

# The spectra and the signless Laplacian spectra of graphs with pockets

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## Abstract

Let  $G[F, V_k, H_v]$  be the graph with  $k$  pockets, where  $F$  is a simple graph of order  $n \geq 1$ ,  $V_k = \{v_1, \dots, v_k\}$  is a subset of the vertex set of  $F$  and  $H_v$  is a simple graph of order  $m \geq 2$ ,  $v$  is a specified vertex of  $H_v$ . Also let  $G[F, E_k, H_{uv}]$  be the graph with  $k$  edge-pockets, where  $F$  is a simple graph of order  $n \geq 2$ ,  $E_k = \{e_1, \dots, e_k\}$  is a subset of the edge set of  $F$  and  $H_{uv}$  is a simple graph of order  $m \geq 3$ ,  $uv$  is a specified edge of  $H_{uv}$  such that  $H_{uv} - u$  is isomorphic to  $H_{uv} - v$ . In this paper, we obtain some results describing the signless Laplacian spectra of  $G[F, V_k, H_v]$  and  $G[F, E_k, H_{uv}]$  in terms of the signless Laplacian spectra of  $F, H_v$  and  $F, H_{uv}$ , respectively. In addition, we also give some results describing the adjacency spectrum of  $G[F, V_k, H_v]$  in terms of the adjacency spectra of  $F, H_v$ . Finally, as an application of these results, we construct infinitely many pairs of signless Laplacian (resp. adjacency) cospectral graphs.

*AMS classification:* 05C50 05C12

*Keywords:* Adjacency matrix; Signless Laplacian matrix; Spectrum; Pockets; Edge-pockets

## 1. Introduction

Throughout this paper, we consider only finite simple graphs. Let  $G = (V, E)$  be a graph with vertex set  $V = \{v_1, v_2, \dots, v_n\}$  and edge set  $E = \{e_1, e_2, \dots, e_n\}$ . The *adjacency matrix*  $A(G)$  of  $G$  is a square matrix of order  $n$ , whose entry  $a_{i,j} = 1$  if  $v_i$  and  $v_j$  are adjacent in  $G$  and 0 otherwise. Let  $D(G)$  be the degree

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diagonal matrix of  $G$ . Then the *Laplacian matrix*  $L(G)$  and *signless Laplacian matrix*  $Q(G)$  are defined as  $D(G) - A(G)$  and  $D(G) + A(G)$ , respectively.

For an  $n \times n$  matrix  $M$  associated to a graph  $G$ , the characteristic polynomial  $\det(xI_n - M)$  of  $M$  is called the *M-characteristic polynomial* of  $G$  and is denoted by  $f_M(x)$ . The eigenvalues of  $M$  (i.e. the zeros of  $\det(xI_n - M)$ ) and the spectrum of  $M$  (which consists of the  $n$  eigenvalues) are also called the *M-eigenvalues* of  $G$  and the *M-spectrum* of  $G$ , respectively. In particular, if  $M$  is the adjacency matrix  $A(G)$  of  $G$ , then the *A-spectrum* of  $G$  is denoted by  $\sigma(A(G)) = (\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G))$ , where  $\lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G)$  are the eigenvalues of  $A(G)$ . If  $M$  is the signless Laplacian matrix  $Q(G)$  of  $G$ , then the *Q-spectrum* of  $G$  is denoted by  $\sigma(Q(G)) = (q_1(G), q_2(G), \dots, q_n(G))$ , where  $q_1(G) \leq q_2(G) \leq \dots \leq q_n(G)$  are the eigenvalues of  $Q(G)$ . Throughout this paper, the *A-spectrum*, *L-spectrum* and *Q-spectrum* denote the adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of  $G$ , respectively. For more review about the *A-spectrum*, *L-spectrum* and *Q-spectrum* of  $G$ , readers may refer to [1, 4, 8, 9, 10, 11, 12, 13, 14, 18, 21] and the references therein.

The following two definitions come from [3] and [20], respectively.

**Definition 1.1**[3] Let  $F, H_v$  be graphs of orders  $n$  and  $m$ , respectively, where  $m \geq 2$ ,  $v$  be a specified vertex of  $H_v$  and  $V_k = \{v_1, \dots, v_k\}$  is a subset of the vertex set of  $F$ . Let  $G = G[F, V_k, H_v]$  be the graph obtained by taking one copy of  $F$  and  $k$  vertex disjoint copies of  $H_v$ , and then attaching the  $i$ th copy of  $H_v$  to the vertex  $u_i$ ,  $i = 1, \dots, k$ , at the vertex  $v$  of  $H$  (identify  $u_i$  with the vertex  $v$  of the  $i$ th copy). Then the copies of the graph  $H_v$  that are attached to the vertices  $u_i$ ,  $i = 1, \dots, k$  are referred to as *pockets*, and  $G$  is described as a *graph with  $k$  pockets*.

**Definition 1.2**[20] Let  $F$  and  $H_{uv}$  be two graphs of orders  $n$  and  $m$ , respectively, where  $n \geq 2, m \geq 3$ ,  $E_k = \{e_1, \dots, e_k\}$  is a subset of the edge set of  $F$  and  $H_{uv}$  has a specified edge  $uv$  such that  $H_{uv} - u$  is isomorphic to  $H_{uv} - v$ . Assume that  $\mathcal{E}_k$  denote the subgraph of  $F$  induced by  $E_k$ . Let  $G = G[F, E_k, H_{uv}]$  be the graph obtained by taking one copy of  $F$  and  $k$  vertex disjoint copies of  $H_{uv}$ , and then pasting the edge  $uv$  in the  $i$ th copy of  $H_{uv}$  with the edge  $e_i \in E_k$ , where  $i = 1, \dots, k$ . Then the copies of the graph  $H_{uv}$  that are pasted to the edges  $e_i$ ,  $i = 1, \dots, k$  are called as *edge-pockets*, and  $G$  is described as a *graph with  $k$  edge-pockets*.

Barik[3] has described the *L-spectrum* of  $G = G[F, V_k, H_v]$  using the *L-spectra* of  $F$  and  $H_v$ , when the specified vertex  $v$  is of degree  $m - 1$  in  $H_v$ . In that case, if a copy of  $H_v$  is attached to every vertex of  $F$ , each at the vertex  $v$  of  $H_v$ , that is, if  $G$  has  $n$  pockets, then the graph  $G = G[F, V_n, H_v]$  is nothing but the corona

$F \circ H$ , where  $H = H_v - v$ . Then the complete  $L$ -spectrum of  $G$  is described using the  $L$ -spectra of  $F$  and  $H$ [2]; and if  $H$  is a regular graph or a complete bipartite graph, then the complete  $A$ -spectrum and  $Q$ -spectrum of  $G$  are also described using the respective  $A$ -spectra and  $Q$ -spectra of  $F$  and  $H$ [2, 5, 6, 19].

Recently, Nath and Paul[20] has described the  $L$ -spectrum of  $G = G[F, E_k, H_{uv}]$  using the  $L$ -spectra of  $F$  and  $H_v$ , when the specified vertices  $u$  and  $v$  are of degree  $m - 1$ , and the subgraph  $\mathcal{E}_k$  of  $F$  induced by  $E_k$  is regular. Similarly, they also describe the  $A$ -spectrum, when  $H_{uv} - \{u, v\}$  is regular. In that case, if a copy of  $H_{uv}$  is pasted to every edge of  $F$ , each at the edge  $uv$  of  $H_{uv}$ , that is, if  $G$  has  $n$  edge-pockets, then the graph  $G = G[F, E_n, H_{uv}]$  is nothing but the edge-corona  $F \diamond H$ , where  $H = H_{uv} - \{u, v\}$ . Then the complete  $L$ -spectrum of  $G$  is described using the  $L$ -spectra of  $F$  and  $H$  when  $F$  is regular[16]; and if  $F$  is a regular graph and  $H$  is also a regular graph or a complete bipartite graph, then the complete  $A$ -spectrum and  $Q$ -spectrum of  $G$  are described using the respective  $A$ -spectra and  $Q$ -spectra of  $F$  and  $H$ [5, 6, 16].

Motivated by these researches, we discuss the  $Q$ -spectrum of  $G = G[F, V_k, H_v]$  and  $G[F, E_k, H_{uv}]$ . We also consider the  $A$ -spectrum of  $G = G[F, V_k, H_v]$  when  $H = H_v - v$  is regular. The rest of this paper is organized as follows. In Section 2, we present some preliminary results, which will be needed to prove our main results. In Section 3, we give the  $A$ -characteristic polynomials and  $Q$ -characteristic polynomials of  $G = G[F, V_k, H_v]$ . Using these results, we describe, except  $n + k$   $A$ -eigenvalues, all the other  $A$ -eigenvalues of  $G[F, V_k, H_v]$  in terms of the  $A$ -eigenvalues of  $H_v$ . We also show that the remaining  $n + k$   $A$ -eigenvalues of  $G[F, V_k, H_v]$  are independent of the graph  $H_v$ . For the  $Q$ -eigenvalues of  $G[F, V_k, H_v]$ , we also obtain the similar results. In Section 4, we give the  $Q$ -characteristic polynomials of  $G[F, E_k, H_{uv}]$  when  $\mathcal{E}_k$  be an  $r$ -regular subgraph of  $F$  induced by  $E_k$  in Definition 1.2. Using this result, we describe, except  $n + k$   $Q$ -eigenvalues, all the other  $Q$ -eigenvalues of  $G[F, E_k, H_{uv}]$  in terms of the  $Q$ -eigenvalues of  $H_{uv}$ . We also show that the remaining  $n + k$   $Q$ -eigenvalues of  $G[F, E_k, H_{uv}]$  are independent of the graph  $H_{uv}$ . In addition, we give a complete description of the  $Q$ -spectrum of  $G[F, E_k, H_{uv}]$  in some particular cases. At the same time, as an application of these results, we also consider to construct infinitely many pairs of  $A$ -cospectral and  $Q$ -cospectral graphs, respectively.

## 2. Preliminaries

In this section, we present some preliminary results which will be needed to prove our main results. In [6], Cui and Tian introduced a new invariant, the  $M$ -coronal  $\Gamma_M(x)$  of a matrix  $M$  of order  $n$  (also see [19]). It is defined to be the sum of the

entries of the matrix  $(xI_n - M)^{-1}$ , that is,

$$\Gamma_M(x) = \mathbf{1}_n^T (xI_n - M)^{-1} \mathbf{1}_n,$$

where  $\mathbf{1}_n$  denotes the column vector of size  $n$  with all the entries equal one and  $I_n$  is the identity matrix of order  $n$ . It is proved[6] that if  $M$  is a matrix of order  $n$  with each row sum equal to a constant  $t$ , then  $\Gamma_M(x) = \frac{n}{x-t}$ .

Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be  $m \times n$  and  $p \times q$  matrices, respectively. Then the *Kronecker product* of  $A$  and  $B$  is defined to the  $mp \times nq$  partition matrix  $(a_{ij}B)$  and is denoted by  $A \otimes B$ . For some properties of the Kronecker product of matrices, see [17].

Let  $G_1$  and  $G_2$  be two graphs with disjoint vertex sets  $V(G_1), V(G_2)$  and edge sets  $E(G_1), E(G_2)$ , respectively. The *join*  $G_1 \vee G_2$ [15] of  $G_1$  and  $G_2$  is the graph union  $G_1 \cup G_2$  together with all the edges joining  $V(G_1)$  and  $V(G_2)$ .

**Theorem 2.1**[12] *Let  $G_i$  be an  $r_i$  regular graph with  $n_i$  vertices, where  $i = 1, 2$ . Then*

$$f_{A(G_1 \vee G_2)}(x) = \frac{f_{A(G_1)}(x)f_{A(G_2)}(x)}{(x - r_1)(x - r_2)} ((x - r_1)(x - r_2) - n_1 n_2).$$

**Theorem 2.2**[7] *Let  $G_i$  be an  $r_i$  regular graph with  $n_i$  vertices, where  $i = 1, 2$ . Then*

$$f_{Q(G_1 \vee G_2)}(x) = \left( 1 - \frac{n_1 n_2}{(x - n_1 - 2r_2)(x - n_2 - 2r_1)} \right) f_{Q(G_1)}(x - n_2) f_{Q(G_2)}(x - n_1).$$

Throughout this paper, assume that  $F$  is a simple graph of order  $n$ ,  $H_v$  and  $H_{uv}$  two simple graphs of order  $m$ , unless mentioned otherwise. We also assume the specified vertex  $v$  in  $H_v$  is of degree  $m - 1$  and the specified vertices  $u$  and  $v$  in  $H_{uv}$  are all of degree  $m - 1$ . Let  $H_1 = H_v - v$ .  $H_2 = H_{uv} - \{u, v\}$ . Then  $H_v = \{v\} \vee H_1$  and  $H_{uv} = (\{u, v\}, \{uv\}) \vee H_2$ .

If  $H_1$  is  $r_1$ -regular, then by Theorems 2.1 and 2.2, we arrive at

$$\sigma(A(H_v)) = (\sigma(A(H_1)) - \{r_1\}) \cup \{\alpha, \beta\}, \quad (1)$$

where  $\alpha, \beta$  are roots of the equation  $x^2 - r_1 x - m + 1 = 0$ , and

$$\sigma(Q(H_v)) = \{q_j(H_1) + 1 | 1 \leq j \leq m - 2\} \cup \{\gamma, \delta\}, \quad (2)$$

where  $\gamma, \delta$  are roots of the equation  $(x - 2r_1 - 1)(x - m + 1) - m + 1 = 0$ .

Similarly, if  $H_2$  is  $r_2$ -regular, then by Theorem 2.2, we obtain

$$\sigma(Q(H_{uv})) = \{q_j(H_2) + 2 \mid 1 \leq j \leq m - 3\} \cup \{m - 2, \zeta, \eta\}, \quad (3)$$

where  $\zeta, \eta$  are roots of the equation  $(x - 2r_2 - 2)(x - m) - 2(m - 2) = 0$ .

### 3. The $A$ -spectrum and $Q$ -spectrum of $G[F, V_k, H_v]$

#### 3.1. The $A$ -spectrum of $G[F, V_k, H_v]$

**Proposition 3.1.** *Let  $G = G[F, V_k, H_v]$  and  $|V_k| = k$ . Then the  $A$ -characteristic polynomial of  $G$  is*

$$f_{A(G)}(x) = (f_{A(H_1)}(x))^k \det(xI_n - M), \quad (4)$$

where  $M = A(F) + \Gamma_{A(H_1)}(x) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix}$ .

**Proof.** With suitable labeling of the vertices of  $G$ , we can write the adjacency matrix of  $G$  to

$$A(G) = \begin{pmatrix} A(F) & \begin{pmatrix} I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \\ (I_k \otimes \mathbf{1}_{m-1} | 0) & I_k \otimes A(H_1) \end{pmatrix}.$$

Then the  $A$ -characteristic polynomial of  $G$  can be calculated as follows:

$$\begin{aligned} f_{A(G)}(x) &= \det \begin{pmatrix} xI_n - A(F) & \begin{pmatrix} -I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \\ (-I_k \otimes \mathbf{1}_{m-1} | 0) & I_k \otimes (xI_{m-1} - A(H_1)) \end{pmatrix} \\ &= \det(xI_{m-1} - A(H_1))^k \det(S_1) \\ &= (f_{A(H_1)}(x))^k \det(S_1), \end{aligned}$$

where

$$\begin{aligned} S_1 &= xI_n - A(F) - \begin{pmatrix} I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \cdot (I_k \otimes (xI_{m-1} - A(H_1)))^{-1} \cdot (I_k \otimes \mathbf{1}_{m-1} | 0) \\ &= xI_n - A(F) - \Gamma_{A(H_1)}(x) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} \end{aligned}$$

is the Schur complement with respect to  $I_k \otimes (xI_{m-1} - A(H_1))$ . This implies the required result.  $\square$

Let  $H_1$  be an  $r_1$ -regular graph and  $|V_k| = k$ . Then, except  $n+k$   $A$ -eigenvalues, we describe all the other  $A$ -eigenvalues of  $G[F, V_k, H_v]$  in terms of the  $A$ -eigenvalues

of  $H_v$ . We also show that the remaining  $n + k$   $A$ -eigenvalues of  $G[F, V_k, H_v]$  are independent of the graph  $H_v$ .

**Theorem 3.2.** *Let  $H_1$  be an  $r_1$ -regular graph, where  $r_1 \geq 1$ . Also let  $\lambda \in \sigma(A(H_v)) \setminus \{\alpha, \beta\}$ , where  $\alpha, \beta$  is described as (1), and  $G = G[F, V_k, H_v]$ . If  $|V_k| = k$ , then  $\lambda \in \sigma(A(G))$  with multiplicity  $k$ . Moreover, the remaining  $n + k$   $A$ -eigenvalues of  $G$  are independent of  $H_v$ .*

**Proof.** Since  $H_1$  is an  $r_1$ -regular graph. Thus  $\Gamma_{A(H_1)}(x) = \frac{m-1}{x-r_1}$  and

$$f_{A(H_1)}(x) = (x - r_1) \prod_{j=1}^{m-2} (x - \lambda_j(H_1)).$$

Now, from Proposition 3.1, one gets

$$f_{A(G)}(x) = (x - r_1)^k \prod_{j=1}^{m-2} (x - \lambda_j(H_1))^k \det(xI_n - M), \quad (5)$$

where  $M = A(F) + \frac{m-1}{x-r_1} \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix}$ . Notice that  $M$  depends on the regularity of  $H_1$  only and not on the structure of  $H_1$ . From (1) and (5), we obtain the required result.  $\square$

As an application of the above results, we may construct many pairs of  $A$ -cospectral graphs.

**Corollary 3.3.** *Let  $H_u$  and  $H_v$  be two disjoint graphs of order  $m$  such that  $H_u - u$  and  $H_v - v$  are  $r_1$ -regular, where  $m \geq 2$  and  $r_1 \geq 1$ . If  $H_u$  and  $H_v$  are  $A$ -cospectral, then  $G[F, V_k, H_u]$  and  $G[F, V_k, H_v]$  are  $A$ -cospectral.*

Let  $H_v^*$  be the  $H_v$  graph such that  $H_1 = G_p^{r_1}$ , where  $G_p^{r_1} = C_p \square K_{r_1-1}$  is the Cartesian product of the cycle  $C_p$  and the complete graph  $K_{r_1-1}$ . Notice that  $G_p^{r_1}$  is  $r_1$ -regular. Theorem 3.2 implies the following result.

**Corollary 3.4.** *Let  $H_1$  be an  $r_1$ -regular graph and  $m - 1 = p(r_1 - 1)$  for some integer  $p$ , where  $p \geq 3$  and  $r_1 \geq 2$ . Let  $G = G[F, V_k, H_v]$  and  $G^* = G[F, V_k, H_v^*]$ . If  $|V_k| = k$ , then  $\sigma(A(G))$  consists of the eigenvalues*

(a)  $\lambda$  with multiplicity  $k$ , for each  $\lambda \in \sigma(A(H_v)) \setminus \{\alpha, \beta\}$ , where  $\alpha, \beta$  is described as (1);

(b)

$$\theta \in \sigma(A(G^*)) \setminus \underbrace{\{\lambda_1(G_p^{r_1}), \dots, \lambda_1(G_p^{r_1})\}}_k \cup \underbrace{\{\lambda_2(G_p^{r_1}), \dots, \lambda_2(G_p^{r_1})\}}_k \cup \dots \cup \underbrace{\{\lambda_{m-2}(G_p^{r_1}), \dots, \lambda_{m-2}(G_p^{r_1})\}}_k.$$

**Proof.** With suitable labeling of the vertices of  $G^*$ , we can write the adjacency matrix of  $G^*$  to

$$A(G^*) = \begin{pmatrix} A(F) & \begin{pmatrix} I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \\ (I_k \otimes \mathbf{1}_{m-1} | 0) & I_k \otimes A(G_p^r) \end{pmatrix}.$$

Thus, from the proofs of Proposition 3.1 and Theorem 3.2, one obtains

$$f_{A(G^*)}(x) = (x - r)^k \prod_{j=1}^{m-2} (x - \lambda_j(G_p^{r_1}))^k \det(xI_n - M), \quad (6)$$

where  $M = A(F) + \frac{m-1}{x-r_1} \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix}$ . It follows from (5) and (6) that

$$f_{A(G)}(x) = \prod_{j=1}^{m-2} (x - \lambda_j(H_1))^k \frac{f_{A(G^*)}(x)}{\prod_{j=1}^{m-2} (x - \lambda_j(G_p^{r_1}))^k}. \quad (7)$$

From (1) and (7), we get  $\lambda \in \sigma(A(G))$  with multiplicity  $k$ , for each  $\lambda \in \sigma(A(H_v)) \setminus \{\alpha, \beta\}$ . Next we only need to prove  $\lambda_j(G_p^{r_1}) \in \sigma(A(G^*))$  with multiplicity  $k$ , for each  $j = 1, \dots, m-2$ .

Let the eigenvalues  $\lambda_1(C_p), \dots, \lambda_p(C_p) = 2$  of  $A(C_p)$  are afforded by the eigenvectors  $X_1, \dots, X_p = \mathbf{1}_p$ , respectively. Also let the eigenvalues  $\lambda_1(K_{r_1-1}), \dots, \lambda_{r_1-1}(K_{r_1-1}) = r_1 - 2$  of  $A(K_{r_1-1})$  are afforded by the eigenvectors  $Y_1, \dots, Y_{r_1-1} = \mathbf{1}_{r_1-1}$ , respectively. Since  $A(G_p^{r_1}) = A(C_p) \otimes I_{r_1-1} + I_p \otimes A(K_{r_1-1})$ . Then  $X_s \otimes Y_t$  is an eigenvector of  $A(G_p^{r_1})$  corresponding to the eigenvalue  $\lambda_s(C_p) + \lambda_t(K_{r_1-1})$ , where  $s = 1, \dots, p$  and  $t = 1, \dots, r_1 - 1$ . Now, for fixed  $s$  and  $t$ , it is easy to verify that

$$\begin{pmatrix} 0 \\ e_1 \otimes X_s \otimes Y_t \end{pmatrix}, \begin{pmatrix} 0 \\ e_2 \otimes X_s \otimes Y_t \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_k \otimes X_s \otimes Y_t \end{pmatrix}$$

are  $k$  linearly independent eigenvectors of  $A(G^*)$  corresponding to the eigenvalue  $\lambda_s(C_p) + \lambda_t(K_{r_1-1})$ , where  $X_s \otimes Y_t \neq X_p \otimes Y_{r_1-1}$  and  $e_i$  denotes the column vector of size  $k$  with the  $i$ -th entry equals one and 0 otherwise. The proof is completed.

□

### 3.2. The $Q$ -spectrum of $G[F, V_k, H_v]$

**Proposition 3.5.** *Let  $G = G[F, V_k, H_v]$  and  $|V_k| = k$ . Then the  $Q$ -characteristic polynomial of  $G$  is*

$$f_{Q(G)}(x) = (f_{Q(H_1)}(x-1))^k \det(xI_n - M), \quad (8)$$

where  $M = Q(F) + (m-1 + \Gamma_{Q(H_1)}(x-1)) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix}$ .

**Proof.** With suitable labeling of the vertices of  $G$ , we can write the signless Laplacian matrix of  $G$  to

$$Q(G) = \begin{pmatrix} Q(F) + (m-1) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \\ (I_k \otimes \mathbf{1}_{m-1} | 0) & I_k \otimes (Q(H_1) + I_{m-1}) \end{pmatrix}.$$

Then the  $Q$ -characteristic polynomial of  $G$  can be calculated as follows:

$$\begin{aligned} f_{Q(G)}(x) &= \det \begin{pmatrix} xI_n - Q(F) - (m-1) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \\ (-I_k \otimes \mathbf{1}_{m-1} | 0) & I_k \otimes ((x-1)I_{m-1} - Q(H_1)) \end{pmatrix} \\ &= \det((x-1)I_{m-1} - Q(H_1))^k \det(S_1) \\ &= (f_{Q(H_1)}(x-1))^k \det(S_1) \end{aligned}$$

where

$$\begin{aligned} S_1 &= xI_n - Q(F) - (m-1) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} (I_k \otimes ((x-1)I_{m-1} - Q(H_1)))^{-1} (I_k \otimes \mathbf{1}_{m-1} | 0) \\ &= xI_n - Q(F) - (m-1) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} - \Gamma_{Q(H_1)}(x-1) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} \end{aligned}$$

is the Schur complement with respect to  $I_k \otimes ((x-1)I_{m-1} - Q(H_1))$ . Hence, the result follows.  $\square$

Let  $H_1$  be a  $r_1$ -regular graph. If  $|V_k| = k$ , then except  $n+k$   $Q$ -eigenvalues, we describe all the other  $Q$ -eigenvalues of  $G[F, V_k, H_v]$  using the  $Q$ -eigenvalues of  $H_v$ . We also show that the remaining  $n+k$   $Q$ -eigenvalues of  $G[F, V_k, H_v]$  are independent of the graph  $H_v$ .

**Theorem 3.6.** *Let  $H_1$  be an  $r_1$ -regular graph, where  $r_1 \geq 1$ . Also let  $q \in \sigma(Q(H_v)) \setminus \{\gamma, \delta\}$ , where  $\gamma, \delta$  is described as (2), and  $G = G[F, V_k, H_v]$ . If  $|V_k| = k$ , then  $q \in \sigma(Q(G))$  with multiplicity  $k$ . Moreover, the remaining  $n+k$*

$Q$ -eigenvalues of  $G$  are independent of  $H_v$ .

**Proof.** If  $H_1$  is an  $r_1$ -regular graph, then  $\Gamma_{Q(H_1)}(x-1) = \frac{m-1}{x-1-2r_1}$  and

$$f_{Q(H_1)}(x-1) = (x-1-2r_1) \prod_{j=1}^{m-2} (x-1-q_j(H_1)).$$

Thus, from Proposition 3.5, one gets

$$f_{Q(G)}(x) = (x-1-2r_1)^k \prod_{j=1}^{m-2} (x-1-q_j(H_1))^k \det(xI_n - M), \quad (9)$$

where  $M = Q(F) + (m-1 + \frac{m-1}{x-1-2r_1}) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix}$ . Notice that  $M$  depends on the regularity of  $H_1$  only and not on the structure of  $H_1$ . From (2) and (9), we obtain the required result.  $\square$

As an application of the above results, we may construct many pairs of  $Q$ -cospectral graphs. The following Corollary 3.7 and above Corollary 3.3 show that there exist two graphs  $G[F, V_k, H_u]$  and  $G[F, V_k, H_v]$  such that they are not only  $A$ -cospectral, but also are  $Q$ -cospectral.

**Corollary 3.7.** *Let  $H_u$  and  $H_v$  be two disjoint graphs of order  $m$  such that  $H_u - u$  and  $H_v - v$  are  $r_1$ -regular, where  $m \geq 2$  and  $r_1 \geq 1$ . If  $H_u$  and  $H_v$  are  $Q$ -cospectral, then  $G[F, V_k, H_u]$  and  $G[F, V_k, H_v]$  are  $Q$ -cospectral.*

**Corollary 3.8.** *Let  $H_1$  be an  $r_1$ -regular graph and  $m-1 = p(r_1-1)$  for some integer  $p$ , where  $p \geq 3$  and  $r_1 \geq 2$ . Let  $G = G[F, V_k, H_v]$  and  $G^* = G[F, V_k, H_v^*]$ . If  $|V_k| = k$ , then  $\sigma(Q(G))$  consists of the eigenvalues*

(a)  $q$  with multiplicity  $k$ , for each  $q \in \sigma(Q(H_v)) \setminus \{\gamma, \delta\}$ , where  $\gamma, \delta$  is described as (2);

(b)

$$\theta \in \sigma(Q(G^*)) \setminus \left\{ \underbrace{q_1(G_p^{r_1}) + 1, \dots, q_1(G_p^{r_1}) + 1}_k, \underbrace{q_2(G_p^{r_1}) + 1, \dots, q_2(G_p^{r_1}) + 1}_k, \dots, \underbrace{q_{m-2}(G_p^{r_1}) + 1, \dots, q_{m-2}(G_p^{r_1}) + 1}_k \right\}.$$

**Proof.** With suitable labeling of the vertices of  $G^*$ , we can write the signless Laplacian matrix of  $G^*$  to

$$Q(G^*) = \begin{pmatrix} Q(F) + (m-1) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} I_k \otimes \mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \\ (I_k \otimes \mathbf{1}_{m-1} | 0) & I_k \otimes (Q(G_p^{r_1}) + I_{m-1}) \end{pmatrix}$$

Thus, from the proofs of Proposition 3.5 and Theorem 3.6, we obtain

$$f_{Q(G^*)}(x) = (x - 1 - 2r_1)^k \prod_{j=1}^{m-2} (x - 1 - q_j(G_p^{r_1}))^k \det(xI_n - M), \quad (10)$$

where  $M = Q(F) + (m - 1 + \frac{m-1}{x-1-2r_1}) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix}$ . It follows from (9) and (10) that

$$f_{Q(G)}(x) = \prod_{j=1}^{m-2} (x - 1 - q_j(H_1))^k \frac{f_{Q(G^*)}(x)}{\prod_{j=1}^{m-2} (x - 1 - q_j(G_p^r))^k} \quad (11)$$

From (2) and (11), we get  $q \in \sigma(Q(G))$  with multiplicity  $k$ , for each  $q \in \sigma(Q(H_v)) \setminus \{\gamma, \delta\}$ . Next we only need to prove  $q_j(G_p^{r_1}) + 1 \in \sigma(Q(G^*))$  with multiplicity  $k$ , for each  $j = 1, \dots, m - 2$ .

Let the eigenvalues  $q_1(C_p), \dots, q_p(C_p) = 4$  of  $Q(C_p)$  are afforded by the eigenvectors  $X_1, \dots, X_p = \mathbf{1}_p$ , respectively, and the eigenvalues  $q_1(K_{r_1-1}), \dots, q_{r_1-1}(K_{r_1-1}) = 2(r_1-2)$  of  $Q(K_{r_1-1})$  are afforded by the eigenvectors  $Y_1, \dots, Y_{r_1-1} = \mathbf{1}_{r_1-1}$ , respectively. Since  $Q(G_p^{r_1}) = Q(C_p) \otimes I_{r_1-1} + I_p \otimes Q(K_{r_1-1})$ . Then  $X_s \otimes Y_t$  is an eigenvector of  $Q(G_p^{r_1})$  corresponding to the eigenvalue  $q_s(C_p) + q_t(K_{r_1-1})$ , where  $s = 1, \dots, p$  and  $t = 1, \dots, r_1 - 1$ . Now, for fixed  $s$  and  $t$ , then

$$\begin{pmatrix} 0 \\ e_1 \otimes X_s \otimes Y_t \end{pmatrix}, \begin{pmatrix} 0 \\ e_2 \otimes X_s \otimes Y_t \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_k \otimes X_s \otimes Y_t \end{pmatrix}$$

are  $k$  linearly independent eigenvectors of  $Q(G^*)$  corresponding to the eigenvalue  $q_s(C_p) + q_t(K_{r_1-1}) + 1$ , where  $X_s \otimes Y_t \neq X_p \otimes Y_{r_1-1}$ . The proof is completed.  $\square$

## 4. The $Q$ -spectrum of $G[F, E_k, H_{uv}]$

**Proposition 4.1.** *Let  $\mathcal{E}_k$  be an  $r$ -regular subgraph of  $F$  induced by  $E_k$  in Definition 1.2. Also let  $G = G[F, E_k, H_{uv}]$  and  $|E_k| = k$ . Then the  $Q$ -characteristic polynomial of  $G$  is*

$$f_{Q(G)}(x) = (f_{Q(H_2)}(x - 2))^k \det(xI_n - M), \quad (12)$$

where  $M = Q(F) + r(m - 2) \begin{pmatrix} I_p & 0^T \\ 0 & 0 \end{pmatrix} + \Gamma_{Q(H_2)}(x - 2) \begin{pmatrix} Q(\mathcal{E}_k) & 0^T \\ 0 & 0 \end{pmatrix}$ .

**Proof.** Notice that  $\mathcal{E}_k$  has  $p = \frac{2k}{r}$  vertices. Let  $Q(\mathcal{E}_k)$  be the signless Laplacian matrix of  $\mathcal{E}_k$ . With suitable labeling of the vertices of  $G$ , we can write the signless Laplacian matrix of  $G$  to

$$Q(G) = \begin{pmatrix} Q(F) + r(m-2) \begin{pmatrix} I_p & 0^T \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{R(\mathcal{E}_k)}{0} \end{pmatrix} \otimes \mathbf{1}_{m-2}^T \\ (R(\mathcal{E}_k)^T | 0) \otimes \mathbf{1}_{m-2} & I_k \otimes (Q(H_2) + 2I_{m-2}) \end{pmatrix}.$$

Then the  $Q$ -characteristic polynomial of  $G$  can be calculated as follows:

$$\begin{aligned} f_{Q(G)}(x) &= \det \begin{pmatrix} xI_n - Q(F) - r(m-2) \begin{pmatrix} I_p & 0^T \\ 0 & 0 \end{pmatrix} & - \begin{pmatrix} \frac{R(\mathcal{E}_k)}{0} \end{pmatrix} \otimes \mathbf{1}_{m-2}^T \\ - (R(\mathcal{E}_k)^T | 0) \otimes \mathbf{1}_{m-2} & I_k \otimes ((x-2)I_{m-2} - Q(H_2)) \end{pmatrix} \\ &= \det((x-2)I_{m-2} - Q(H_2))^k \det(S_1) \\ &= (f_{Q(H_2)}(x-2))^k \det(S_1), \end{aligned}$$

where

$$\begin{aligned} S_1 &= xI_n - Q(F) - r(m-2) \begin{pmatrix} I_p & 0^T \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} \frac{R(\mathcal{E}_k)}{0} \end{pmatrix} \otimes \mathbf{1}_{m-2}^T \cdot (I_k \otimes ((x-2)I_{m-2} - Q(H_2)))^{-1} \cdot (R(\mathcal{E}_k)^T | 0) \otimes \mathbf{1}_{m-2} \\ &= xI_n - Q(F) - r(m-2) \begin{pmatrix} I_p & 0^T \\ 0 & 0 \end{pmatrix} - \Gamma_{Q(H_2)}(x-2) \begin{pmatrix} Q(\mathcal{E}_k) & 0^T \\ 0 & 0 \end{pmatrix} \end{aligned}$$

is the Schur complement with respect to  $I_k \otimes ((x-2)I_{m-2} - Q(H_2))$ . This implies the required result.  $\square$

Let  $H_2$  be a  $r_2$ -regular graph. If  $|E_k| = k$ , then except  $n+k$   $Q$ -eigenvalues, we describe all the other  $Q$ -eigenvalues of  $G[F, E_k, H_{uv}]$  in term of the  $Q$ -eigenvalues of  $H_{uv}$ . We also show that the remaining  $n+k$   $Q$ -eigenvalues of  $G[F, E_k, H_{uv}]$  are independent of the graph  $H_{uv}$ .

**Theorem 4.2.** *Let  $H_2$  be an  $r_2$ -regular graph, where  $r_2 \geq 2$ . Also let  $q \in \sigma(Q(H_{uv})) \setminus \{m-2, \zeta, \eta\}$ , where  $\zeta, \eta$  is described as (3), and  $G[F, E_k, H_{uv}]$ . Assume that  $\mathcal{E}_k$  is an  $r$ -regular subgraph of  $F$  induced by  $E_k$ . If  $|E_k| = k$ , then  $q \in \sigma(Q(G))$  with multiplicity  $k$ . Moreover, the remaining  $n+k$   $Q$ -eigenvalues of  $G$  are independent of  $H_{uv}$ .*

**Proof.** Since  $H_2$  is an  $r_2$ -regular graph. Then  $\Gamma_{Q(H_2)}(x-2) = \frac{m-2}{x-2-2r_2}$  and

$$f_{Q(H_2)}(x-2) = (x-2-2r_2) \prod_{j=1}^{m-3} (x-2-q_j(H_2)).$$

Thus by Proposition 4.1,

$$f_{Q(G)}(x) = (x - 2 - 2r_2)^k \prod_{j=1}^{m-3} (x - 2 - q_j(H_2))^k \det(xI_n - M), \quad (13)$$

where  $M = Q(F) + (m - 2) \begin{pmatrix} rI_p + \frac{Q(\mathcal{E}_k)}{x-2-2r_2} & 0^T \\ 0 & 0 \end{pmatrix}$ . Notice that  $M$  depends on the regularity of  $H_2$  only and not on the structure of  $H_2$ . From (3) and (13), we obtain the required result.  $\square$

As an application of the above results, we may construct many pairs of  $Q$ -cospectral graphs.

**Corollary 4.3.** *Let  $H_{uv}$  and  $H_{xy}$  be two disjoint graphs of order  $m$  such that  $H_{uv} - \{u, v\}$  and  $H_{xy} - \{x, y\}$  are  $r_2$ -regular, where  $m \geq 3$  and  $r_2 \geq 2$ . Let  $\mathcal{E}_k$  be an  $r$ -regular subgraph of  $F$  induced by  $E_k$  in Definition 1.2. If  $H_{uv}$  and  $H_{xy}$  are  $Q$ -cospectral, then  $G[F, E_k, H_{uv}]$  and  $G[F, E_k, H_{xy}]$  are  $Q$ -cospectral.*

Let  $H_{uv}^{**}$  be the  $H_{uv}$  graph such that  $H_2 = G_p^{r_2}$ , where  $G_p^{r_2} = C_p \square K_{r_2-1}$  is the Cartesian product of the cycle  $C_p$  and the complete graph  $K_{r_2-1}$ . Notice that  $G_p^{r_2}$  is  $r_2$ -regular. Theorem 4.2 implies the following result.

**Corollary 4.4.** *Let  $H_2$  be an  $r_2$ -regular graph and  $m - 2 = p(r_2 - 1)$  for some integer  $p$ , where  $p \geq 3$  and  $r_2 \geq 2$ . Let  $G = G[F, E_k, H_{uv}]$  and  $G^{**} = G[F, E_k, H_{uv}^{**}]$ . Assume that  $\mathcal{E}_k$  is an  $r$ -regular subgraph of  $F$  induced by  $E_k$ . If  $|E_k| = k$ , then  $\sigma(Q(G))$  consists of the eigenvalues*

(a)  $q$  with multiplicity  $k$ , for each  $q \in \sigma(Q(H_{uv})) \setminus \{m - 2, \zeta, \eta\}$ , where  $\zeta, \eta$  is described as (3);

$$(b) \theta \in \sigma(Q(G^{**})) \setminus \left\{ \underbrace{q_1(G_p^{r_2}) + 2, \dots, q_1(G_p^{r_2}) + 2}_k, \underbrace{q_2(G_p^{r_2}) + 2, \dots, q_2(G_p^{r_2}) + 2}_k, \dots, \underbrace{q_{m-3}(G_p^{r_2}) + 2, \dots, q_{m-3}(G_p^{r_2}) + 2}_k \right\}.$$

**Proof.** With suitable labeling of the vertices of  $G^{**}$ , we can write the signless Laplacian matrix of  $G^{**}$  to

$$Q(G^{**}) = \begin{pmatrix} Q(F) + r(m - 2) \begin{pmatrix} I_p & 0^T \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} R(\mathcal{E}_k) \\ 0 \end{pmatrix} \otimes \mathbf{1}_{m-2}^T \\ (R(\mathcal{E}_k)^T | 0) \otimes \mathbf{1}_{m-2} & I_k \otimes (Q(G_p^{r_2}) + 2I_{m-2}) \end{pmatrix}.$$

Thus, from the proofs of Proposition 4.1 and Theorem 4.2, we obtain

$$f_{Q(G^{**})}(x) = (x - 2 - 2r_2)^k \prod_{j=1}^{m-3} (x - 2 - q_j(G_p^{r_2}))^k \det(xI_n - M), \quad (14)$$

where  $M = Q(F) + (m-2) \begin{pmatrix} rI_p + \frac{Q(\mathcal{E}_k)}{x-2-2r_2} & 0^T \\ 0 & 0 \end{pmatrix}$ . It follows from (13) and (14) that

$$f_{Q(G)}(x) = \prod_{j=1}^{m-3} (x - 2 - q_j(H_2))^k \frac{f_{Q(G^{**})}(x)}{\prod_{j=1}^{m-3} (x - 2 - q_j(G_p^{r_2}))^k} \quad (15)$$

From (3) and (15), we get  $q \in \sigma(Q(G))$  with multiplicity  $k$ , for each  $q \in \sigma(Q(H_{uv})) \setminus \{m-2, \zeta, \eta\}$ . Next, using the similar technique to the proof of Corollary 3.8, we may prove that  $q_j(G_p^{r_2}) + 2 \in \sigma(Q(G^{**}))$  with multiplicity  $k$ , for each  $j = 1, \dots, m-3$ . The proof is completed.  $\square$

In the rest of this paper, we consider many special cases. Assume that  $\mathcal{E}_k$  is an  $r$ -regular spanning subgraph of  $F$ . From the proof of Proposition 4.1, we easily obtain

**Proposition 4.4.** *Let  $\mathcal{E}_k$  be an  $r$ -regular spanning subgraph of  $F$ . Also let  $G = G[F, E_k, H_{uv}]$  and  $|E_k| = k$ . Then the  $Q$ -characteristic polynomial of  $G$  is*

$$f_{Q(G)}(x) = (f_{Q(H_2)}(x-2))^k \det(xI_n - M), \quad (16)$$

where  $M = r((m-2) + \Gamma_{Q(H_2)}(x-2))I_n + Q(F) + \Gamma_{Q(H_2)}(x-2)A(\mathcal{E}_k)$ .

A  $k$ -matching in  $G$  is a disjoint union of  $k$ -edges. If  $2k$  is the order of  $G$ , then a  $k$ -matching of  $G$  is called a *perfect matching* of  $G$  ([15]).

**Theorem 4.5.** *Let  $F = K_{2k}$ ,  $\mathcal{E}_k$  be a perfect matching of  $F$ . Also let  $H_2$  be an  $r_2$ -regular graph, where  $r_2 \geq 2$  and  $G = G[F, E_k, H_{uv}]$ . Then the  $Q$ -spectrum of  $G$  is given by*

(i)  $q$  with multiplicity  $k$ , for each  $q \in \sigma(Q(H_{uv})) \setminus \{m-2, \zeta, \eta\}$ , where  $\zeta, \eta$  is described as (3);

(ii)  $m + 2k - 4$  with multiplicity  $k$ ;

(iii) two roots of the equation  $(x - m - 4k + 4)(x - 2r_2 - 2) - 2(m - 2) = 0$ , each with multiplicity 1;

(iv) two roots of the equation  $(x - m - 2k + 4)(x - 2r_2 - 2) - 2(m - 2) = 0$ , each with multiplicity  $k - 1$ .

**Proof.** From Proposition 4.4, we obtain

$$f_{Q(G)}(x) = (x - 2 - 2r_2)^k \prod_{j=1}^{m-3} (x - 2 - q_j(H_2))^k \det(xI_{2k} - M), \quad (17)$$

where

$$\begin{aligned} M &= ((m - 2) + \Gamma_{Q(H_2)}(x - 2))I_{2k} + (2k - 2)I_{2k} + J_{2k} + \Gamma_{Q(H_2)}(x - 2)(I_k \otimes A(K_2)) \\ &= \left(m - 2 + \frac{m-2}{x-2-2r_2} + 2k - 2\right) I_{2k} + \frac{m-2}{x-2-2r_2}(I_k \otimes A(K_2)) + J_{2k}. \end{aligned}$$

Take  $M_1 = \left(m - 2 + \frac{m-2}{x-2-2r_2} + 2k - 2\right) I_{2k} + \frac{m-2}{x-2-2r_2}(I_k \otimes A(K_2))$ . By a simple computation, one gets

$$\begin{aligned} \det(xI_{2k} - M) &= \det(xI_{2k} - M_1 - J_{2k}) \\ &= \det(xI_{2k} - M_1) \cdot (1 - \Gamma_{M_1}(x)) \\ &= (x - m - 2k + 4)^k \left(\frac{(x-m-2k+4)(x-2-2r_2)-2(m-2)}{x-2-2r_2}\right)^{k-1} \left(\frac{(x-m-4k+4)(x-2-2r_2)-2(m-2)}{x-2-2r_2}\right). \end{aligned} \quad (18)$$

From (17) and (18), we obtain the required result.  $\square$

**Theorem 4.6.** *Let  $F = K_n$ ,  $\mathcal{E}_k = C_n$ . Also let  $H_2$  be an  $r_2$ -regular graph, where  $r_2 \geq 2$  and  $G = G[F, E_k, H_{uv}]$ . Then the  $Q$ -spectrum of  $G$  is given by*

(i)  $q$  with multiplicity  $n$ , for each  $q \in \sigma(Q(H_{uv})) \setminus \{m - 2, \zeta, \eta\}$ , where  $\zeta, \eta$  is described as (3);

(ii) two roots of the equation  $(x - 2m - 2n + 6)(x - 2r_2 - 2) - 4(m - 2) = 0$ , each with multiplicity 1;

(iii) two roots of the equation  $(x - 2m - n + 6)(x - 2r_2 - 2) - 2(m - 2)(1 + \cos \frac{2\pi l}{n}) = 0$ , for each  $l = 1, 2, \dots, n - 1$ .

**Proof.** The proof is similar to that of Theorem 4.5, omitted.  $\square$

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