Hypercomplex Fock States for Discrete Electromagnetic Schrödinger Operators: A Bayesian Probability Perspective

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Abstract

We present and study a new class of Fock states underlying to discrete electromagnetic Schrödinger operators from a multivector calculus perspective. This naturally lead to hypercomplex versions of Poisson-Charlier polynomials, Meixner polynomials, among other ones. The foundations of this work are based on the exploitation of the quantum probability formulation 'à la Dirac' to the setting of Bayesian probabilities, on which the Fock states arise as discrete quasi-probability distributions carrying a set of *independent and identically distributed* (i.i.d) random variables. By employing Mellin-Barnes integrals in the complex plane we obtain counterparts for the well-known multidimensional Poisson and hypergeometric distributions, as well as quasi-probability distributions that may take negative or complex values on the lattice $h\mathbb{Z}^n$.

Keywords: Clifford algebras, Fock states, generalized Mittag-Leffler functions, generalized Wright functions, quasi-probability distributions *2010 MSC:* 26A33, 30G35, 33C20, 62F15, 81Q60

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1. Introduction

Discrete electromagnetic Schrödinger operators correspond to a subclass of (doubly) Jacobi operators. They are ubiquitous in several fields of mathematics, physics and beyond, as is witnessed by the papers [16, 18, 35, 38, 31, 8, 6, 37, 24, 1] and monograph [36]. Here, the factorization method is the cornerstone in the study of the quasi-exact solvability of such kind of operators since it avoids non-perturbative arguments that appear under the discretization of its *continuum* counterpart, the quantum harmonic oscillator $-\frac{1}{2m}\Delta + V(x)$ with mass m and potential V(x) (cf. [16, 34]). In case of crystallographic root systems are involved, the discrete electromagnetic Schrödinger operators may be described as discrete (pseudo) Laplacians (cf. [38, Section 6]), whose origin goes back to the works of Macdonald [21, 22]. As it was shown in Ruijsenaars's seminal work [33], Macdonald's theory may be obtained as a special case of integrable lattice models of Calogero-Moser type that exhibit factorized scattering. For further details, we refer to [32].

The main objective of this paper is to show the feasibility of special functions of hypercomplex variable, with values on the Clifford algebra of signature (0, n), as Fock states of a certain multidimensional Schrödinger operator L_h acting on the lattice

$$h\mathbb{Z}^n = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \frac{x_j}{h} \in \mathbb{Z} , \text{ for all } j = 1, 2, \dots, n \right\},\$$

with mesh width h > 0.

In the series of papers [13, 14, 15] the author developed a framework to compute, in a direct manner, quasi-monomials of discrete hypercomplex variable from the knowledge of a underlying set of Lie-algebraic symmetries. The methods and techniques employed are closely related with Wigner's quantal systems and go far beyond the symmetries of the Weyl-Heisenberg algebra, mentioned in many textbooks as the underlying symmetries encoded by Hermite polynomials/functions (cf. [9]).

In this paper we center our analysis on questions regarding the quasi-exact solvability associated to a discretization L_h of a Sturm-Liouville type operator. This essentially corresponds to the problem formulation:

Problem 1.1. Given a pair of Clifford-vector-valued operators (A_h^+, A_h^-) satisfying

$$L_{h} = \frac{1}{2} \left(A_{h}^{+} A_{h}^{-} + A_{h}^{-} A_{h}^{+} \right),$$

can we recover the discrete electric and magnetic potentials of L_h , $\Phi_h(x)$ and $\mathbf{a}_h(x)$ respectively, from the knowledge of its k-Fock states $\psi_k(x;h)$ ($k \in \mathbb{N}_0$)? Here, the construction of the pair (A_h^+, A_h^-) was inspired on Spiridonov-Vinet-Zhedanov approach [34] and roughly follows the same order of ideas used on Odake-Sasaki's papers [27, 28, 29, 30] to generate one-dimensional 'discrete' quantum systems through the Supersymmetric Quantum Mechanics (SUSY QM)² framework.

We are not concerned here with a SUSY QM extension/generalization to hypercomplex variables in the way that the k-Fock states are eigenfunctions of one of the Hamiltonians, $A_h^- A_h^+$ and $A_h^+ A_h^-$ respectively, neither with an exploitation of the commutation method (cf. [36, Chapter 11]). On the context of this paper, the k-Fock states $\psi_k(x; h)$ shall be understood as basis functions with membership in a certain linear subspace \mathcal{F}_h of the Hilbert module $\ell_2(h\mathbb{Z}^n; C\ell_{0,n}) = \ell_2(h\mathbb{Z}^n) \otimes C\ell_{0,n}$, generated from (A_h^+, A_h^-) - the so-called Fock space \mathcal{F}_h , to be defined later on this paper.

Of particular importance for the development of this approach will be the connection with Bayesian probabilities that results from the observation that, for a given ground state $\psi_0(x;h)$ satisfying $\langle \psi_0, \psi_0 \rangle_h = 1$, the quantity

$$\Pr\left(\sum_{j=1}^{n} \mathbf{e}_j X_j = x\right) = h^n \psi_0(x;h)^{\dagger} \psi_0(x;h) \tag{1}$$

may be regarded as a discrete quasi-probability law on $h\mathbb{Z}^n$, carrying a set of *independent* and *identically distributed* (i.i.d.) random variables X_1, X_2, \ldots, X_n .

This quasi-probability formulation is reminiscent of a similar probability formulation, considered in the context of transition probabilities (cf. [7, 26]). In that scope, the Bayesian scheme is achieved to determine the expectation values of quantum observables, which are essentially the Landau levels attached to the discrete electromagnetic Schrödinger operator (5) when one considers the minimization problem

$$\psi = \operatorname{argmin}_{\widetilde{\psi}} \frac{\langle \widetilde{\psi}, L_h \widetilde{\psi} \rangle_h}{\langle \widetilde{\psi}, \widetilde{\psi} \rangle_h}$$

to seek the quantum state ψ with 'best energy concentration' in $h\mathbb{Z}^n$.

Accordingly to the general theory, in case that L_h is real-valued and symmetric – the so-called Hermitian condition – is sufficient to guarantee that L_h is quasi-exactly solvable (cf. [35, Proposition 1.4]). That's indeed the case of the characterization provided through the formulation of **Problem 1.1** (see also Appendix A.2). Surprisingly enough, Bender and its collaborators have been stressed in a series of papers (see [2, 3, 4] and the references given there) that such condition is not necessary³ and may be replaced with a most general one, involving a space-time reflection symmetry (shorty, a \mathcal{PT} symmetry) constraint. Thus, it may happen that the right-hand side of (1) may also take complex values (cf. [5]).

To be in accordance with Dirac's insight [11] on quantum probabilities, we will consider throughout this paper the \dagger -operation provided by (3), also for bound states that take values in the complexified Clifford algebra $\mathbb{C} \otimes C\ell_{0,n}$. We turn next to the content and the organization of the subsequent sections:

²The fundamentals of SUSY QM can be traced back to the seminal work of Cooper-Khare-Sukhatme [10], where the interest lies essentially in the solution of Pauli and Dirac equations (cf. [10, Section 10. & Section 11.]).

³See also the examples treated in [10, Subsection 12.1.] and in [1, Section 6].

- In Section 2 we will introduce the basic setting that will be used throughout the paper, namely the multivector calculus embody in the Clifford algebra $C\ell_{0,n}$ in the spirit of Sturm-Liouville theory. We will also introduce some basic features in the context of Fock spaces (cf. [17]) to describe the Fock states of the discrete electromagnetic Schrödinger operator L_h on $h\mathbb{Z}^n$.
- In Section 3 we will take into account the factorization of L_h and the vacuum vector $\psi_0(x;h)$ of A_h^+ to display a correspondence between the Fock states of the form $\psi_k(x;h) = (A_h^-)^k \psi_0(x;h)$ and the quasi-monomials, encoded by the pair (D_h^+, M_h) , where D_h^+ stands the finite difference Dirac operator of forward type. That corresponds to Lemma 3.1. Moreover, with Proposition 3.1 and Proposition 3.2 we hereby provide an answer to **Problem 1.1**.
- In Section 4 we will make use of the Bayesian probability framework beyond Dirac's insight [11] to compute some examples involving the well-known Poisson and hypergeometric distributions, likewise quasi-probability distributions involving the generalized Mittag-Leffler/Wright functions.
- In Section 5 we conclude with a more detailed discussion of Bayesian probabilities with *imaginary* bias, towards the regularization of the Mittag-Leffler distribution.
- In Section 6 we will outlook the main contributions obtained and will raise some problems/questions to be investigated afterwards.

2. The Setting

We start this section by collecting some basic facts about Clifford algebras that will be used on the sequel. We refer [19, Chapter 1] for further details.

Recall that $C\ell_{0,n}$ is the algebra generated by the set of vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ that satisfy, for each $j, k = 1, 2, \ldots, n$, the set of anti-commuting relations

$$\mathbf{e}_j \mathbf{e}_k + \mathbf{e}_k \mathbf{e}_j = -2\delta_{jk}.\tag{2}$$

The Clifford algebra $C\ell_{0,n}$ is an associative algebra with identity 1 and dimension 2^n , that contains \mathbb{R} and \mathbb{R}^n as subspaces. This in particular means that for two given n-tuples (x_1, x_2, \ldots, x_n) and (y_1, y_2, \ldots, y_n) of \mathbb{R}^n , represented on $C\ell_{0,n}$ through the linear combinations

$$x = \sum_{j=1}^{n} x_j \mathbf{e}_j$$
 and $y = \sum_{j=1}^{n} y_j \mathbf{e}_j$,

respectively, the anti-commutator $xy + yx = -2\sum_{j=1}^{n} x_j y_j$ is scalar-valued.

We will use throughout this paper the notations $\mathcal{B}(x,y) = -\frac{1}{2}(xy + yx)$ to denote the bilinear form of \mathbb{R}^n and $x \pm h\mathbf{e}_j$ to denote the underlying forward/backward shifts $(x_1, x_2, \ldots, x_j \pm h, \ldots, x_n)$ on $h\mathbb{Z}^n$. Generally speaking, on $C\ell_{0,n}$ one may consider for a subset $J = \{j_1, j_2, \ldots, j_r\}$ of $\{1, 2, \ldots, n\}$, with $1 \leq j_1 < j_2 < \ldots < j_r \leq n$, r-multivector bases of the form $\mathbf{e}_J = \mathbf{e}_{i_1} \mathbf{e}_{i_2} \dots \mathbf{e}_{i_r}$, and moreover, Clifford-vector-valued functions $\mathbf{f}(x)$ as linear combinations of the above form

$$\mathbf{f}(x) = \sum_{r=0}^{n} \sum_{|J|=r} f_J(x) \, \mathbf{e}_J, \text{ with } f_J(x) \text{ scalar-valued.}$$

Hereby |J| denotes the cardinality of J. The \dagger -conjugation operation $\mathbf{f}(x) \mapsto \mathbf{f}(x)^{\dagger}$, defined as

$$\mathbf{f}(x)^{\dagger} = \sum_{r=0}^{n} \sum_{|J|=r} f_J(x) \, \mathbf{e}_J^{\dagger}, \quad \text{with} \quad \mathbf{e}_J^{\dagger} = (-1)^r \mathbf{e}_{j_r} \dots \mathbf{e}_{j_2} \mathbf{e}_{j_1} \tag{3}$$

is an automorphism of $\mathcal{C}\ell_{0,n}$ satisfying, for each $\mathbf{f}(x)$ and $\mathbf{g}(x)$, the conjugation properties

$$\left(\mathbf{f}(x)^{\dagger}\right)^{\dagger} = \mathbf{f}(x) \text{ and } \left(\mathbf{f}(x)\mathbf{g}(x)\right)^{\dagger} = \mathbf{g}(x)^{\dagger}\mathbf{f}(x)^{\dagger}.$$
 (4)

The conjugation properties on $C\ell_{0,n}$ are two-fold since they correspond to a generalization of the standard conjugation in the field of complex numbers and to the multivector extension of the Hermitian conjugation operation in the scope of matrix theory. In particular, it follows from the property $\mathbf{e}_{i}^{\dagger} = -\mathbf{e}_{j}$ and from the anti-commutator relations (2) that the quantities $\mathbf{f}(x)^{\dagger}\mathbf{f}(x)$ and $\mathbf{f}(x)\mathbf{f}(x)^{\dagger}$ are scalar-valued and coincide. Being $\mathbf{f}(x) =$ $\sum_{j=1}^{n} f_j(x) \mathbf{e}_j$ a Clifford vector representation of the vector-field $(f_1(x), f_2(x), \dots, f_n(x))$ of

 \mathbb{R}^n , one readily has

$$\mathbf{f}(x)^{\dagger}\mathbf{f}(x) = \mathbf{f}(x)\mathbf{f}(x)^{\dagger} = \sum_{j=0}^{n} f_j(x)^2,$$

which is nothing else than the square of the Euclidean norm on \mathbb{R}^n .

Moreover, if we use the \dagger -conjugation operator also for functions $\mathbf{f}(x)$ with values in the complexified Clifford algebra $\mathbb{C} \otimes C\ell_{0,n}$, the resulting quantity $\mathbf{f}(x)^{\dagger}\mathbf{f}(x)$ must be interpreted, on a wide generality, as a quasi-probability in Dirac's sense, i.e. a probability density function that encompasses negative values (cf. [11, p. 4]).

From the above characterization, one can thereafter define the set of Clifford-vectorvalued functions $\mathbf{f}(x)$ on $h\mathbb{Z}^n$ with membership on the Hilbert module $\ell_2(h\mathbb{Z}^n; C\ell_{0,n}) =$ $\ell_2(h\mathbb{Z}^n)\otimes C\ell_{0,n}$, as the linear space endowed by the sesquilinear form

$$\langle \mathbf{f}, \mathbf{g} \rangle_h = \sum_{x \in h\mathbb{Z}^n} h^n \mathbf{f}(x)^{\dagger} \mathbf{g}(x).$$

The class of discrete electromagnetic Schrödinger operators on $h\mathbb{Z}^n$ that we will consider throughout this paper are defined viz

$$L_h \mathbf{f}(x) = \frac{1}{2\mu} \sum_{j=1}^n \left(\frac{2}{qh} \mathbf{f}(x) - a_h(x_j) \mathbf{f}(x+h\mathbf{e}_j) - a_h(x_j-h) \mathbf{f}(x-h\mathbf{e}_j) \right) + q \Phi_h(x) \mathbf{f}(x).$$
(5)

Hereby $\Phi_h(x)$ (scalar-valued function) denotes the discrete analogue of the electric potential whereas $\mathbf{a}_h(x) = \sum_{j=1}^n \mathbf{e}_j a_h(x_j)$ (vector-valued function) denotes the discrete analogue of the magnetic potential. The parameters μ and q denote the mass and the electric charge of the electron, respectively. In case of $\Phi_h(x)$ and $\mathbf{a}_h(x)$ satisfy the set of constraints

$$q\Phi_{h}(x) + \frac{1}{2\mu} \sum_{j=1}^{n} \left(\frac{2}{qh} - a_{h}(x_{j}) - a_{h}(x_{j} - h) \right) = V\left(\frac{x}{h}\right) + O\left(h^{3}\right)$$
$$\mathbf{a}_{h}(x) = \sum_{j=1}^{n} \mathbf{e}_{j} \ w\left(\frac{1}{q} \ \frac{x_{j}}{h}\right) (1 + O\left(h\right)),$$

where $w\left(\frac{x_j}{qh}\right)$ stands the leading term of $a_h(x_j)$, one gets (cf. [34, p. 64])

$$L_{h}\mathbf{f}(x) = -\frac{h^{2}}{2\mu} \sum_{j=1}^{n} \frac{\partial}{\partial x_{j}} \left(w\left(\frac{x_{j}}{qh}\right) \frac{\partial \mathbf{f}}{\partial x_{j}}(x) \right) + V\left(\frac{x}{h}\right) \mathbf{f}(x) + O\left(h^{3}\right).$$
(6)

The above asymptotic model may be seen as a discrete counterpart of the Sturm-Liouville operator. Here one notice that the exact solvability of the right-hand side of (6) was studied in detail on [10, Section 2]. For the case of $a_h(x_j) \sim \frac{1}{qh}$ (i.e. $w\left(\frac{1}{q} \frac{x_j}{h}\right) = \frac{1}{qh}$), L_h is asymptotically equivalent to the discrete harmonic oscillator $-\frac{1}{2m}\Delta_h + q\Phi_h(x)$ with mass $m \sim \frac{\mu q}{h}$, whose kinetic term is written in terms of the star Laplacian (cf. [38, p. 423], [13, p. 1967] & [31, p. 4])

$$\Delta_h \mathbf{f}(x) = \sum_{j=1}^n \frac{\mathbf{f}(x+h\mathbf{e}_j) + \mathbf{f}(x-h\mathbf{e}_j) - 2\mathbf{f}(x)}{h^2}.$$

Other interesting examples may arise if $w\left(\frac{1}{q} \frac{x_j}{h}\right)$ has polynomial behavior, exponential behaviour or even if one takes a e.g. quantum deformation of the constant polynomial $w\left(\frac{1}{q} \frac{x_j}{h}\right) = \frac{\mu}{qh}$ (cf. [40, Section 2.]). For the particular choice $a_h(x_j) = \frac{1}{q} \left(\frac{1}{h} + \mu \frac{x_j}{h} + \frac{\mu}{2}\right)$ it readily follows from the set of identities

$$\mathbf{f}(x+h\mathbf{e}_j) = \mathbf{f}(x) + h\left(\frac{\mathbf{f}(x+h\mathbf{e}_j) - \mathbf{f}(x)}{h}\right)$$

$$\mathbf{f}(x-h\mathbf{e}_j) = \mathbf{f}(x) - h\left(\frac{\mathbf{f}(x) - \mathbf{f}(x-h\mathbf{e}_j)}{h}\right)$$

$$(j = 1, 2, \dots, n)$$

that the asymptotic expansion of the discrete electromagnetic Schrödinger operator (5) reduces to

$$L_h \mathbf{f}(x) = -\frac{1}{2\mu q} \left(E_h^+ \mathbf{f}(x) - E_h^- \mathbf{f}(x) \right) + V\left(\frac{x}{h}\right) \mathbf{f}(x),$$

with $V\left(\frac{x}{h}\right) = -\sum_{j=1}^{n} \frac{x_j}{h} + q\Phi_h(x)$. Hereby $E_h^{\pm} \mathbf{f}(x) = \sum_{j=1}^{n} \left(\frac{1}{\mu} + x_j \pm \frac{h}{2}\right) \frac{\mathbf{f}(x \pm h\mathbf{e}_j) - \mathbf{f}(x)}{\pm h}$ 6 corresponds to the forward/backward counterpart⁴ of the so-called Euler operator $E = \sum_{j=1}^{n} x_j \partial_{x_j}$, carrying the polynomial $w(x_j) = 1 + \mu x_j$ (cf. [14, p. 3]). That gives us an alternative way to discretize the quantum harmonic oscillator $-\frac{1}{2m}\Delta + V\left(\frac{x}{h}\right)$ with mass term $m \sim \frac{\mu q}{h}$.

Next, we will consider a faithful adaptation of the Fock space formalism [17], already considered in [9], to discrete hypercomplex variables. We introduce the Fock space structure over $h\mathbb{Z}^n$ as a linear subspace \mathcal{F}_h of $\ell_2(h\mathbb{Z}^n; C\ell_{0,n})$ encoded by the pair (A_h^+, A_h^-) of Clifford-vector-valued operators. To be more precise, we say that \mathcal{F}_h defines a Fock space over $h\mathbb{Z}^n$ if the following conditions are satisfied:

1. Duality condition: For two given lattice functions $\mathbf{f}(x)$ and $\mathbf{g}(x)$ with membership in \mathcal{F}_h , the pair of Clifford-vector-valued operators (A_h^+, A_h^-) satisfy

$$\langle A_h^+ \mathbf{f}, \mathbf{g} \rangle_h = \langle \mathbf{f}, A_h^- \mathbf{g} \rangle_h$$

2. Vacuum vector condition: There exists a lattice function $\psi_0(x;h)$ with membership in \mathcal{F}_h such that

$$A_h^+\psi_0(x;h) = 0.$$

3. Energy condition: The vacuum vector ψ_0 satisfies

$$\langle \psi_0, \psi_0 \rangle_h = 1.$$

From direct application of the quantum field lemma (cf. [17]) the resulting Fock space \mathcal{F}_h is thus generated by the k-Fock states

$$\psi_k(x;h) = (A_h^-)^k \psi_0(x;h).$$
(7)

It readily follows from the \dagger -conjugation property (4) that the left representation $\Lambda(\mathbf{s}) : \mathbf{f}(x) \mapsto \mathbf{sf}(x)$ is an isometry on $\ell_2(h\mathbb{Z}^n; C\ell_{0,n})$ whenever $\mathbf{ss}^{\dagger} = \mathbf{s}^{\dagger}\mathbf{s} = 1$, i.e.

$$\langle \mathbf{sf}(x), \mathbf{sg}(x) \rangle_h = \langle \mathbf{f}(x), \mathbf{g}(x) \rangle_h.$$
 (8)

Regarding the above isometry property one may consider the Lie groups O(n) and SO(n). Here O(n) is the group of linear transformations of \mathbb{R}^n which leave invariant the bilinear form $\mathcal{B}(x,y) = -\frac{1}{2}(xy + yx)$ and SO(n) (the so-called *special orthogonal group*) is the group of linear transformations with determinant 1. These groups have natural transitive actions on the (n-1)-sphere

$$S^{n-1} = \left\{ x = \sum_{j=1}^{n} x_j \mathbf{e}_j \in C\ell_{0,n} : x^{\dagger}x = xx^{\dagger} = 1 \right\}.$$

⁴The formulation of E_h^{\pm} introduced in [14] is seemingly close to the formulation of number operator in quantum mechanics and it goes beyond the standard discretizations of the Euler operator E.

Namely, through the action of SO(n) we can rewrite every $x \in \mathbb{R}^n$ as $x = \rho \mathbf{s}$, with $\rho = \frac{x}{|x|}$ and $\mathbf{s} \in S^{n-1}$. Using the fact that the group stabilizer of the Clifford vector $\mathbf{e}_n \in C\ell_{0,n}$ is isomorphic to SO(n-1), the points of \mathbf{s} of S^{n-1} can be identified with the homogeneous space SO(n)/SO(n-1) through the isomorphism property $SO(n)/SO(n-1) \cong S^{n-1}$. In terms of the main involution operation $\mathbf{s} \mapsto \mathbf{s}'$, defined in $C\ell_{0,n}$ as

$$\mathbf{s}' = \sum_{r=0}^{n} \sum_{|J|=r} s_J \mathbf{e}'_J \quad \text{with} \quad \mathbf{e}'_J = (-1)^r \mathbf{e}_{j_1} \mathbf{e}_{j_2} \dots \mathbf{e}_{j_r}$$

we find that the Pin group

$$\operatorname{Pin}(n) = \left\{ \mathbf{s} = \prod_{p=1}^{q} \mathbf{s}_{p} : \mathbf{s}_{1}, \mathbf{s}_{2}, \dots, \mathbf{s}_{q} \in S^{n-1}, \ q \in \mathbb{N} \right\}$$

and the Spin group

$$\operatorname{Spin}(n) = \left\{ \mathbf{s} = \prod_{p=0}^{2q} \mathbf{s}_p : \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{2q} \in S^{n-1}, \ q \in \mathbb{N} \right\}$$

may be regarded as the underlying double-covering sheets for the groups O(n) and SO(n), respectively, endowed by homomorphism action $\chi(\mathbf{s}) : \mathbf{f}(x) \mapsto \mathbf{sf}(x)(\mathbf{s}')^{-1}$ (cf. [19, Chapter 3]).

Since Spin(n) is a subgroup of Pin(n), it remains natural to look throughout for *vacuum* vectors $\psi_0(x;h)$ of the form $\psi_0(x;h) = \phi(x;h)\mathbf{s}$, where $\phi(x;h)$ is scalar-valued and $\mathbf{s} \in \text{Pin}(n)$. From now on we will always use the bold notation \mathbf{s} when we are referring to an element of Pin(n) or Spin(n).

3. Main Results

3.1. Factorization Approach

We now turn to the factorization question posed in **Problem 1.1**. To do so, we consider the set of operators, A_h^+ and A_h^- , defined *viz*

$$A_{h}^{+} = \sum_{j=1}^{n} \mathbf{e}_{j} A_{h}^{+j} \quad \text{with} \quad A_{h}^{+j} = \sqrt{\frac{qh}{4\mu}} \left(a_{h}(x_{j}) T_{h}^{+j} - \frac{2}{qh} I \right)$$

$$A_{h}^{-} = \sum_{j=1}^{n} \mathbf{e}_{j} A_{h}^{-j} \quad \text{with} \quad A_{h}^{-j} = \sqrt{\frac{qh}{4\mu}} \left(\frac{2}{qh} I - a_{h}(x_{j} - h) T_{h}^{-j} \right).$$
 (9)

Here we recall that in terms of the identity operator $I : \mathbf{f}(x) \mapsto \mathbf{f}(x)$ and the forward/backward shifts $T_h^{\pm j} \mathbf{f}(x) = \mathbf{f}(x \pm h\mathbf{e}_j)$ on the x_j -axis, the action $\mathbf{f}(x) \mapsto L_h \mathbf{f}(x)$ corresponds to

$$L_h = \frac{1}{2\mu} \sum_{j=1}^n \left(\frac{2}{qh} I - a_h(x_j) T_h^{+j} - a_h(x_j - h) T_h^{-j} \right) + q \Phi_h(x) I.$$

It is straightforward to verify that $A_h^{+j}A_h^{-j} + A_h^{-j}A_h^{+j}$ equals

$$-\frac{2}{\mu q h}I + \frac{1}{\mu}a_h(x_j)T_h^{+j} + \frac{1}{\mu}a_h(x_j - h)T_h^{-j} - \frac{q h}{4\mu}\left(a_h(x_j)^2 + a_h(x_j - h)^2\right)I.$$

Then, for $\Phi_h(x) = \frac{h}{8\mu} \sum_{j=1}^n \left(a_h(x_j)^2 + a_h(x_j - h)^2 \right)$ one obtains (see Appendix A.1)

$$\frac{1}{2}\left(A_{h}^{+}A_{h}^{-}+A_{h}^{-}A_{h}^{+}\right)-q\Phi_{h}(x)I=\frac{1}{2\mu}\sum_{j=1}^{n}\left(\frac{2}{qh}I-a_{h}(x_{j})T_{h}^{+j}-a_{h}(x_{j}-h)T_{h}^{-j}\right),$$

Thereby, the discrete electric potential $\Phi_h(x)$ is uniquely determined by the factorization property $L_h = \frac{1}{2} \left(A_h^+ A_h^- + A_h^- A_h^+ \right)$. On the other hand, based on the summation formulae (cf. [25, Subsection 1.5])

$$\sum_{x \in h\mathbb{Z}^n} h^n \mathbf{f}(x \pm h\mathbf{e}_j)^{\dagger} \mathbf{g}(x) = \sum_{x \in h\mathbb{Z}^n} h^n \mathbf{f}(x)^{\dagger} \mathbf{g}(x \mp h\mathbf{e}_j)$$

one easily recognize the following adjoint relations, written in terms of the shift operators $T_h^{\pm j}$:

$$\left\langle a_h(x_j)T_h^{+j}\mathbf{f},\mathbf{g} \right\rangle_h = \left\langle \mathbf{f}, a_h(x_j-h)T_h^{-j}\mathbf{g} \right\rangle_h \left\langle a_h(x_j-h)T_h^{-j}\mathbf{f},\mathbf{g} \right\rangle_h = \left\langle \mathbf{f}, a_h(x_j)T_h^{+j}\mathbf{g} \right\rangle_h.$$
 (10)

This yield A_h^+ and A_h^- as Hermitian conjugates one of the other with respect to the Hilbert module $\ell_2(h\mathbb{Z}^n; C\ell_{0,n})$, as required by the **Duality condition** underlying to the Fock space \mathcal{F}_h over $h\mathbb{Z}^n$ (see Appendix A.2). Since the *vacuum* vector $\psi_0(x;h) = \phi(x;h)$ s is Pin(n)-valued we can make use of the method of separation of variables to compute $\phi(x;h)$ from the set of functional equations

$$\phi(x+h\mathbf{e}_j;h) = \frac{2}{qh} \frac{1}{a_h(x_j)} \phi(x;h)$$
 for each $j = 1, 2, \dots, n.$ (11)

Indeed, (11) is equivalent to the set of equations $A_h^{+j}\phi(x;h) = 0$ (j = 1, 2, ..., n), and hence, to $A_h^+\psi_0(x;h) = (A_h^+\phi(x;h))\mathbf{s} = 0$. Henceforth we make use of the conjugation property $(\mathbf{sf}(x))^{\dagger} = \mathbf{f}(x)^{\dagger}\mathbf{s}^{\dagger}$ to get rid of the Pinor/Spinor element \mathbf{s} on the quasi-probability formulation (1) of the **Energy condition** $\langle \psi_0, \psi_0 \rangle_h = 1$. Indeed, for $\psi_0(x;h) = \phi(x;h)\mathbf{s}$, the quasi-probability law (1) carrying a set of *independent and identically distributed* (i.i.d) random variables X_1, X_2, \ldots, X_n thus becomes

$$\Pr\left(\sum_{j=1}^{n} \mathbf{e}_j X_j = x\right) = h^n \phi(x; h)^2.$$
(12)

3.2. Intertwining Properties

With the aim of obtaining a recovery for the discrete electric and magnetic potentials, $\Phi_h(x)$ and $\mathbf{a}_h(x)$ respectively, from the knowledge of the k-bound states (7) of L_h with membership in the Fock space \mathcal{F}_h , we are going now to establish a general framework involving a generalization of the quasi-monomiality principle obtained in author's recent paper [15]. For their construction we shall employ intertwining properties between A_h^{\pm} and the set of ladder Clifford-vector-valued operators

$$D_h^+ = \sum_{j=1}^n \mathbf{e}_j \partial_h^{+j}$$

$$M_h = \sum_{j=1}^n \mathbf{e}_j \left(ha_h (x_j - h)^2 T_h^{-j} - \frac{4}{q^2 h} I \right).$$

As usual, $\partial_h^{+j} \mathbf{f}(x) = \frac{\mathbf{f}(x + h\mathbf{e}_j) - \mathbf{f}(x)}{h}$ (j = 1, 2, ..., n) denote the forward finite difference operators on $h\mathbb{Z}^n$ (cf. [15, Subsection 2.1.]).

First, recall that the *vacuum* vector $\psi_0(x;h) = \phi(x;h)\mathbf{s}$ annihilated by A_h^+ , may be computed from the set of functional equations (11). More generally, the set of constraints (11) provide us a scheme to derive an intertwining property between the degree-lowering type operator A_h^+ and the finite difference Dirac operator D_h^+ , seemingly close to the Rodrigues type formula involving the Clifford-Hermite polynomials/functions (cf. [9, Lemma 3.1]). For every Clifford-vector-valued function $\mathbf{f}(x)$ we thus have the set of relations

$$\begin{aligned} A_h^+(\phi(x;h)\mathbf{f}(x)) &= \sum_{j=1}^n \mathbf{e}_j \ \sqrt{\frac{qh}{4\mu}} \left(a_h(x_j)\phi(x+h\mathbf{e}_j;h)\mathbf{f}(x+h\mathbf{e}_j) - \frac{2}{qh}\phi(x;h)\mathbf{f}(x) \right) \\ &= \sqrt{\frac{h}{\mu q}} \ \sum_{j=1}^n \mathbf{e}_j \ \phi(x;h)\frac{\mathbf{f}(x+h\mathbf{e}_j) - \mathbf{f}(x)}{h} \\ &= \sqrt{\frac{h}{\mu q}} \ \phi(x;h) \ D_h^+\mathbf{f}(x) \end{aligned}$$

that in turn yields the operational formula

$$\phi(x;h)^{-1}A_h^+\left(\phi(x;h)\mathbf{f}(x)\right) = \sqrt{\frac{h}{\mu q}}D_h^+\mathbf{f}(x)$$

In a similar manner one can derive an intertwining property, involving the operators A_h^- and M_h if we reformulate the set of functional equations (11) in terms of the backward

shifts $T_h^{-j} \mathbf{f}(x) = \mathbf{f}(x - h\mathbf{e}_j)$. Thereby, the set of relations

$$\begin{aligned} A_h^-(\phi(x;h)\mathbf{f}(x)) &= \sum_{j=1}^n \mathbf{e}_j \sqrt{\frac{qh}{4\mu}} \left(\frac{2}{qh}\phi(x;h)\mathbf{f}(x) - a_h(x_j - h)\phi(x - h\mathbf{e}_j;h)\mathbf{f}(x - h\mathbf{e}_j)\right) \\ &= -\sqrt{\frac{qh}{4\mu}} \sum_{j=1}^n \mathbf{e}_j \phi(x;h) \left(\frac{qh}{2}a_h(x_j - h)^2 \mathbf{f}(x - h\mathbf{e}_j) - \frac{2}{qh}\mathbf{f}(x)\right) \\ &= -\frac{1}{4}\sqrt{\frac{q^3h}{\mu}} \phi(x;h)M_h\mathbf{f}(x), \end{aligned}$$

that hold for an arbitrary Clifford-vector-valued function $\mathbf{f}(x),$ yield as a direct consequence of

$$\phi(x - h\mathbf{e}_j; h) = \frac{qh}{2} a_h(x_j - h) \phi(x; h) \quad (j = 1, 2, \dots, n).$$

This implies

$$\phi(x;h)^{-1}A_h^-(\phi(x;h)\mathbf{f}(x)) = -\frac{1}{4}\sqrt{\frac{q^3h}{\mu}} \ M_h \ \mathbf{f}(x).$$

Furthermore, induction over $k \in \mathbb{N}_0$ shows that the k-bound states (7) are thus characterized by the operational formula

$$\psi_k(x;h) = \frac{(-1)^k}{4^k} \frac{q^{\frac{3k}{2}}h^{\frac{k}{2}}}{\mu^{\frac{k}{2}}} \phi(x;h) \ (M_h)^k \mathbf{s}.$$
 (13)

On the other hand, combination of the previously obtained relations give rive to

$$\phi(x;h)^{-1}A_{h}^{-}A_{h}^{+}(\phi(x;h)\mathbf{f}(x)) = -\frac{qh}{4\mu} M_{h}D_{h}^{+}\mathbf{f}(x)$$

$$\phi(x;h)^{-1}A_{h}^{+}A_{h}^{-}(\phi(x;h)\mathbf{f}(x)) = -\frac{qh}{4\mu} D_{h}^{+}M_{h}\mathbf{f}(x).$$

This results into the following lemma:

Lemma 3.1. Let $\mathbf{s} \in Pin(n)$, $\phi(x;h)$ a scalar-valued function satisfying (11), $\psi_k(x;h)$ the k-bound states defined viz equation (7) and

$$\mathbf{m}_k(x;h) = (M_h)^k \mathbf{s}$$

be quasi-monomials of order $k \ (k \in \mathbb{N}_0)$. Then we have the following:

1. The quasi-monomials $\mathbf{m}_k(x;h)$ may be determined through the formula

$$\mathbf{m}_{k}(x;h) = (-1)^{k} 4^{k} \frac{\mu^{\frac{k}{2}}}{q^{\frac{3k}{2}} h^{\frac{k}{2}}} \frac{\psi_{k}(x;h)}{\phi(x;h)}.$$

2. The quasi-monomials $\mathbf{m}_k(x;h)$ and the Fock states $\psi_k(x;h)$ are interrelated by the isospectral formula

$$M_h D_h^+ \mathbf{m}_k(x;h) + D_h^+ M_h \mathbf{m}_k(x;h) = (-1)^{k+1} 4^{k+1} \frac{\mu^{\frac{k}{2}+1}}{q^{\frac{3k}{2}+1}h^{\frac{k}{2}+1}} \phi(x;h)^{-1} L_h \psi_k(x;h).$$

Regardless the formal computation of the \mathbf{m}'_k s, we would like to stress that the operator $(M_h)^{2r}$ (k = 2r) is scalar-valued operator whereas $(M_h)^{2r+1}$ is vector-valued (k = 2r + 1). To fill this gap, the computation of the quasi-monomials $\mathbf{m}_k(x;h)$ of even and odd orders separately⁵. For the even orders (k = 2r) we use the multinomial formula

$$\mathbf{m}_{2r}(x;h) = \left((M_h)^2 \right)^r \mathbf{s}$$

= $\sum_{|\sigma|=r} (-1)^r \frac{r!}{\sigma!} \prod_{j=1}^n \left(ha_h (x_j - h)^2 T_h^{-j} - \frac{4}{q^2 h} I \right)^{2\sigma_j} \mathbf{s}$ (14)

that results from the operational identity

$$(M_h)^2 = -\sum_{j=1}^n \left(ha_h (x_j - h)^2 T_h^{-j} - \frac{4}{q^2 h} I \right)^2,$$

whereas for the odd orders (k = 2r + 1), we take into account the recursive formula

$$\mathbf{m}_{2r+1}(x;h) = M_h \mathbf{m}_{2r}(x;h).$$
(15)

Here and elsewhere, for a given multi-index $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n), |\sigma| = \sum_{j=1}^n \sigma_j$ denotes

the multi-index degree and $\sigma! = \prod_{j=1}^{n} \sigma_j!$ the multi-index factorial.

From the construction considered previously it follows from a short computation that for a given *vacuum* vector of the form $\psi_0(x;h) = \phi(x;h)\mathbf{s}$ ($\mathbf{s} \in \operatorname{Pin}(n)$) the discrete electric and magnetic potentials, $\Phi_h(x)$ and $\mathbf{a}_h(x)$ respectively, are uniquely determined by the formulae

$$\Phi_h(x) = \frac{h}{8\mu} \sum_{j=1}^n \frac{4}{q^2 h^2} \left(\frac{\phi(x;h)^2}{\phi(x+h\mathbf{e}_j;h)^2} + \frac{\phi(x-h\mathbf{e}_j;h)^2}{\phi(x;h)^2} \right)$$
(16)

$$\mathbf{a}_{h}(x) = \sum_{j=1}^{n} \mathbf{e}_{j} \frac{2}{qh} \frac{\phi(x;h)}{\phi(x+h\mathbf{e}_{j};h)}.$$
(17)

This readily solves part of the question posed in **Problem 1.1**. More generally, statement 1. of Lemma 3.1 allows us to give a faithful answer to **Problem 1.1** as a some sort of inverse problem:

 $^{^{5}}$ This is similarly to what was done in [15, Example 3.2] and in [15, Example 3.3] to compute hypercomplex versions for the falling factorials and Poisson-Charlier polynomials, respectively.

Proposition 3.1. Let us assume that the k-Fock states $\psi_k(x;h)$ of the discrete electromagnetic Schrödinger operator L_h are Pin(n)-valued.

If for a given sequence $\{\mathbf{m}_k(x;h) : k \in \mathbb{N}_0\}$ of quasi-monomials the statement 2. of Lemma 3.1 is fulfilled, then the vacuum vector $\psi_0(x;h) = \phi(x;h)\mathbf{s}$ ($\mathbf{s} \in Pin(n)$) may be recovered from the projection-based formula

$$\phi(x;h) = (-1)^k 4^k \frac{\mu^k}{q^{\frac{3k}{2}} h^{\frac{k}{2}}} \frac{\mathbf{m}_k(x;h)^{\dagger} \psi_k(x;h)}{\mathbf{m}_k(x;h)^{\dagger} \mathbf{m}_k(x;h)}$$

Moreover, the discrete electric and magnetic potentials, $\Phi_h(x)$ resp. $\mathbf{a}_h(x)$, are uniquely determined by inserting the right-hand side of $\phi(x;h)$ on the formulae (16) and (17), respectively.

On the above characterizations, the scalar-valued potential $\Phi_h(x)$ is determined from the components of the discrete magnetic potential $\mathbf{a}_h(x)$ or from the knowledge of the *vacuum* vector. Concerning the construction of the quasi-monomials we would like to stress that for each $\mathbf{s} \in \text{Pin}(n) \mathbf{m}_k(x;h)\mathbf{s}^{\dagger}$ is scalar-valued for k even and vector-valued for k odd. In particular, for k = 1

$$\mathbf{m}_1(x+h\mathbf{e};h)\mathbf{s}^{\dagger} = \sum_{j=1}^n \mathbf{e}_j \left(ha_h(x_j)^2 - \frac{4}{q^2h} \right), \quad \text{with} \quad \mathbf{e} = \sum_{j=1}^n \mathbf{e}_j. \tag{18}$$

A short computation involving the bilinear form $\mathcal{B}(x,y) = -\frac{1}{2}(xy + yx)$ gives

$$\mathcal{B}\left(\frac{1}{h}\mathbf{m}_1(x;h)\mathbf{s}^{\dagger},\mathbf{e}_j\right) = a_h(x_j-h)^2 - \frac{4}{q^2h^2} \quad \text{and} \quad \mathcal{B}\left(\frac{1}{h}\mathbf{m}_1(x+h\mathbf{e};h)\mathbf{s}^{\dagger},\mathbf{e}_j\right) = a_h(x_j)^2 - \frac{4}{q^2h^2}.$$

That allows us to formulate the following proposition:

Proposition 3.2. Let us assume that $\mathbf{m}_1(x; h)$ is a $\mathbb{C} \otimes Pin(n)$ -valued quasi-monomial of order 1. Then, the discrete electric and magnetic potentials, $\Phi_h(x)$ and $\mathbf{a}_h(x)$ respectively, may be recovered from the formulae

$$\Phi_h(x) = \frac{1}{8\mu} \mathcal{B}\left(\frac{1}{h}\mathbf{m}_1(x+h\mathbf{e};h)\mathbf{s}^{\dagger},\mathbf{e}\right) + \frac{1}{8\mu} \mathcal{B}\left(\frac{1}{h}\mathbf{m}_1(x;h)\mathbf{s}^{\dagger},\mathbf{e}\right) + \frac{n}{\mu q^2 h^2}$$
$$\mathbf{a}_h(x) = \sum_{j=1}^n \mathbf{e}_j \sqrt{\mathcal{B}\left(\frac{1}{h}\mathbf{m}_1(x+h\mathbf{e};h)\mathbf{s}^{\dagger},\mathbf{e}_j\right) + \frac{4}{q^2 h^2}}.$$

Proposition 3.2 provides an alternative way to recover the electric and magnetic potentials from the knowledge of the quasi-monomial of order k = 1, in interplay with the recursive formula (15) for r = 0.

4. The Bayesian Probability Insight

4.1. Poisson and Hypergeometric Distributions

Our next step is to study the quasi-exact solvability of the multidimensional discrete electromagnetic Schrödinger operator (5) through the connection between the bound

states (7) and the discrete electric and magnetic potentials. In view of Proposition 3.1 we will restrict ourselves to the construction of the discrete electric and magnetic potentials, $\Phi_h(x)$ and $\mathbf{a}_h(x)$ respectively, from the knowledge of the ground state $\psi_0(x;h) = \phi(x;h)\mathbf{s}$.

Based on the descriptions (16) and (17) obtained in Section 3 for $\Phi_h(x)$ and $\mathbf{a}_h(x)$ respectively, it remains natural to exploit the Fock space \mathcal{F}_h from the Bayesian probability side (cf. [7, 26]) by means of the *likelihood* function $x \mapsto h^n \phi(x; h)^2$ encoded by the quasiprobability law (12). Particular examples arising this construction are⁶:

1. The multi-variable Poisson-Charlier polynomials, determined from the multi-variable Poisson distribution with parameter $\lambda > 0$ (cf. [16, p. 335]):

$$h^{n}\phi(x;h)^{2} = \begin{cases} \prod_{j=1}^{n} \exp(-\lambda) \ \frac{\lambda^{\frac{x_{j}}{h}}}{\Gamma\left(\frac{x_{j}}{h}+1\right)} & \text{,if } x \in h\mathbb{Z}_{\geq 0}^{n} \\ 0 & \text{, otherwise} \end{cases}$$

2. The multi-variable Meixner polynomials, determined from the multivariable hypergeometric distribution, defined as (cf. [16, pp. 337-338])

$$h^{n}\phi(x;h)^{2} = \begin{cases} \prod_{j=1}^{n} \frac{\Gamma\left(\beta + \frac{x_{j}}{h}\right)}{\Gamma(\beta)} \frac{\lambda^{\frac{x_{j}}{h}}(1-\lambda)^{\beta}}{\Gamma\left(\frac{x_{j}}{h}+1\right)} & \text{, if } x \in h\mathbb{Z}_{\geq 0}^{n} \\ 0 & \text{, otherwise} \end{cases}$$

carrying the parameters $\beta > 0$ and $0 < \lambda < 1$.

For the multi-variable Poisson distribution with parameter $\lambda = \frac{4}{qh^2}$

$$\Phi_h(x) = \frac{h}{8\mu} \sum_{j=1}^n \left(\frac{2}{q} \ \frac{x_j}{h} + \frac{1}{q}\right) \text{ and } \mathbf{a}_h(x) = \sum_{j=1}^n \mathbf{e}_j \ \sqrt{\frac{1}{q} \ \frac{x_j}{h} + \frac{1}{q}}$$

are the underlying discrete electric and magnetic potentials, respectively, defined for $x \in h\mathbb{Z}_{\geq 0}^n$. Thus, the Clifford-vector-valued polynomials obtained through the operational action of the multiplication operator

$$M_h = \sum_{j=1}^n \mathbf{e}_j \frac{1}{q} \left(x_j T_h^{-j} - \frac{4}{qh} I \right)$$

are of Poisson-Charlier type (cf. [15, Example 3.3]). Such families of quasi-monomials are encoded on the pair (D_h^+, M_h) , by means of Fischer/Fourier duality (cf. [13, 14]).

For the case where electric charge satisfies the condition $h > \frac{2}{q}$, the above hypergeometric distribution with parameters $\lambda = \frac{4}{q^2 h^2}$ and $\beta > 0$ endow the discrete electric and

 $^{^6 \}mathrm{See}$ also [26, Section 4].

magnetic potentials

$$\Phi_{h}(x) = \begin{cases} \frac{h}{8\mu} \sum_{j=1}^{n} \left(\frac{\frac{1}{q}}{\frac{x_{j}}{h}} + \frac{1}{q}}{\frac{1}{q}} + \frac{\frac{1}{q}}{\frac{x_{j}}{h}} + \frac{\beta}{q} \right), \text{ if } x \in h\mathbb{Z}_{\geq 0}^{n} \\ 0, \text{ otherwise} \end{cases}$$
$$\mathbf{a}_{h}(x) = \begin{cases} \sum_{j=1}^{n} \mathbf{e}_{j} \sqrt{\frac{\frac{1}{q}}{\frac{x_{j}}{h}} + \frac{1}{q}}{\frac{1}{q}}, \text{ if } x \in h\mathbb{Z}_{\geq 0}^{n} \\ 0, \text{ otherwise} \end{cases}$$

that in turn yields $M_h = \sum_{j=1}^n \mathbf{e}_j h \left(\frac{x_j}{x_j + (\beta - 1)h} T_h^{-j} - \frac{4}{q^2 h^2} I \right)$ as multiplication operator, acting on $h\mathbb{Z}_{\geq 0}^n$.

4.2. Mittag-Leffler Distributions

Let us specialize our results in the case where generalized Mittag-Leffler functions $E_{\alpha,\beta}$ are involved. The multivariable *likelihood* function that generalizes the Poisson⁷ distribution is thus given by

$$h^{n}\phi(x;h)^{2} = \begin{cases} \prod_{j=1}^{n} E_{\alpha,\beta} \left(\frac{4}{q^{2-\alpha}h^{2}}\right)^{-1} \frac{4^{\frac{x_{j}}{h}}q^{\frac{(2-\alpha)x_{j}}{h}}h^{-\frac{2x_{j}}{h}}}{\Gamma\left(\beta+\alpha\frac{x_{j}}{h}\right)} , \text{if } x \in h\mathbb{Z}^{n}_{\geq 0} \\ 0 , \text{otherwise} \end{cases}$$
(19)

As a matter of fact,

$$E_{\alpha,\beta}(\lambda) = \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma(\beta + \alpha m)}$$

is well defined for $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$ (cf. [23, p. 8]). A short computation involving the equations (16) and (17) show that the discrete electric and magnetic fields, $\Phi_h(x)$ and $\mathbf{a}_h(x)$ respectively, are given by the general formulae

$$\Phi_{h}(x) = \frac{h}{8\mu} \sum_{j=1}^{n} \frac{1}{q^{\alpha}} \left(\frac{\Gamma\left(\alpha + \beta + \alpha \frac{x_{j}}{h}\right)}{\Gamma\left(\beta + \alpha \frac{x_{j}}{h}\right)} + \frac{\Gamma\left(\beta + \alpha \frac{x_{j}}{h}\right)}{\Gamma\left(\beta - \alpha + \alpha \frac{x_{j}}{h}\right)} \right)$$
$$\mathbf{a}_{h}(x) = \sum_{j=1}^{n} \mathbf{e}_{j} \sqrt{\frac{1}{q^{\alpha}} \frac{\Gamma\left(\alpha + \beta + \alpha \frac{x_{j}}{h}\right)}{\Gamma\left(\beta + \alpha \frac{x_{j}}{h}\right)}}.$$

⁷i.e. the *likelihood* function determined from the coefficients of the exponential function $\exp(\lambda) = E_{1,1}(\lambda)$.

or equivalently,

$$\Phi_h(x) = \frac{h}{8\mu} \sum_{j=1}^n \frac{1}{q^\alpha} \left(\left(\beta + \alpha \frac{x_j}{h} \right)_\alpha + \left(\beta - \alpha + \alpha \frac{x_j}{h} \right)_\alpha \right) \quad \text{and} \quad \mathbf{a}_h(x) = \sum_{j=1}^n \mathbf{e}_j \sqrt{\frac{1}{q^\alpha} \left(\beta + \alpha \frac{x_j}{h} \right)_\alpha},$$

where $(a)_{\alpha} = \frac{\Gamma(\alpha + a)}{\Gamma(a)}$ denotes the Pochhammer symbol.

Moreover, the multiplication operator M_h is a polynomial-type operator of order α , given by

$$M_h = \sum_{j=1}^n \mathbf{e}_j \left(\frac{h}{q^{\alpha}} \left(\beta - \alpha + \alpha \frac{x_j}{h} \right)_{\alpha} T_h^{-j} - \frac{4}{q^2 h} I \right).$$
(20)

It is clear from a straightforward application of the generalized Stirling's formula

$$\Gamma(s+z) \sim \exp(s\log(z)) \ \Gamma(z) \quad \text{as} \quad |z| \to \infty$$
 (21)

that $\Phi_h(x)$ and $\mathbf{a}_h(x)$ admit, for $h \to 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$, the asymptotic expansions

$$\Phi_h(x) \sim \frac{h}{4\mu} \sum_{j=1}^n \exp\left(\alpha \log\left(\frac{\alpha}{q} \frac{x_j}{h}\right)\right) \quad \text{and} \quad \mathbf{a}_h(x) \sim \sum_{j=1}^n \mathbf{e}_j \exp\left(\frac{\alpha}{2} \log\left(\frac{\alpha}{q} \frac{x_j}{h}\right)\right)$$

so that we are under the conditions of eq. (6). The the discrete electromagnetic Schrödinger operator is thus asymptotically equivalent to the Sturm-Liouville operator

$$\mathbf{f}(x) \mapsto -\frac{h^2}{2\mu} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\exp\left(\frac{\alpha}{2} \log\left(\frac{\alpha}{q} \frac{x_j}{h}\right)\right) \frac{\partial \mathbf{f}}{\partial x_j}(x) \right) + V\left(\frac{x}{h}\right) \mathbf{f}(x),$$

with potential

$$V\left(\frac{x}{h}\right) = \frac{1}{2\mu} \sum_{j=1}^{n} \left[\frac{2}{qh} + \frac{h}{2} \exp\left(\alpha \log\left(\frac{\alpha}{q} \frac{x_j}{h}\right)\right) - \exp\left(\frac{\alpha}{2} \log\left(\frac{\alpha}{q} \frac{x_j}{h}\right)\right) - \exp\left(\frac{\alpha}{2} \log\left(\frac{\alpha}{q} \frac{x_j}{h} - \frac{\alpha}{q}\right)\right)\right].$$

4.3. Generalized Wright distributions

Widely speaking, one can construct generalizations of the Mittag-Leffler's distribution (19) by means of the following Mellin-Barnes integral representation

$${}_{p}\Psi_{t}\left[\begin{array}{c}(a_{k},\alpha_{k})_{1,p}\\(b_{l},\beta_{l})_{1,t}\end{array}\middle|\lambda\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\prod_{k=1}^{p}\Gamma(a_{k}-\alpha_{k}s)}{\prod_{l=1}^{t}\Gamma(b_{l}-\beta_{l}s)}(-\lambda)^{-s} ds.$$
(22)

Such kind of integral representation formulae correspond to H-function representations of a generalized Wright function, with parameters $\lambda \in \mathbb{C}$, $a_k, b_l \in \mathbb{C}$ and $\alpha_k, \beta_l \in \mathbb{R} \setminus \{0\}$ (k = 1, 2, ..., p; l = 1, 2, ..., t) – see, for instance, [12, Subsection 1.19], [20, Subsection 5] and [23, Section 1.9]. Here we notice that in case of the closed path joining the endpoints $c - i\infty$ and $c + i\infty$ (0 < c < 1) contains the simple poles s = -m ($m \in \mathbb{N}_0$) on the left, from standard arguments of residue theory, there holds⁸

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(1-s)(-\lambda)^{-s}}{\Gamma(\beta-\alpha s)} \, ds = \sum_{m=0}^{\infty} \lim_{s \to -m} (s+m) \frac{\Gamma(s)\Gamma(1-s)(-\lambda)^{-s}}{\Gamma(\beta-\alpha s)}$$
$$= \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma(\beta+\alpha m)},$$

that is $E_{\alpha,\beta}(\lambda) = {}_{1}\Psi_{1} \begin{bmatrix} (1,1) \\ (\beta,\alpha) \end{bmatrix} \lambda$ (cf. [23, Example 1.4]).

More generally, one can compute generalized multivariable probability distributions of Wright type, by rewritting (22) as a series representation with coefficients (cf. [20, Section 4] and [23, Subsection 1.9.1])

$$\mu_m = \frac{\prod_{k=1}^p \Gamma(a_k + \alpha_k m)}{\prod_{l=1}^t \Gamma(b_l + \beta_l m)} \ \frac{\lambda^m}{\Gamma(m+1)}$$

Assuming that for $\alpha_k > 0$, $\beta_l > 0$ $(k = 1, 2, \dots, p; l = 1, 2, \dots, t)$ the intersection between the simple poles $b_l = -m$ ($m \in \mathbb{N}_0$) of $\Gamma(s)$ and the simple poles $\frac{a_k+m}{\alpha_k}$ ($k = 1, \ldots, p; m \in \mathbb{N}_0$) of $\Gamma(a_k - \alpha_k s)$ ($k = 1, \ldots, p$) yields an empty set, i.e. $\frac{a_k+m}{\alpha_k} \neq -m$, the Mallin Derma integral (20) Mellin-Barnes integral (22) admits series expansion

$$\Psi_t \begin{bmatrix} (a_k, \alpha_k)_{1,p} \\ (b_l, \beta_l)_{1,t} \end{bmatrix} = \sum_{m=0}^{\infty} \mu_m$$
(23)

whenever $\sum_{l=1}^{t} \beta_l - \sum_{k=1}^{p} \alpha_k \ge -1$ (cf. [20, Theorem 1]). Moreover:

- In case of ∑_{l=1}^t β_l ∑_{k=1}^p α_k > -1, the series expansion (23) is absolutely convergent for all λ ∈ C.
 In case of ∑_{l=1}^t β_l ∑_{k=1}^p α_k = -1, the series expansion (23) is absolutely convergent for all values of |z| < ρ and of |z| = ρ, Re(μ) > ¹/₂, with

$$\rho = \frac{\prod_{l=1}^{t} |\beta_l|^{\beta_l}}{\prod_{k=1}^{p} |\alpha_k|^{\alpha_k}} \quad \text{and} \quad \mu = \sum_{l=1}^{t} b_l - \sum_{k=1}^{p} a_k + \frac{p-t}{2}.$$

Therefore, the *likelihood* function $x \mapsto h^n \phi(x;h)^2$, defined viz

$$\begin{cases} \prod_{j=1}^{n} {}_{p}\Psi_{t} \begin{bmatrix} (a_{k}, \alpha_{k})_{1,p} \\ (b_{l}, \beta_{l})_{1,t} \end{bmatrix}^{-1} \frac{\prod_{k=1}^{p} \Gamma\left(a_{k} + \alpha_{k} \frac{x_{j}}{h}\right)}{\prod_{l=1}^{t} \Gamma\left(b_{l} + \beta_{l} \frac{x_{j}}{h}\right)} \frac{\lambda^{\frac{x_{j}}{h}}}{\Gamma\left(\frac{x_{j}}{h} + 1\right)} &, \text{if } x \in h\mathbb{Z}_{\geq 0} \\ 0 &, \text{otherwise} \end{cases}$$
(24)

⁸See also [20, Section 6].

corresponds to a Bayesian probability distribution of Wright type.

Such construction is far beyond the Mittag-Leffler's distribution (19) since it also encompasses the multi-variable hypergeometric distribution considered in Subsection 4.1 (take, for instance, $p = 1, t = 1, a_1 = b_1 = \beta, \alpha_1 = 1$ and $\beta_1 = 0$ on the above formula). Widely speaking, a wise application of Gauss-Legendre multiplication formula (cf. [12, p. 4])

$$\prod_{r=0}^{s-1} \Gamma\left(\frac{r}{s} + z\right) = (2\pi)^{\frac{s-1}{2}} s^{\frac{1}{2} - sz} \Gamma(sz)$$
(25)

allows us to amalgamize Mittag-Leffler and hypergeometric distributions as MacRobert's E-functions in disguise (cf. [12, p. 203]).

Let us now take, for each $k = 1, 2, ..., \gamma$ $(p = \gamma)$ and $l = 1, 2, ..., \alpha$ $(t = \alpha)$, the substitutions $\alpha_k = \beta_l = 1$, $a_k = \frac{k-1+\delta}{\gamma}$ and $b_l = \frac{l-1+\beta}{\alpha}$. A straightforward application of (25) shows that the coefficients μ_m of (23) are equal to

$$\mu_m = \sqrt{\frac{\gamma}{\alpha} (2\pi)^{\gamma - \alpha}} \frac{\alpha^{\beta + \alpha m} \Gamma \left(\delta + \gamma m\right)}{\gamma^{\delta + \gamma m} \Gamma \left(\beta + \alpha m\right)} \frac{\lambda^m}{\Gamma \left(m + 1\right)}$$

Thus, under the condition $\alpha - \gamma > -1$ the *likelihood* function (24), carrying the parameter $\lambda = \frac{\gamma^{\gamma}}{\alpha^{\alpha}} \frac{4}{q^{1+\gamma-\alpha}h^2}$ simplifies to

The above *likelihood* function is also well defined⁹ for $h^2 > \frac{\gamma^{2\gamma}}{\alpha^{2\alpha}} \frac{4}{q^{1+\gamma-\alpha}}$ and of $h^2 = \frac{\gamma^{2\gamma}}{\alpha^{2\alpha}} \frac{4}{q^{1+\gamma-\alpha}}$, $\operatorname{Re}(\beta) - \operatorname{Re}(\delta) > \frac{1}{2}$ whenever $\alpha - \gamma = -1$. In particular: 1. For $h^2 = \frac{\gamma^{2\gamma}}{\alpha^{2\alpha}} \frac{4}{q^{1+\gamma-\alpha}}$ and $0 < q^{1+\gamma-\alpha} < 4\frac{\gamma^{2\gamma}}{\alpha^{2\alpha}}$, the above set of formulae are also true under the choice $\delta = \frac{\beta}{2} = \frac{\gamma^{2\gamma}}{\alpha^{2\alpha}} \frac{2}{q^{1+\gamma-\alpha}}$.

2. For $\gamma = \delta = 1$, the *likelihood* function (26) is the Mittag-Leffler distribution (19) in

⁹In case of $\alpha - \gamma = -1$, the Wright series $_{1}\Psi_{1}\begin{bmatrix} (\delta, \gamma) \\ (\beta, \alpha) \end{bmatrix} \lambda$ is also absolutely convergent for $|\lambda| < \frac{\alpha^{\alpha}}{\gamma^{\gamma}}$ and of $|\lambda| = \frac{\alpha^{\alpha}}{\gamma^{\gamma}}$, $\operatorname{Re}(\beta) - \operatorname{Re}(\delta) > \frac{1}{2}$.

disguise. Moreover, if $\alpha = \operatorname{Re}(\alpha) > 0$, $\alpha \to 0^+$ and $h > \frac{2}{q}$, (19) simplifies to

$$h^{n}\phi(x;h)^{2} = \begin{cases} \prod_{j=1}^{n} \left(1 - \frac{4}{q^{2}h^{2}}\right)^{-1} q^{-\frac{2x_{j}}{h}} h^{-\frac{2x_{j}}{h}} , \text{if } x \in h\mathbb{Z}_{\geq 0}^{n} \\ 0 , \text{otherwise} \end{cases}$$
(27)

3. For $\beta = \delta$, the *likelihood* function (26) amalgamates the Poisson distribution $(\alpha = \gamma = 1)$ as well as the orthogonal measure that gives rise, up to the constant $\left(1 - \frac{4}{q^2 h^2}\right)^{-\beta n}$, to the hypergeometric distribution on $h\mathbb{Z}^n_{\geq 0}$, carrying the parameter $\lambda = \frac{4}{q^2 h^2}$ $(\alpha \to 0^+, \gamma = 1 \text{ and } h > \frac{2}{q})$ (see Subsection 4.1).

Let us now turn our attention to the construction of quasi-monomials carrying (26). A direct consequence of (14) shows that the even powers represented through the operational formula

$$\mathbf{m}_{2r}(x;h) = (-1)^r \left(\sum_{j=1}^n \left(\frac{h}{q^{1+\alpha-\gamma}} \frac{x_j}{h} \frac{\left(\beta-\alpha+\alpha\frac{x_j}{h}\right)_{\alpha}}{\left(\delta-\gamma+\gamma\frac{x_j}{h}\right)_{\gamma}} T_h^{-j} - \frac{4}{q^2h} I \right)^2 \right)^r \mathbf{s} \quad (\mathbf{s} \in \operatorname{Pin}(n))$$

have the hypergeometric series representation (cf. Appendix B.1)

$$\mathbf{m}_{2r}(x;h) = (-1)^r \frac{16^r}{q^{2r(1+\alpha-\gamma)}h^{2r}} \sum_{|\sigma|=r} \frac{r!}{\sigma!} \prod_{j=1}^n \mathbf{w}_{\sigma_j}(x;h) \mathbf{s},$$

with

$$\mathbf{w}_{\sigma_j}(x;h) = {}_{2+\alpha}F_{\gamma}\left(-2\sigma_j, -\frac{x_j}{h}, \left(\frac{k-1+\beta}{\alpha}-1+\frac{x_j}{h}\right)_{1,\alpha}; \left(\frac{l-1+\delta}{\gamma}-1+\frac{x_j}{h}\right)_{1,\gamma}; -\frac{\alpha^{\alpha}}{\gamma^{\gamma}}\frac{q^{1+\gamma-\alpha}h^2}{4}\right).$$

Here and elsewhere $_{2+\alpha}F_{\gamma}$ denotes the generalized hypergeometric series expansion (cf. [12, Chapter IV])

$${}_{\alpha+1}F_{\gamma}\left(a,b,(c_{k})_{1,\alpha};(d_{l})_{1,\gamma};\lambda\right) = \sum_{p=0}^{\infty} (a)_{p} (b)_{p} \frac{\prod_{k=1}^{\alpha} (c_{k})_{p}}{\prod_{l=1}^{\gamma} (d_{l})_{p}} \frac{\lambda^{p}}{p!}.$$

By applying (25) to the product of Pochhammer coefficients of $_{\alpha+1}F_{\gamma}$, there holds

$$\prod_{k=0}^{\alpha-1} \left(\frac{k+\beta}{\alpha} - 1 + \frac{x_j}{h}\right)_p = \prod_{k=0}^{\alpha-1} \frac{\Gamma\left(\frac{\beta+k}{\alpha} - 1 + \frac{x_j}{h} + p\right)}{\Gamma\left(\frac{k+\beta}{\alpha} - 1 + \frac{x_j}{h}\right)}$$
$$= \alpha^{-\alpha p} \frac{\Gamma\left(\beta - \alpha + \alpha \frac{x_j}{h} + \alpha p\right)}{\Gamma\left(\beta - \alpha + \alpha \frac{x_j}{h}\right)}$$

and analogously

$$\prod_{l=0}^{\gamma-1} \left(\frac{l+\delta}{\gamma} - 1 + \frac{x_j}{h} \right)_p = \gamma^{-\gamma p} \frac{\Gamma\left(\delta - \gamma + \gamma \frac{x_j}{h} + \gamma p\right)}{\Gamma\left(\delta - \gamma + \gamma \frac{x_j}{h}\right)}$$
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Thus, from the identities $(-2\sigma_j)_p = (-1)^p \frac{\Gamma(2\sigma_j+1)}{\Gamma(2\sigma_j+1-p)}$ and $\left(-\frac{x_j}{h}\right) = (-1)^p \frac{\Gamma\left(\frac{x_j}{h}+1\right)}{\Gamma\left(\frac{x_j}{h}+1\right)}$ it results the formal series representation

$${}_{2+\alpha}F_{\gamma}\left(-2\sigma_{j},-\frac{x_{j}}{h},\left(\frac{k-1+\beta}{\alpha}-1+\frac{x_{j}}{h}\right)_{1,\alpha};\left(\frac{l-1+\delta}{\gamma}-1+\frac{x_{j}}{h}\right)_{1,\gamma};-\frac{\alpha^{\alpha}}{\gamma^{\gamma}}\frac{q^{1+\gamma-\alpha}h^{2}}{4}\right) = \\ = \frac{\Gamma\left(\delta-\gamma+\gamma\frac{x_{j}}{h}\right)\Gamma\left(2\sigma_{j}+1\right)\Gamma\left(\frac{x_{j}}{h}+1\right)}{\Gamma\left(\beta-\alpha+\alpha\frac{x_{j}}{h}\right)} \times \\ \times {}_{1}\Psi_{3}\left[\left.\begin{pmatrix}\beta-\alpha+\alpha\frac{x_{j}}{h},\alpha\\(\delta-\gamma+\gamma\frac{x_{j}}{h},\gamma)\end{pmatrix}\right.\left(2\sigma_{j}+1,-1\right)\left.\begin{pmatrix}\frac{x_{j}}{h}+1,-1\end{pmatrix}\right|\right.-\frac{q^{1+\gamma-\alpha}h^{2}}{4}\right].$$

Here we would like to stress that the right-hand side of the above expansion shall be understood as a $2\sigma_j$ -term truncation of the generalized Wright series expansion (23).

Thus, for r > 0 the even quasi-monomials $\mathbf{m}_{2r}(x;h)$ are given by

$$\mathbf{m}_{2r}(x;h) = (-1)^r \frac{16^r}{q^{2r(1+\alpha-\gamma)}h^{2r}} \sum_{|\sigma|=r} \frac{r!}{\sigma!} \prod_{j=1}^n \frac{\Gamma\left(\delta - \gamma + \gamma \frac{x_j}{h}\right) \Gamma\left(2\sigma_j + 1\right)}{\Gamma\left(\beta - \alpha + \alpha \frac{x_j}{h}\right)} \times {}_{1}\Psi_3 \left[\begin{array}{c} \left(\beta - \alpha + \alpha \frac{x_j}{h}, \alpha\right) \\ \left(\delta - \gamma + \gamma \frac{x_j}{h}, \gamma\right) & \left(2\sigma_j + 1, -1\right) \end{array} \right| \left(\frac{x_j}{h} + 1, -1\right) \left(\frac{q^{1+\gamma-\alpha}h^2}{4}\right] \mathbf{s}, \text{ with } \mathbf{s} \in \operatorname{Pin}(n).$$

This surprisingly subtle characterization may be seen as an hypercomplex extension of Gauss's hypergeometric representation (cf. [12, Subsection 2.1.3.] & [23, p. 24]) when $\alpha \to 0^+$ and $\gamma = 1$. In the case of $\gamma \to 0^+$, the above representation becomes a Kummer's type representation (cf. [23, p. 24]) that amalgamizes the hypercomplex extension of the Poisson-Charlier polynomials ($\alpha = \delta = 1$) of even order (cf. [15, Example 3.3]) as well as an hypercomplex counterpart for the Meixner polynomials, up to the constant $\left(1 - \frac{4}{q^2h^2}\right)^{-\beta n}$ ($\alpha = 1 \& \delta = \beta$).

5. Further remarks on quasi-probabilities

Although the examples treated throughout Section 4.2 and Section 4.3 involve Bayesian probabilities in the classical sense, as the ones considered by the papers [7, 26], the framework developed provides us a goal-oriented guide to extend to Bayesian quasi-probabilities with *imaginary* bias such as the one obtained via Bender-Hook-Meisinger-Wang's approach (cf. [5, Section 3]).

In this section we illustrate the case of complex-valued *likelihood* functions that give rise to $\mathbb{C} \otimes \operatorname{Pin}(n)$ -valued *vacuum* vectors $\psi_0(x;h) = \phi(x;h)$ s towards the regularization of Mittag-Leffler distribution (19). To illustrate that let us define for each $\varepsilon > 0$ the family of complex-valued *likelihood* functions $x \mapsto h^n \phi_{\varepsilon}(x;h)^2$ on $h\mathbb{Z}^n$, carrying the parameter

$$\lambda = \left(\frac{4}{q^{2-\alpha}h^{2}}\right)^{1-\varepsilon} e^{\frac{i\pi\varepsilon}{2}}, \text{ as a quasi-probability distribution}$$

$$\begin{cases} \prod_{j=1}^{n} 3\Theta_{3} \left((q^{2}h^{2+\alpha})^{\varepsilon-1}e^{\frac{i\pi\varepsilon}{2}}\right)^{-1} \frac{\sin\left(\frac{\pi\varepsilon x_{j}}{2h}\right)}{\alpha^{\alpha}\sin\left(\frac{\pi\varepsilon x_{j}}{2\alpha^{\alpha}h}\right)} \frac{4^{\frac{(1-\varepsilon)x_{j}}{h}}q^{\frac{-(2-\alpha)(1-\varepsilon)x_{j}}{h}}h^{\frac{-2(1-\varepsilon)x_{j}}{h}}}{\Gamma\left(\beta e^{i\pi\varepsilon} + \alpha\frac{(1-\varepsilon)x_{j}}{h}\right)} e^{\frac{i\pi\varepsilon x_{j}}{2h}}, \quad x \in h\mathbb{Z}_{\geq 0}^{n} \\ \prod_{j=1}^{n} 3\Theta_{3} \left((q^{2}h^{2+\alpha})^{\varepsilon-1}e^{\frac{i\pi\varepsilon}{2}}\right)^{-1} \frac{\sin\left(\frac{\pi\varepsilon x_{j}}{2h}\right)}{\alpha^{\alpha}\sin\left(\frac{\pi\varepsilon x_{j}}{2\alpha^{\alpha}h}\right)} \frac{\varepsilon^{\frac{x_{j}}{h}}q^{\frac{-(1-\varepsilon)x_{j}}{h}}q^{\frac{(2-\alpha)(1-\varepsilon)x_{j}}{h}}h^{\frac{2(1-\varepsilon)x_{j}}{h}}}{\Gamma\left(\beta e^{i\pi\varepsilon} + \alpha\frac{(1-\varepsilon)x_{j}}{h}\right)} e^{\frac{i\pi\varepsilon x_{j}}{2h}}, \quad \text{otherwise} \end{cases}$$

$$(28)$$

endowed by the Laurent series expansion

$${}_{3}\Theta_{3}(\lambda) = {}_{3}\Psi_{3} \begin{bmatrix} (1,1) & (1,-\frac{\varepsilon}{2\alpha^{\alpha}}) & (1,\frac{\varepsilon}{2\alpha^{\alpha}}) \\ (\beta e^{i\pi\varepsilon},\alpha-\varepsilon\alpha) & (1,-\frac{\varepsilon}{2}) & (1,\frac{\varepsilon}{2}) \end{bmatrix} \begin{vmatrix} \lambda \end{bmatrix} + \\ + {}_{3}\Psi_{3} \begin{bmatrix} (1,1) & (1,-\frac{\varepsilon}{2\alpha^{\alpha}}) & (1,\frac{\varepsilon}{2\alpha^{\alpha}}) \\ (\beta e^{i\pi\varepsilon},\alpha-\alpha\varepsilon) & (1,-\frac{\varepsilon}{2}) & (1,\frac{\varepsilon}{2}) \end{vmatrix} \begin{vmatrix} \varepsilon \\ \lambda \end{bmatrix} - \frac{1}{\Gamma(\beta e^{i\pi\varepsilon})}.$$

On the above construction we make use of the formulae $\Gamma(1 + z) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ (cf. [12, p. 4]). Here we recall that the *likelihood* function (28) is well defined:

1. For every $\lambda = \left(\frac{4}{q^{2-\alpha}h^2}\right)^{1-\varepsilon} e^{\frac{i\pi\varepsilon}{2}} \in \mathbb{C}$ whenever¹⁰ $0 < \varepsilon < 1$. 2. For¹¹ $\beta \notin \mathbb{N}_0$, in case of $\varepsilon \to 1^-$.

From the limit property $\lim_{\epsilon \to 0^+} \frac{\sin\left(\frac{\pi \epsilon x_j}{2h}\right)}{\alpha^{\alpha} \sin\left(\frac{\pi \epsilon x_j}{2\alpha^{\alpha}h}\right)} = 1$ there holds

$$\lim_{\varepsilon \to 0^+} {}_{3}\Psi_{3} \begin{bmatrix} (1,1) & (1,-\frac{\varepsilon}{2\alpha^{\alpha}}) & (1,\frac{\varepsilon}{2\alpha^{\alpha}}) \\ (\beta e^{i\pi\varepsilon},\alpha-\varepsilon\alpha) & (1,-\frac{\varepsilon}{2}) & (1,\frac{\varepsilon}{2}) \end{bmatrix} \begin{bmatrix} \left(\frac{4}{q^{2-\alpha}h^{2}}\right)^{1-\varepsilon}e^{\frac{i\pi\varepsilon}{2}} \end{bmatrix} = {}_{1}\Psi_{1} \begin{bmatrix} (1,1) & \left(\frac{1}{q^{2-\alpha}h^{2}}\right) \\ (\beta,\alpha) & \left(\frac{1}{q^{2-\alpha}h^{2}}\right) \end{bmatrix}$$
$$\lim_{\varepsilon \to 0^{+}} {}_{3}\Psi_{3} \begin{bmatrix} (1,1) & \left(1,-\frac{\varepsilon}{2\alpha^{\alpha}}\right) & \left(1,\frac{\varepsilon}{2\alpha^{\alpha}}\right) \\ (\beta e^{i\pi\varepsilon},\alpha-\varepsilon\alpha) & \left(1,-\frac{\varepsilon}{2}\right) & \left(1,\frac{\varepsilon}{2}\right) \end{bmatrix} \varepsilon \left(\frac{4}{q^{2-\alpha}h^{2}}\right)^{\varepsilon-1}e^{-\frac{i\pi\varepsilon}{2}} \end{bmatrix} = \frac{1}{\Gamma(\beta)}.$$

Thus, for $0 < \varepsilon < 1$ (28) is a complex-valued regularization of the Mittag-Leffler distribution (19), since

$$\lim_{\varepsilon \to 0^+} {}_{3}\Theta_3\left(\left(\frac{4}{q^{2-\alpha}h^2}\right)^{1-\varepsilon}e^{\frac{i\pi\varepsilon}{2}}\right) = {}_{1}\Psi_1\left[\begin{array}{c}(1,1)\\(\beta,\alpha)\end{array}\right|\frac{1}{q^{2-\alpha}h^2}\right].$$

¹⁰The case $0 < \varepsilon < 1$ follows from the fact that in case of $(1 - \varepsilon)\alpha > 0$ the Laurent series ${}_{3}\Theta_{3}(\lambda)$ is absolutely convergent in \mathbb{C} .

¹¹We can rid the condition $\operatorname{Re}(-\beta) > \frac{3}{2}$ that yields from application of [20, Theorem 1] because $\Gamma\left(\beta e^{i\pi\varepsilon} + \alpha \frac{(1-\varepsilon)x_j}{h}\right)$ equals to the constant $\Gamma(-\beta)$ in case of $\varepsilon = 1$.

In case of $\varepsilon \to 1^-$, it follows that

$$\lim_{\varepsilon \to 1^{-}} h^{n} \phi_{\varepsilon}(x;h)^{2} = \prod_{j=1}^{n} {}_{3}\Theta_{2}(\lambda)^{-1} \frac{\sin\left(\frac{\pi x_{j}}{2h}\right)}{\alpha^{\alpha} \sin\left(\frac{\pi x_{j}}{2\alpha^{\alpha}h}\right)} e^{\frac{i\pi x_{j}}{2h}} (x \in h\mathbb{Z}^{n}),$$

with

$${}_{3}\Theta_{2}(\lambda) = {}_{3}\Psi_{2} \left[\begin{array}{ccc} (1,1) & \left(1,-\frac{1}{2\alpha^{\alpha}}\right) & \left(1,\frac{1}{2\alpha^{\alpha}}\right) \\ (1,-\frac{1}{2}) & \left(1,\frac{1}{2}\right) \end{array} \right] + \\ + {}_{3}\Psi_{2} \left[\begin{array}{ccc} (1,1) & \left(1,-\frac{1}{2\alpha^{\alpha}}\right) & \left(1,\frac{1}{2\alpha^{\alpha}}\right) \\ (1,-\frac{1}{2}) & \left(1,\frac{1}{2}\right) \end{array} \right] - 1$$

This quasi-probability like distribution is no longer a regularization for the Mittag-Leffler distribution (19). On the other hand, it exhibits a *space-time symmetry*¹² due to the invariance property

$$\lim_{\varepsilon \to 1^-} h^n \overline{\phi_{\varepsilon}(-x;h)^2} = \lim_{\varepsilon \to 1^-} h^n \phi_{\varepsilon}(x;h)^2.$$

Interesting enough is that the resulting discrete magnetic potential

$$\mathbf{a}_{h}(-ix) = \sum_{j=1}^{n} -i\mathbf{e}_{j} \frac{\sinh\left(\frac{\pi x_{j}}{2\alpha^{\alpha}h} + \frac{\pi}{2\alpha^{\alpha}}\right)}{qh\sinh\left(\frac{\pi x_{j}}{2\alpha^{\alpha}h}\right)} \tanh\left(\frac{\pi x_{j}}{2h}\right)$$

obtained from the transformation $x \mapsto -ix$ on the formula (17) is closely related with the hyperbolic potentials of Macdonald-Ruijnaars type (cf. [38, 40]).

6. Conclusions

Emphasizing how the use of quasi-probabilities may be useful in the construction of Fock spaces over lattices, we have developed a framework on which the k-Fock states $\psi_k(x;h)$ of L_h , and moreover, the quasi-monomials $\mathbf{m}_k(x;h)$ can be determined from a general *vacuum* vector of the form $\psi_0(x;h) = \phi(x;h)\mathbf{s}$ ($\mathbf{s} \in \operatorname{Pin}(n)$), encoded by the quasi-probability law $\operatorname{Pr}\left(\sum_{j=1}^n \mathbf{e}_j X_j = x\right) = h^n \phi(x;h)^2$. The main novelty here against

[13, 14, 15] stems into the description of families of special functions of hypercomplex variable through the Bayesian quasi-probability formulation rather than seeking through the set of underlying symmetries.

We make use of Mellin-Barnes integrals to get in touch with Dirac's framework on quasi-probabilities [11]. In the shed of the H-Fox framework, it is not surprising that applications in statistics may be considered in the context of the presented approach (cf. [23, Chapter 4]). On the other hand, since the Lagrangian operators from relativistic wave mechanics encompass conserved current densities that may be interpreted as quasiprobabilities (cf. [11, pp. 5-8]), we expect that the Bayesian quasi-probability formalism

¹²A \mathcal{PT} -symmetry, accordingly to nomenclature adopted on the papers [2, 3, 4, 5].

developed throughout this paper may be useful to investigate questions in lattice quantum mechanics towards gauge fields, fermion fields and Quantum Cromodynamics (cf. [25, Chapter 3, Chapter 4 & Chapter 5]), beyond the applications already considered in [8, 37, 24, 26]. We also believe that this approach may be useful to establish a deep and thorough analysis of quantum field models that exhibit axial anomalies such as the ones considered in [3].

The examples involving H-Fox functions – in concrete, the generalized Mittag-Leffler $E_{\alpha,\beta}(\lambda)$ and Wright functions ${}_{p}\Psi_{t}\begin{bmatrix} (a_{k},\alpha_{k})_{1,p} \\ (b_{l},\beta_{l})_{1,t} \end{bmatrix} \lambda - \text{displays also a tangible interplay}$ between Mellin-Barnes type integrals and fractional calculus (cf. [23, Chapter 3]). Such connection seems to have been somehow overlooked by several authors when they are dealing with families of orthogonal polynomials beyond the known ones within the Askey-Wilson scheme (cf. [29, 40, 39]).

Due to the lack of applications on the literature concerning the interplay between Bayesian quasi-probabilities with *imaginary* bias and \mathcal{PT} -symmetric quantum mechanics (cf. [2, 3, 4, 5]) we believe that this topic deserves a closer inspection, beyond the simplest examples considered in [5, Section 4] and in Section 5. Further applications of this approach to crystallographic root systems towards Macdonald-Ruijnaars (pseudo) Laplacians (cf. [38]) will also be investigated in depth in a future research.

Appendix A. Technical Results used in Section 3

Lemma Appendix A.1. For the pair of Clifford-vector-valued ladder operators (A_h^+, A_h^-) defined as

$$A_{h}^{+} = \sum_{j=1}^{n} \mathbf{e}_{j} A_{h}^{+j} \quad and \quad A_{h}^{-} = \sum_{j=1}^{n} \mathbf{e}_{j} A_{h}^{-j},$$
(A.1)

the anti-commutator $A_h^- A_h^+ + A_h^+ A_h^-$ is scalar-valued whenever $[A_h^{-k}, A_h^{+j}] = 0$ for $j \neq k$. Moreover, we have

$$A_h^- A_h^+ + A_h^+ A_h^- = -\sum_{j=1}^n \left(A_h^{-j} A_h^{+j} + A_h^{+j} A_h^{-j} \right).$$

Proof: Starting from the definition, we obtain from (2)

$$A_{h}^{-}A_{h}^{+} + A_{h}^{+}A_{h}^{-} = \sum_{j,k=1}^{n} \left(\mathbf{e}_{j}\mathbf{e}_{k}A_{h}^{-j}A_{h}^{+k} + \mathbf{e}_{k}\mathbf{e}_{j}A_{h}^{+k}A_{h}^{-j} \right)$$
$$= \sum_{j,k=1}^{n} \left(-2\delta_{jk}A_{h}^{-j}A_{h}^{+k} + \mathbf{e}_{k}\mathbf{e}_{j}[A_{h}^{+k}, A_{h}^{-j}] \right)$$

We see therefore that the bivector summands $\mathbf{e}_k \mathbf{e}_j[A_h^{+k}, A_h^{-j}]$ of $A_h^- A_h^+ + A_h^+ A_h^-$ vanish only in case of $[A_h^{+k}, A_h^{-j}] = 0$ hold for every j, k = 1, 2, ..., n, with $j \neq k$. Thus, we have

$$A_h^- A_h^+ + A_h^+ A_h^- = -2\sum_{\substack{j=1\\23}}^n A_h^{-j} A_h^{+j} - \sum_{j=1}^n [A_h^{+j}, A_h^{-j}].$$

Finally, from the expression $\left[A_h^{+j}, A_h^{-j}\right] = A_h^{+j}A_h^{-j} - A_h^{-j}A_h^{+j}$ we can see that $-2A_h^{-j}A_h^{+j} - [A_h^{+j}, A_h^{-j}]$ equals to $-A_h^{-j}A_h^{+j} - A_h^{+j}A_h^{-j}$, and hence, the above relation may also be rewritten as

$$A_h^- A_h^+ + A_h^+ A_h^- = -\sum_{j=1}^n \left(A_h^{-j} A_h^{+j} + A_h^{+j} A_h^{-j} \right).$$

Lemma Appendix A.2. For every **f** and **g** with membership in $\ell_2(h\mathbb{Z}^n; C\ell_{0,n})$ there holds

$$\langle A_h^+ \mathbf{f}, \mathbf{g} \rangle_h = \langle \mathbf{f}, A_h^- \mathbf{g} \rangle_h \quad and \quad \langle A_h^- \mathbf{f}, \mathbf{g} \rangle_h = \langle \mathbf{f}, A_h^+ \mathbf{g} \rangle_h.$$

Moreover

$$\langle L_h \mathbf{f}, \mathbf{g} \rangle_h = \langle \mathbf{f}, L_h \mathbf{g} \rangle_h = \frac{1}{2} \langle A_h^+ \mathbf{f}, A_h^+ \mathbf{g} \rangle_h + \frac{1}{2} \langle A_h^- \mathbf{f}, A_h^- \mathbf{g} \rangle_h$$

Proof: Recall that from the \dagger - conjugation properties $\left(\mathbf{e}_{j}A_{h}^{\pm j}\mathbf{f}(x)\right)^{\dagger} = -(A_{h}^{\pm j}\mathbf{f}(x))^{\dagger}\mathbf{e}_{j}$, that follow from (4), we obtain for each j = 1, 2, ..., n, the conjugation formula

$$\left(A_h^{\pm}\mathbf{f}(x)\right)^{\dagger} = -\sum_{j=1}^n \left(A_h^{\pm j}\mathbf{f}(x)\right)^{\dagger} \mathbf{e}_j$$

On the other hand, from (10) we find that the ladder operators $A_h^{\pm j}$ defined viz (A.1) satisfy $\langle A_h^{+j} \mathbf{f}, \mathbf{g} \rangle_h = -\langle \mathbf{f}, A_h^{-j} \mathbf{g} \rangle_h$ and $\langle A_h^{-j} \mathbf{f}, \mathbf{g} \rangle_h = -\langle \mathbf{f}, A_h^{+j} \mathbf{g} \rangle_h$. Combination of the above properties results, for each $j = 1, 2, \ldots, n$, into the sequence

of relations:

$$\langle \mathbf{e}_j A_h^{\pm j} \mathbf{f}(x), \mathbf{g}(x) \rangle_h = -\langle A_h^{\pm j} \mathbf{f}(x), \mathbf{e}_j \mathbf{g}(x) \rangle_h = \langle \mathbf{f}(x), \mathbf{e}_j A_h^{\pm j} \mathbf{g}(x) \rangle_h.$$

Hence, the Hermitian conjugation properties

$$\langle A_h^+ \mathbf{f}, \mathbf{g} \rangle_h = \langle \mathbf{f}, A_h^- \mathbf{g} \rangle_h$$
 and $\langle A_h^- \mathbf{f}, \mathbf{g} \rangle_h = \langle \mathbf{f}, A_h^+ \mathbf{g} \rangle_h$

in $\ell_2(h\mathbb{Z}^n; C\ell_{0,n})$, and moreover, the set of identities

$$\langle L_h \mathbf{f}, \mathbf{g} \rangle_h = \langle \mathbf{f}, L_h \mathbf{g} \rangle_h = \frac{1}{2} \langle A_h^+ \mathbf{f}, A_h^+ \mathbf{g} \rangle_h + \frac{1}{2} \langle A_h^- \mathbf{f}, A_h^- \mathbf{g} \rangle_h$$

follow straightforwardly from the factorization formula $L_h = \frac{1}{2}(A_h^+A_h^- + A_h^-A_h^+)$.

Appendix B. Technical Results used in Section 4

Lemma Appendix B.1. In case of $x \mapsto h^n \phi(x;h)^2$ corresponds to the likelihood function (26), we thus have

$$\mathbf{m}_{2r}(x;h) = (-1)^r \frac{16^r}{q^{2r(1+\alpha-\gamma)}h^{2r}} \sum_{|\sigma|=r} \frac{r!}{\sigma!} \prod_{j=1}^n \mathbf{w}_{\sigma_j}(x;h) \mathbf{s}$$

with $\mathbf{w}_{\sigma_j}(x;h) = 2 + \alpha F_{\gamma}\left(-2\sigma_j, -\frac{x_j}{h}, \left(\frac{k-1+\beta}{\alpha}-1+\frac{x_j}{h}\right)_{1,\alpha}; \left(\frac{l-1+\delta}{\gamma}-1+\frac{x_j}{h}\right)_{1,\gamma}; -\frac{\alpha^{\alpha}}{\gamma^{\gamma}}\frac{q^{1+\gamma-\alpha}h^2}{4}\right).$

Proof: A direct computation involving the binomial identity shows that

$$\begin{pmatrix} \frac{h}{q^{1+\alpha-\gamma}} \frac{x_j}{h} \frac{(\beta-\alpha+\alpha\frac{x_j}{h})_{\alpha}}{(\delta-\gamma+\gamma\frac{x_j}{h})_{\gamma}} T_h^{-j} - \frac{4}{q^2h} I \end{pmatrix}^{2\sigma_j} = \left(\frac{h}{q^{1+\alpha-\gamma}}\right)^{2\sigma_j} \sum_{p=0}^{2\sigma_j} \left(\frac{2\sigma_j}{p}\right) \left(-\frac{4}{q^{1+\gamma-\alpha}h^2}\right)^{2\sigma_j-p} \times \\ \times \left(\frac{x_j}{h} \frac{(\beta-\alpha+\alpha\frac{x_j}{h})_{\alpha}}{(\delta-\gamma+\gamma\frac{x_j}{h})_{\gamma}} T_h^{-j}\right)^p \\ = \left(\frac{4}{q^{2(1+\alpha-\gamma)}h}\right)^{2\sigma_j} \sum_{p=0}^{2\sigma_j} \left(\frac{2\sigma_j}{p}\right) \left(-\frac{q^{1+\gamma-\alpha}h^2}{4}\right)^p \times \\ \times (-1)^p \left(-\frac{x_j}{h}\right)_p \frac{\prod_{k=0}^{p-1} \left(\alpha\frac{x_j}{h}+\beta-(k+1)\alpha\right)_{\alpha}}{\prod_{l=0}^{p-1} \left(\gamma\frac{x_j}{h}+\delta-(k+1)\gamma\right)_{\alpha}}.$$

Here we recall $\begin{pmatrix} 2\sigma_j \\ p \end{pmatrix} = (-1)^p \frac{(-2\sigma_j)_p}{p!}$ for $p \le 2\sigma_j$ and $\begin{pmatrix} 2\sigma_j \\ p \end{pmatrix} = 0$ for $p > 2\sigma_j$.

On the other hand, the product $\prod_{k=1}^{p} \left(\beta + \alpha \frac{x_j}{h} - k \alpha\right)_{\alpha}$ may be rewritten as

$$\begin{split} \prod_{k=0}^{p-1} \left(\beta + \alpha \frac{x_j}{h} - (k+1)\alpha \right)_{\alpha} &= \prod_{k=0}^{p-1} \prod_{s=0}^{\alpha-1} \left(\alpha \left(\frac{\beta}{\alpha} - k - 1 + \frac{x_j}{h} \right) + s \right) \\ &= \alpha^{\alpha p} \prod_{s=0}^{\alpha-1} \left(\frac{s+\beta}{\alpha} - 1 + \frac{x_j}{h} \right)_p, \end{split}$$

and analogously,

$$\prod_{l=0}^{p-1} \left(\delta + \gamma \frac{x_j}{h} - (l+1)\gamma\right)_{\gamma} = \gamma^{\gamma p} \prod_{s=0}^{\gamma-1} \left(\frac{s+\delta}{\gamma} - 1 + \frac{x_j}{h}\right)_{p}.$$

This implies

$$\left(\frac{h}{q^{1+\alpha-\gamma}} \frac{x_j}{h} \frac{\left(\beta-\alpha+\alpha\frac{x_j}{h}\right)_{\alpha}}{h^{\gamma} \left(\delta-\gamma+\gamma\frac{x_j}{h}\right)_{\gamma}} T_h^{-j} - \frac{4}{q^2 h} I\right)^{2\sigma_j} = \left(\frac{4}{q^{2(1+\alpha-\gamma)}h}\right)^{2\sigma_j} \sum_{p=0}^{\infty} \frac{1}{p!} \left(-\frac{\alpha^{\alpha}}{\gamma^{\gamma}} \frac{q^{1+\gamma-\alpha}h^2}{4}\right)^p \times \left(-2\sigma_j\right)_p \left(-\frac{x_j}{h}\right)_p \frac{\prod_{k=0}^{\alpha-1} \left(\frac{k+\beta}{\alpha}-1+\frac{x_j}{h}\right)_p}{\prod_{l=0}^{\gamma-1} \left(\frac{l+\delta}{\gamma}-1+\frac{x_j}{h}\right)_p} = \left(\frac{4}{q^{2(1+\alpha-\gamma)}h}\right)^{2\sigma_j} \mathbf{w}_{\sigma_j}(x;h),$$

with $\mathbf{w}_{\sigma_j}(x;h) = {}_{2+\alpha}F_{\gamma}\left(-2\sigma_j, -\frac{x_j}{h}, \left(\frac{k-1+\beta}{\alpha}-1+\frac{x_j}{h}\right)_{1,\alpha}; \left(\frac{l-1+\delta}{\gamma}-1+\frac{x_j}{h}\right)_{1,\gamma}; -\frac{\alpha^{\alpha}}{\gamma^{\gamma}}\frac{q^{1+\gamma-\alpha}h^2}{4}\right).$ By inserting the above relation on the right-hand side of (14), we obtain for $|\sigma| = r$

the desired result. \blacksquare

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