}(t)\,p_n(x,t) + c_n^{(1)}(t)\,p_{n-1}(x,t), \tag{47}$$

$$x\,q_n(x,t) = q_{n+1}(x,t) + d_n^{(2)}(t)\,q_n(x,t) + c_n^{(2)}(t)\,q_{n-1}(x,t), \tag{48}$$

with $p_{-1}(x,t)=0,\ p_0(x,t)=1,\ q_{-1}(x,t)=0,\ q_0(x,t)=1$. Following Theorem 1, the coefficients $c_n^{(i)}(t),\ d_n^{(i)}(t)$ for $i=1,2$, satisfy the univariate Toda system (1) and (2), with initial conditions $c_n^{(i)}(0)=c_n,\ d_n^{(i)}(0)=d_n$.

Define the bivariate polynomials

$$P_{n-k,k}(x,y,t)=p_{n-k}(x,t)\,q_k(y,t),$$

for $k=0,1,\dots,n$ and $n\geq 0$, and define the PS $\{\mathbb{P}_n\}_{n\geq 0}$, as in Definition 1,

$$\mathbb{P}_n=\mathbb{P}_n(t)=\mathbb{P}_n(x,y,t)=(P_{n,0}(x,y,t),P_{n-1,1}(x,y,t),\dots,P_{0,n}(x,y,t))^T.$$

Then $\{\mathbb{P}_n(t)\}_{n\geq 0}$ is a monic OPS associated with the modified measure

$$d\tilde{\mu}(x,y,t)=e^{-(x+y)t}\omega_1(x)\,\omega_2(y)\,dx\,dy=e^{-xt}\,\omega_1(x)\,e^{-yt}\,\omega_2(y)\,dx\,dy.$$

Using the three term recurrence relation for the univariate orthogonal polynomials, we can give the explicit expression for the matrix coefficients of the three term relations for the bivariate orthogonal polynomials. In fact, the monic OPS $\{\mathbb{P}_n(t)\}_{n\geq 0}$ satisfy the three term relations (12) and (13) with

$$\begin{aligned}
D_{n,1}(t)&=\operatorname{diag}\{d_{n-k}^{(1)}(t),\ k=0,1,\dots,n\},\\
D_{n,2}(t)&=\operatorname{diag}\{d_{k}^{(2)}(t),\ k=0,1,\dots,n\},\\
C_{n,1}(t)&=L_{n-1,1}^T\operatorname{diag}\{c_{n-k}^{(1)}(t),\ k=0,1,\dots,n-1\},\\
C_{n,2}(t)&=L_{n-1,2}^T\operatorname{diag}\{c_{k}^{(2)}(t),\ k=0,1,\dots,n-1\}.
\end{aligned}$$

Applying Theorems 4 and 5, the matrix coefficients of the three term relations satisfy the 2D Toda lattice (15) and (16). Using the explicit expressions for the matrices, we have to observe that (16) can be write for the entries of $\dot{C}_{n,1}(t)$, obtaining

$$\begin{aligned}
\dot{c}_n^{(1)}(t)&=c_n^{(1)}(t)[d_{n-1}^{(1)}(t)+d_0^{(2)}(t)]-[d_n^{(1)}(t)+d_0^{(2)}(t)]c_n^{(1)}(t),\\
\dot{c}_{n-1}^{(1)}(t)&=c_{n-1}^{(1)}(t)[d_{n-2}^{(1)}(t)+d_1^{(2)}(t)]-[d_{n-1}^{(1)}(t)+d_1^{(2)}(t)]c_{n-1}^{(1)}(t),\\
&\ \ \vdots\\
\dot{c}_1^{(1)}(t)&=c_1^{(1)}(t)[d_0^{(1)}(t)+d_{n-1}^{(2)}(t)]-[d_0^{(1)}(t)+d_n^{(2)}(t)]c_1^{(1)}(t),
\end{aligned}$$

that is,

$$\dot{c}_n^{(1)}(t)=c_n^{(1)}(t)\,[d_{n-1}^{(1)}(t)-d_n^{(1)}(t)],\quad n=1,2,\dots.$$

In the same way, Eq. (15) means that

$$\dot{d}_n^{(1)}(t)=c_n^{(1)}(t)-c_{n+1}^{(1)}(t),\quad n=1,2,\dots.$$

This recovers the Toda lattice in one variable for the MOPS $\{p_n(t)\}_{n\,\geq\,0}$.

Analogously, using $i=2$ in (15) and (16), we recover Toda system for the second MOPS $\{q_n(t)\}_{n\,\geq\,0}$

$$\begin{aligned}
\dot{c}_n^{(2)}(t)&=c_n^{(2)}(t)\,(d_{n-1}^{(2)}(t)-d_n^{(2)}(t)),\\
\dot{d}_n^{(2)}(t)&=c_n^{(2)}(t)-c_{n+1}^{(2)}(t).
\end{aligned}$$

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