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A note on "Convergence radius of Osada's method under Hölder continuous condition"

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Abstract

In this paper we revise the proofs of the results obtained in "Convergence radius of Osada's method under Hölder continuous condition" [4], because the remainder of the Taylor's expansion used for the obtainment of the local convergence radius is not correct. So we perform the complete study in order to modify the equation for getting the local convergence radius, the uniqueness radius and the error bounds. Moreover a dynamical study for the third order Osada's method is also developed.

Key words: Nonlinear equations, iterative methods, Multiple roots, Local convergence; Dynamics

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1. Introduction

In the last years some of the studies concerning on iterative methods for approximating roots of nonlinear equations have focused on multiple roots. It is a special case where some particular aspects must be taken into account. Some real applications give this problem special interest, see [8], with a study of the multipactor effect, analyzing the trajectory equation of an electron in the air gap between two parallel plates results in a

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nonlinear equation with a multiple root. This also happens in the Van der Waals equation of state among other phenomenons.

Specially interesting from a mathematical point of view is paper [1] where a complete local convergence study has been performed, obtaining the convergence radius of the well-known modified Newton's method for multiple zeros, when the involved function satisfies a Hölder or a center-Hölder continuity condition. This result is improved in [2]. Similar results for the third order method due to Halley are obtained in [3].

We are now interested in this kind of local convergence studies for third order methods for multiple roots. So we center our attention in papers [3] and [4], where the authors analyze the local convergence for Osada and Halleys's method under Hölder and center-Hölder continuity conditions.

We consider the third order method of Osada [4] to find a multiple zero x^* of multiplicity m of a nonlinear equation $f(x) = 0$, $f : D \subseteq: \mathbb{R} \longrightarrow: \mathbb{R}$, given by:

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)}. \tag{1}$$

We say that r is the radius of the local convergence ball if the sequence x_n generated by this iterative method, starting from any initial point in the open ball $B(x^*, r)$ converges to x^* and remains in the ball. In these studies it is interesting to obtain the largest possible value of r, but obviously, this depends on the conditions that the nonlinear function verifies. Here we consider that f satisfies the following Hölder continuous conditions $\forall x, y \in D$,

$$|f^{(m)}(x^*)^{-1}(f^{(m+1)}(x) - f^{(m+1)}(y))| \leq K_0|x-y|^p, \quad K_0 > 0, p \in]0,1]. \tag{2}$$

$$|f^{(m)}(x^*)^{-1}f^{(m+1)}(x)| \leq K_m, \quad \forall x \in D, \quad K_m > 0. \tag{3}$$

Unfortunately, the Taylor's expansion used by the authors of [4] in the proof of lemma 1 is not correct. The same authors use the correct version of the remainder in Taylor's expansion in the paper "On the convergence radius of the modified Newton method for multiple roots under the center-Hölder condition", see lemma 1 of [2].

In [4], the authors consider the following formula for Taylor's expansion with integral remainder:

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}(x-a)^2 f''(a) + \frac{1}{3!}(x-a)^3 f'''(a) + \cdots \\ + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{1}{n!}\int_a^x (f^{(n+1)}(t) - f^{(n+1)}(a))(x-t)^n dt. \end{aligned}$$

It is well know that for a Taylor expansion of order n, the derivative evaluated in the remainder is of order $n+1$, but if one uses the integral form remainder, this derivative is of order n. That is, the Taylor's expansion with integral form remainder [7] has the form

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + E_n(x),$$

where

$$E_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

Another way to express the error is

$$E_n(x) = \frac{1}{(n-1)!} \int_a^x (f^{(n)}(t) - f^{(n)}(a))(x-t)^{n-1} dt,$$

where one can check the last equality by writing the last integral as $\int_\alpha^x u dv$ with $u = f^{(n)}(t) - f^n(\alpha)$ and $dv = (x-t)^{n-1} dt$.

In order to correct the results obtained in paper [4], we use different results involving divided differences that are introduced in the following section.

2. Preliminaries

We recall the definitions of divided differences and their properties.

Definition 2.1 [5] The divided differences $f[a_0, a_1, \ldots, a_k]$, on $k+1$ different points $a_0, a_1, \ldots, a_k$ of a function $f(x)$ are defined by

$$\begin{aligned} f[a_0] &= f(a_0), \\ f[a_0, a_1] &= \frac{f[a_0] - f[a_1]}{a_0 - a_1}, \\ &\vdots \\ f[a_0, a_1, \ldots, a_k] &= \frac{f[a_0, a_1, \ldots, a_{k-1}] - f[a_1, a_2, \ldots, a_k]}{a_0 - a_k}. \end{aligned}$$

If the function f is sufficiently differentiable, then its divided differences $f[a_0, a_1, \ldots, a_k]$ can be defined if some of the arguments a_i coincide. For instance, if $f(x)$ has k-th derivative at a_0, then it makes sense to define

$$f[\underbrace{a_0, a_0, \ldots, a_0}_{k+1}] = \frac{f^{(k)}(a_0)}{k!}. \tag{4}$$

Lemma 1 *[5] The divided differences $f[a_0, a_1, \ldots, a_k]$ are symmetric functions of their arguments, i.e., they are invariant under permutations of $[a_0, a_1, \ldots, a_k]$.*

Lemma 2 *[6] If the function f has k-th derivative, and $f^{(k)}(x)$ is continuous in the interval $I_x = [min(x_0, x_1, \ldots, x_k), max(x_0, x_1, \ldots, x_k)]$ then*

$$f[x_0, x_1, \ldots, x_k] = \int_0^1 \ldots \int_0^1 t_1^{k-1} t_2^{k-2} \ldots t_{k-1} f^{(k)}(t) dt_1 \ldots dt_k,$$

$$\textit{where} \quad t = x_0 + (x_1 - x_0)t_1 + (x_2 - x_1)t_1 t_2 + \ldots + (x_k - x_{k-1}) t_1 \ldots t_k.$$

Lemma 3 *If the function f has $(k+1)th$ derivative, then for every argument x, the following interpolation formula holds*

$$f(x) = f[a_0] + \sum_{i=1}^{k} f[a_0, a_1, \ldots, a_i] \prod_{j=0}^{i-1} (x - a_j) + f[a_0, a_1, \ldots, a_k, x] \prod_{i=0}^{k} (x - a_i).$$

Lemma 4 *Assume the function f has an $(m+1)th$ derivative, and x^* is a zero of multiplicity m, then for every argument x, we define functions $g(x)$, $p(x)$ and $q(x)$ as*

$$
\begin{aligned}
g(x) &= f[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x], \quad p(x) = f[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x, x], \\
q(x) &= f[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x, x, x].
\end{aligned} \tag{5}
$$

then

$$
g'(x) = p(x), \quad g''(x) = 2q(x). \tag{6}
$$

Lemma 5 *If the function f has $(m+1)th$ derivative, and x^* is a zero of multiplicity m, then for every argument x, the following formulae hold*

$$
f(x) = f[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x](x - x^*)^m = g(x)(x - x^*)^m. \tag{7}
$$

$$
\begin{aligned}
f'(x) &= f[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x, x](x - x^*)^m + mf[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x](x - x^*)^{m-1} \\
&= p(x)(x - x^*)^m + mg(x)(x - x^*)^{m-1}.
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
f''(x) &= 2f[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x, x, x](x - x^*)^m + 2mf[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x, x](x - x^*)^{m-1} \\
&+ m(m-1)[\underbrace{x^*, x^*, \ldots, x^*}_{m}, x](x - x^*)^{m-2} \\
&= 2q(x)(x - x^*)^m + 2mp(x)(x - x^*)^{m-1} + m(m-1)g(x)(x - x^*)^{m-2}.
\end{aligned} \tag{9}
$$

where $g(x)$, $p(x)$ and $q(x)$ are defined in (5).

Proof Applying Lemma 2 to the case of the multiple zero x^* of multiplicity m, and using (4) and (5), we get (7). Differentiating both sides of (7) gives (8), and differentiating both sides of (8) again, one obtains (9).

Lemma 6 *Let $r_0 = \frac{m+1}{K_m}$, and $T_{m,p} = \prod_i^{m+1}(p+i)$. Then, under conditions* (2) *and* (3) *for all $x_0 \in]x^* - r_0, x^* + r_0[= I_0$ and $e_0 = x_0 - x^*$ functions defined in Lemma 4 verify the following bounds:*

$$
\begin{aligned}
&(B1) \quad \left|g(x^*)^{-1}p(x_0)\right| \le \frac{K_m}{m+1} \\
&(B2) \quad \left|g(x_0)^{-1}g(x^*)\right| \le \frac{m+1}{m+1-K_m|e_0|} \\
&(B3) \quad \left|g(x^*)^{-1}q(x_0)e_0\right| \le \frac{K_0 m!|e_0|^p}{T_{m,p}} \\
&(B4) \quad \left|g(x_0)^{-1}p(x_0)\right| \le \frac{K_m}{m+1-K_m|e_0|} \\
&(B5) \quad \left|g(x_0)^{-1}q(x_0)e_0\right| \le \frac{m+1}{m+1-K_m|e_0|}\frac{k_0 m!}{T_{m,p}}|e_0|^p.
\end{aligned}
$$

Proof First of all, by (4) and (5) we get

$$
g(x^*) = f[\underbrace{x^*, x^*, \ldots, x^*}_{m+1}] = \frac{f^{(m)}(x^*)}{m!}. \tag{10}
$$

Using (4), (5), conditions (2), (3) and Lemma 2, we have (B1) as follows

$$\begin{aligned}
&|g(x^*)^{-1}p(x_0)| \\
&= |g(x^*)^{-1}\int_0^1 \dots \int_0^1 t_1^m t_2^{m-1} \dots t_m f^{(m+1)}(x^* + e_0 t_1 \dots t_m) dt_1 \dots dt_{m+1}| \\
&\le \frac{m! K_m}{(m+1)!} = \frac{K_m}{m+1},
\end{aligned} \tag{11}$$

Using condition (3), the mean value theorem, with the same reasoning as in (B1) and the definition of r_0, we obtain

$$|1 - g(x^*)^{-1}g(x_0)| = \left|g(x^*)^{-1}(g(x^*) - g(x_0))\right| \le \left|g(x^*)^{-1}g'(\xi_0)e_0\right| \le \frac{K_m}{m+1}|e_0| < 1.$$

From this relation we get that $g(x_0)^{-1}$ exists by Banach's lemma, we have (B2) as follows:

$$|g(x_0)^{-1}g(x^*)| \le \frac{1}{1 - \frac{K_m|e_0|}{m+1}} = \frac{m+1}{m+1-K_m|e_0|}. \tag{12}$$

Now we get (B3) by using lemma 2

$$\begin{aligned}
&|g(x^*)^{-1}q(x_0)e_0| \\
&= |g(x^*)^{-1}\int_0^1 \dots \int_0^1 t_1^m t_2^{m-1} \dots t_m [f^{(m+1)}(x^* + e_0 t_1 \dots t_{m-1}) \\
&\quad - f^{(m+1)}(x^* + e_0 t_1 \dots t_m)] dt_1 \dots dt_{m+1}| \\
&\le \int_0^1 t_1^m t_2^{m-1} \dots t_m |g(x^*)^{-1}[f^{(m+1)}(x^* + e_0 t_1 \dots t_{m-1}) \\
&\qquad\qquad - f^{(m+1)}(x^* + e_0 t_1 \dots t_m)]| dt_1 \dots dt_{m+1} \\
&\le m! \int_0^1 t_1^m t_2^{m-1} \dots t_m K_0 (t_1 \dots t_{m-1}(1-t_m))^p |e_0|^p dt_1 \dots dt_{m+1} \\
&= K_0|e_0|^p m! \int_0^1 t_1^{m+p} dt_1 \int_0^1 t_2^{m+p-1} dt_2 \dots \int_0^1 t_{m-1}^{2+p} dt_{m-1} \\
&\qquad \int_0^1 t_m(1-t_m)^p dt_m \int_0^1 dt_{m+1} \\
&= K_0|e_0|^p m! \frac{1}{m+p+1} \frac{1}{m+p} \cdots \frac{1}{p+3} \int_0^1 t_m(1-t_m)^p dt_m \\
&= K_0|e_0|^p m! \frac{1}{m+p+1} \frac{1}{m+p} \cdots \frac{1}{p+3} \frac{1}{(p+2)(p+1)} \\
&= \frac{K_0 m! |e_0|^p}{T_{m,p}}.
\end{aligned} \tag{13}$$

(B4) and (B5) are easily deduced from (B1), (B2) and (B3).

3. Main results

In this section we obtain the local convergence radius for Osada's method under Holder continuous conditions given by (2) and (3).

Theorem 1 *Let $D \subset \mathbb{R}$ an open convex and non-empty set where $f : D \longrightarrow \mathbb{R}$ in $\mathcal{C}^m(D)$ with x^* a root with multiplicity m for the nonlinear equation $f(x) = 0$ and Hölder conditions (2) and (3) are verified. Let $r_0 = \frac{m+1}{k_m}$, and let r_1 be the smallest positive root of the function:*

$$h_1(t) = 2m^2(m-1) - \frac{2m(m^2+2m-1)K_m}{m+1}t - \frac{4mK_0m!}{T_{m,p}}t^{p+1} - \frac{4mK_m^2}{(m+1)(m+1-K_mt)}t^2 - \frac{4K_mK_0m!}{T_{m,p}(m+1-K_mt)}t^{p+2}, \tag{14}$$

and let r_2 be the smallest positive root of the function:

$$h_2(t) = 8(m+1)K_mK_0m!t^{p+2} + (m^2+6m+1)K_m^2T_{m,p}t^2 + 2m(m+1)^2(m+1-K_mt)K_0m!t^{p+1} + 2m(m^2+2m-1)(m+1-K_mt)K_mT_{m,p}t - 2m^2(m^2-1)(m+1-K_mt)T_{m,p},$$

Then, for any initial point $x_0 \in]x^*-r, x^*+r[= I$, where $r = min\{r_0, r_1, r_2\}$, the sequence $\{x_n\}$, $n \geq 0$ generated by Osada's method (1) is well defined and converges at a rate of order $p+2$ to the unique solution $x^* \in I_0$. Moreover, the following error bound holds for all $n \geq 0$

$$|e_{k+1}| \leq \frac{|e_k|^{p+2}}{r_2^{p+1}} \tag{15}$$

Proof First of all, we justify the existence of value r_1 due to the fact that h_1 is continuous in the interval $]0, r_0[$, with $h_1(0) = 2m^2(m-1) > 0$ and $h_1(r_0) \to -\infty$. Analogously h_2 is continuous function in the interval $]0, r_0[$ with $h_2(0) > 0$ and $h_2(r_0) < 0$. Then, there exists at least a positive root in this interval and we take r_2 be the smallest one.

For $n = 0$, Osada's iteration is written as:

$$x_1 = x_0 - \frac{1}{2}m(m+1)\frac{f(x_0)}{f'(x_0)} + \frac{1}{2}(m-1)^2\frac{f'(x_0)}{f''(x_0)}. \tag{16}$$

Then, assuming that x^* is a zero of multiplicity m of function f, we have, by Lemma 5, that

$$f(x) = g(x)(x-x^*)^m, \tag{17}$$

where $g(x)$ has been defined in Lemma 4. By taking $x_0 \in I$ and by using (7), (8) and (9), we have

$$\begin{aligned} f(x_0) &= g(x_0)e_0^m, \\ f'(x_0) &= p(x_0)e_0^m + g(x_0)me_0^{m-1}, \\ f''(x_0) &= 2q(x_0)e_0^m + 2p(x_0)me_0^{m-1} + g(x_0)m(m-1)e_0^{m-2}. \end{aligned} \tag{18}$$

Substituting (18) into (16), we get

$$e_1 = \frac{4p(x_0)q(x_0)e_0^4 + p^2(x_0)(m+1)^2e_0^3 + 2g(x_0)q(x_0)m(1-m)e_0^3}{4p(x_0)q(x_0)e_0^3 + 4p^2(x_0)me_0^2 + 2g(x_0)p(x_0)m(3m-1)e_0 + 4mg(x_0)q(x_0)e_0^2 + 2g^2(x_0)m^2(m-1)}.$$

By taking A equal to the numerator divided by e_0 and B the denominator, dividing both by $g(x^*)$ and $g(x_0)$, and denoting the new terms by $\hat{A}$ and $\hat{B}$, we have

$$e_1 = \frac{A}{B} e_0 = \frac{g(x^*)^{-1} g(x_0)^{-1} A}{g(x^*)^{-1} g(x_0)^{-1} B} e_0 = \frac{\hat{A}}{\hat{B}} e_0. \tag{19}$$

Now, in order to bound the quotient $|e_1|$, we obtain an upper bound for the numerator and a lower bound for the denominator using Hölder continuos conditions (2) and (3) and the bounds obtained in Lemma 6 as follows:

$$\begin{aligned}
|\hat{A}| &= \left| g(x^*)^{-1} g(x_0)^{-1} \left(4p(x_0)q(x_0)e_0^3 + p^2(x_0)(m+1)^2 e_0^2 + 2g(x_0)q(x_0)m(1-m)e_0^2\right)\right| \\
&\leq 4\left| g(x^*)^{-1} p(x_0) g(x_0)^{-1} q(x_0) e_0 \right| e_0^2 + (m+1)^2 \left| g(x^*)^{-1} p(x_0) g(x_0)^{-1} p(x_0) \right| e_0^2 \\
&\quad + 2m(1-m) \left| g(x^*)^{-1} q(x_0) g(x_0)^{-1} g(x_0) e_0 \right| e_0 \\
&\leq 4 \frac{K_m}{m+1} \frac{m+1}{m+1-K_m|e_0|} \frac{K_0 m!}{T_{m,p}} |e_0|^{p+2} + (m+1)^2 \frac{K_m}{m+1} \frac{K_m}{m+1-K_m|e_0|} |e_0|^2 \\
&\quad + 2m(m-1) \frac{K_0 m!}{T_{m,p}} |e_0|^{p+1}.
\end{aligned}$$

Then, we have obtained $|\hat{A}| \leq \varphi(|e_0|)$ with

$$\varphi(t) = \frac{4K_m K_0 m!}{T_{m,p}(m+1-K_m t)} t^{p+2} + \frac{(m+1)K_m^2}{m+1-K_m t} t^2 + \frac{2m(m-1)K_0 m!}{T_{m,p}} t^{p+1},$$

an increasing function.

Now we bound the denominator by using the bounds obtained in Lemma 6 and the property $|a+b| \geq |a| - |b|$,

$$\begin{aligned}
|\hat{B}| &= |g(x^*)^{-1} g(x_0)^{-1} (4p(x_0)q(x_0)e_0^3 + 4p^2(x_0) m e_0^2 + 2g(x_0)p(x_0)m(3m-1)e_0 \\
&\quad + 4mg(x_0)q(x_0)e_0^2 + 2g^2(x_0)m^2(m-1))| \\
&= |(4g(x^*)^{-1} p(x_0) g(x_0)^{-1} q(x_0) e_0 e_0^2 + 4mg(x^*)^{-1} p(x_0) g(x_0)^{-1} p(x_0) e_0^2 \\
&\quad + 2m(3m-1) g(x^*)^{-1} p(x_0) g(x_0)^{-1} g(x_0) e_0 + 4mg(x^*)^{-1} q(x_0) g(x_0)^{-1} g(x_0) e_0^2 \\
&\quad + 2m^2(m-1) g(x^*)^{-1} g(x_0) g(x_0)^{-1} g(x_0))| \\
&= |(2m^2(m-1) g(x^*)^{-1} g(x_0) + 2m(3m-1) g(x^*)^{-1} p(x_0) e_0 + 4mg(x^*)^{-1} q(x_0) e_0^2 \\
&\quad + 4mg(x^*)^{-1} p(x_0) g(x_0)^{-1} p(x_0) e_0^2 + 4g(x^*)^{-1} p(x_0) g(x_0)^{-1} q(x_0) e_0 e_0^2)| \\
&\geq 2m^2(m-1) - 2m^2(m-1) \left| g(x^*)^{-1} (g(x^*) - g(x_0)) \right| \\
&\quad - 2m(3m-1) \left| g(x^*)^{-1} p(x_0) \right| e_0 - 4m \left| g(x^*)^{-1} q(x_0) e_0 \right| e_0 \\
&\quad - 4m \left| g(x^*)^{-1} p(x_0) g(x_0)^{-1} p(x_0) \right| e_0^2 - 4 \left| g(x^*)^{-1} p(x_0) g(x_0)^{-1} q(x_0) e_0 \right| e_0^2 \\
&\geq 2m^2(m-1) - 2m^2(m-1) \frac{K_m}{m+1} |e_0| - 2m(3m-1) \frac{K_m}{m+1} |e_0| - 4m \frac{K_0 m!}{T_{m,p}} |e_0|^{p+1} \\
&\quad - 4m \frac{K_m}{m+1} \frac{K_m}{m+1-K_m|e_0|} |e_0|^2 - 4 \frac{K_m}{m+1} \frac{m+1}{m+1-K_m|e_0|} \frac{K_0 m!}{T_{m,p}} |e_0|^{p+2} \\
&\geq 2m^2(m-1) - \frac{2m(m^2+2m-1)K_m}{m+1} |e_0| - \frac{4mK_0 m!}{T_{m,p}} |e_0|^{p+1} \\
&\quad - \frac{4mK_m^2}{(m+1)(m+1-K_m|e_0|)} |e_0|^2 - \frac{4K_m K_0 m!}{T_{m,p}(m+1-K_m|e_0|)} |e_0|^{p+2} = h_1(|e_0|).
\end{aligned}$$

Then, by using that function h_1 defined by (14) is decreasing in $]0, r_0[$ and that $|e_0| < r_0$, we have $|\hat{B}| \geq h_1(|e_0|) > h_1(r_1) = 0$. Thus by using the definition $r = min\{r_0, r_1, r_2\}$, and noting that φ and h_1 are respectively increasing and decreasing functions, from (19) we get

$$|e_1| = |\frac{\hat{A}}{\hat{B}} e_0| \leq \frac{\varphi(|e_0|)}{h_1(|e_0|)}|e_0| < \frac{\varphi(r)}{h_1(r)}|e_0| \leq \frac{\varphi(r_2)}{h_1(r_2)}|e_0| = |e_0|.$$

So, we conclude that $x_1 \in]x^* - r, x^* + r[= I$. Now the same process holds starting from x_1 and getting x_2 and, by utilizing an inductive procedure, one has $x_k \in]x^* - r, x^* + r[= I$ for all $k > 0$ as follows:

$$|e_{k+1}| \leq \frac{\varphi(r)}{h_1(r)}|e_k| \leq \left(\frac{\varphi(r)}{h_1(r)}\right)^2 |e_{k-1}| \leq \cdots \leq \left(\frac{\varphi(r)}{h_1(r)}\right)^{k+1} |e_0|. \tag{20}$$

Then, by taking limits in the last expression and using that $\lim_{k\to+\infty} \left(\frac{\varphi(r)}{h_1(r)}\right)^{k+1} = 0$, we get that $\lim_{k\to+\infty} x_k, = x^*$ and so we conclude the convergence proof.

Moreover, we can obtain the rate of convergence turning to (20) and using the definitions of functions φ and h_1, as follows:

$$|e_1| \leq |e_0| \frac{\frac{4K_m K_0 m!}{T_{m,p}(m+1-K_m\ |e_0|)}\ |e_0|^{p+2} + \frac{2m(1-m)K_0 m!}{T_{m,p}}|e_0|^{p+1} + \frac{(m+1)K_m^2}{(m+1-K_m\ |e_0|)}|e_0|^2}{h_1(e_0)}.$$

Now, we multiply and divide by $|e_0|^{p+1}$ obtaining

$$|e_1| \leq |e_0|^{p+2} \frac{\frac{4K_m K_0 m!}{T_{m,p}(m+1-K_m\ |e_0|)}\ |e_0| + \frac{2m(1-m)K_0 m!}{T_{m,p}} + \frac{(m+1)K_m^2}{(m+1-K_m\ |e_0|)}|e_0|^{1-p}}{h_1(e_0)}.$$

Observe that the function obtained in the numerator is increasing in I_0, so that by the definition of r_2 one gets:

$$\begin{aligned}|e_1| &\leq \frac{|e_0|^{p+2}}{r_2^{p+1}} \frac{\left(\frac{4K_m K_0 m!}{T_{m,p}(m+1-K_m r_2)}\ r_2 + \frac{2m(1-m)K_0 m!}{T_{m,p}} + \frac{(m+1)K_m^2}{(m+1-K_m r_2)} r_2^{1-p}\right) r_2^{p+1}}{h_1(r_2)} \\ &= \frac{|e_0|^{p+2}}{r_2^{p+1}}.\end{aligned}$$

An by an induction procedure we have $|e_{k+1}| \leq \dfrac{|e_k|^{2+p}}{r_2^{p+1}}$, that is, the sequence x_k converges to x^* with order at least $p+2$.

To show the uniqueness, we assume that there exists a second solution $y^* \in]x^* - r_0, x^* + r_0[$. By (17) we have

$$f(y^*) = g(y^*)(y^* - x^*)^m = 0. \tag{21}$$

By using the mean value theorem, there exists φ in I such that

$$\left|1 - g(x^*)^{-1} g(y^*)\right| = \left|g(x^*)^{-1}(g(x^*) - g(y^*))\right| = \left|g(x^*)^{-1} g'(\varphi)|e_0|\right| \leq \frac{K_m|x^* - y^*|}{m+1} < 1,$$

so, we deduce that $g(y^*) \neq 0$ and then, by (21), we have that $y^* = x^*$.

4. Numerical examples

In this section, we compare the local convergence radius r_H of Halley's methods considered in [3] with those obtained in this paper, r_0, r_1 and r_2.
For that, we give some examples in order to apply the theoretical results obtained and then we can correct the results of [4]. The examples are taken from [3] and [4] to show the comparison of our results with theirs.

4.1. *Example 1*

Let $D = (-\frac{\pi}{2}, \frac{\pi}{2})$, and define function f on D by

$$f(x) = cos(x) - 1$$

Obviously $x^* = 0$ is a root of multiplicity m=2. Then, the values for the constants defined in (2) and (3) are: $K_m = 1$ and $K_0 = 1$ with $p = 1$. Using [3], one gets $r_H \approx 1.2679$ and by using the results obtained in Theorem 1, we get $r_0 \approx 3.0000$, $r_1 \approx 0.7418$, $r_2 = r \approx 0.6997$.

4.2. *Example 2*

Let $D = \mathbb{R}$, and define function f on D by

$$f(x) = x^2(x^2 - 1).$$

We have that $x^* = 0$ is a zero of f with multiplicity $m = 2$. Then by taking $p = 1$, $K_m = 12$ and $K_0 = 12$, and using [3], one gets $r_H \approx 0.1152$. By using the results obtained in Theorem 1, we get $r_0 \approx 0.2500$, $r_1 \approx 0.0646$, $r_2 = r \approx 0.0598$.

4.3. *Example 3*

Let $D = \mathbb{R}$, and define a function f on D by

$$f(x) = \int_0^x G(x)dx,$$

where

$$G(x) = \int_0^x (x + cos(\pi x^2))dx,$$

We have that $x^* = 0$ is a zero of f with multiplicity $m = 2$. So, we have $m = 2$, $p = 1$, $K_m = 1+2\pi$ and $K_0 = 2\pi$. Using [3], one gets $r_H \approx 0.1892$ and using the results obtained in Theorem 1 we get $r_0 \approx 0.4119$, $r_1 \approx 0.1063$, $r_2 = r \approx 0.0984$.

4.4. *Example 4*

Let $D = \mathbb{R}$, and define a function f on D by

$$f(x) = x + cos(x) - \frac{\pi}{2}.$$

We have that $x^* = \frac{\pi}{2}$ is a zero of f with multiplicity $m = 3$. Then we have $p = 1$, $K_m = 1$ and $K_0 = 1$. Using [3], one gets $r_H \approx 1.9720$ and using the results obtained in Theorem 1 we get $r_0 \approx 4$, $r_1 \approx 1.5064$, $r_2 = r \approx 1.4283$.

4.5. *Example 5*

Let $D = \mathbb{R}$, and define a function f on D by

$$f(x) = x^2 e^x - sin(x) + x.$$

We have that $x^* = 1$ is a zero of f with multiplicity $m = 2$. Then we have $p = 1$, $K_m = 3.9$ and $K_0 = 5.9$. Using [3], one gets $r_H \approx 0.3439$ and using the results obtained in Theorem 1 we get $r_0 \approx 0.7692$, $r_1 \approx 0.1958$, $r_2 \approx 0.1824$.

4.6. *Example 6*

Let $D = \mathbb{R}$, and define a function f on D by

$$f(x) = x^5 - 8x^4 + 24x^3 - 34x^2 + 23x - 6.$$

We have that $x^* = 1$ is a zero of f with multiplicity $m = 3$. Then we have $p = 1$, $K_m = 4$ and $K_0 = 10$. Using [3], one gets $r_H \approx 0.5091$ and using the results obtained in Theorem 1 we get $r_0 \approx 1$, $r_1 \approx 0.3822$, $r_2 = r \approx 0.3584$.

As conclusion of these numerical test we obtain the radius of local convergence intervals for some examples by using the third order Osada's iterative method correcting the ones obtained in [4].

5. Dynamics of Osada's method

The behavior of iterative methods has been examined from a global point of view by using ideas of dynamical systems, see for example [10,11,12,13,14,15,16,17]. Complex dynamics is the most used tool for the study of the global iterative methods, no only because of the good properties of the analytic functions in the complex domain, but also because they provide good pictorial representations in two dimensions.

Let us recall some basic concepts of discrete dynamics, in order to fix the notation (see [14]. Consider a function $G : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$. The set of successive images of a point z_0 by G: $z_0, G(z_0), G(G(z_0)), \ldots$, is the orbit of z_0. A point $z_0 \in \widehat{\mathbb{C}}$ is called a fixed point of G, if $G(z_0) = z_0$. A fixed point z_0 is attracting if the orbits of all the points in a neighborhood of z_0 tend to z_0. Moreover, z_0 is called a periodic point of period $p > 1$ if it is a point such that $G^p(z_0) = z_0$ but $G^k(z_0) \neq z_0$, for each $k < p$. Moreover, a point z_0 is called pre-periodic if it is not periodic but there exists a $k > 0$ such that $G^k(z_0)$ is periodic. There exist different types of fixed points depending on its associated multiplier $|G'(z_0)|$. Taking into account the associated multiplier, a fixed point z_0 is called:
– superattractor if $|G'(z_0)| = 0$,
– attractor if $|G'(z_0)| < 1$,
– parabolic if $|G'(z_0)| = 1$ and

– repulsor if $|G'(z_0)| > 1$.
On the other hand, a critical point z_0 is a point such that $G'(z_0) = 0$.

The basin of attraction of an attracting point α is formed by the points whose orbit tends to α. The Fatou set of the rational function G is the set of points $z \in \widehat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\widehat{\mathbb{C}}$ is the Julia set. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

Here we consider $G(z)$ the iteration function obtained by applying Osada's method to a polynomial $p(z)$. The fixed points that do not correspond to the roots of the polynomial $p(z)$ are called strange fixed points. Moreover, we call free critical point those critical points which are no roots of the polynomial $p(z)$.

The basins of attraction of the different fixed points of G are graphically represented by coloring each basin in a different color, forming the so called dynamical plane. If G is the iteration function of a numerical method for solving equations, the attraction basins of G give an idea of the behavior of the method and its sensitivity to the initial guess. Moreover, convergence speed is represented by using darker colors as the number of iterations required to converge starting from a given point increases.

The Osada's method is intended for fast convergence to roots of multiplicity m, and its iteration function is are given by

$$G_f(z) = u - \frac{m(m+1)}{2}\frac{f(z)}{f'(z)} + \frac{(m-1)^2}{2}\frac{f(z)}{f''(z)}. \tag{22}$$

Let us consider a polynomial with d different roots of multiplicity m each,

$$f(z) = (z^d - 1)^m, \tag{23}$$

with $d, m \geq 2$. The iteration function of Osada's method (22) applied to a polynomial of the form (23) is

$$G_f(z) = \frac{(m+1)\left(z^d-1\right)^2 + 2d^2z^d\left(mz^d-1\right) - d\left(z^d-1\right)\left((3m+1)\,z^d - m - 1\right)}{2dz^{d-1}\left(1-d+(dm-1)\,z^d\right)}$$

The fixed points, $G_f(z) = z$, are the solutions of

$$\frac{\left(z^d-1\right)\left((3md-m-d-1)z^d-(m+1)(d-1)\right)}{2dz^{d-1}\left(1-d+(dm-1)z^d\right)} = 0.$$

Thus, besides the roots of the polynomial, the d-th roots of 1, G_f has another d fixed points, called strange fixed points, the d-th roots of

$$\frac{(m+1)(d-1)}{3md-m-d-1}.$$

The character of the fixed points depends on the value of the derivative of the iteration function,

$$G_f'(z) = \frac{(d-1)\left(z^d-1\right)^2\left(\left(m^2d(2d-1)-3md+m+1\right)z^d-(m+1)(d-1)^2\right)}{2dz^{-d}\left(1-z^d+d\left(-1+mz^d\right)\right)^2}.$$

Hence, the roots z of the polynomial are superattracting fixed points, because $G_f'(z) = 0$, whereas the d strange fixed points are repelling because

$$G_f'\left(\left(\frac{(m+1)(d-1)}{3md-m-d-1}\right)^{\frac{1}{d}}\right) = \frac{4md-2m-2d}{(m-1)(d-1)}$$
$$= 2 + \frac{2md-2}{(m-1)(d-1)} > 1,$$

for $m \geq 2, d \geq 2$.

Working in $\widehat{\mathbb{C}}$, infinity is also a fixed point. By using the conjugation $w = \frac{1}{z}$, we find that the derivative of $F(w) = 1/G_f(\frac{1}{w})$ at $w = 0$,

$$F'(0) = 1 + \frac{2(md-1)+(m-1)(d-1)}{(2m(d-1)+m-1)(d-1)}$$

is greater than 1 for $m \geq 2, d \geq 2$, so that the point at infinity is repelling.

The connectedness of Julia set is related to the number of repelling fixed points of G_f (see [18]). The dynamical planes in the figures show that it is not connected, so that there are at least two repelling fixed points. In fact, we have shown that G_f has d repelling fixed points.

The figures corresponding to polynomials with d roots of multiplicity m are quite similar to the dynamical planes of Newton's method for polynomials with d simple roots, which is very remarkable for a method of higher order and multiple roots.

In summary, Osada's method has a good dynamical behavior for the considered polynomials, because the roots are superattracting fixed points of the iteration function, and its other fixed points, including infinity, are repelling.

6. Conclusions

In this work we give a note on the already published paper: " Convergence radius of Osada's method under center-Hölder continuous condition", Applied Mathematics and Computation 243 (2014) 809-816. We obtain a new value for the convergence local radius correcting the one obtained in this publication. We complete the paper performing a dynamical study of this iterative method.

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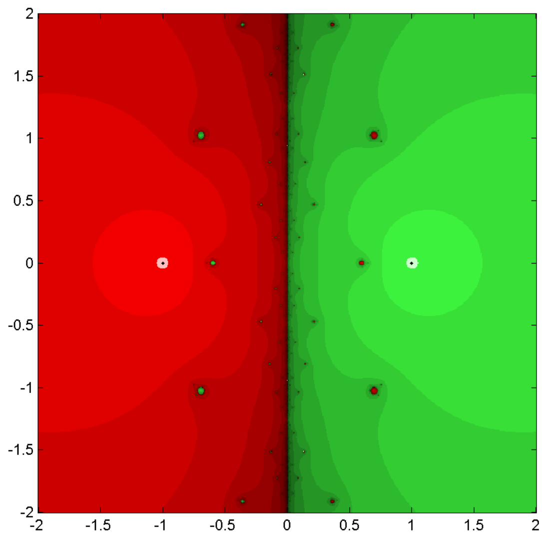

Figure 1. Dynamical plane of G_f for $d = 2, m = 2$. Each basin contains small inclusions of the other.

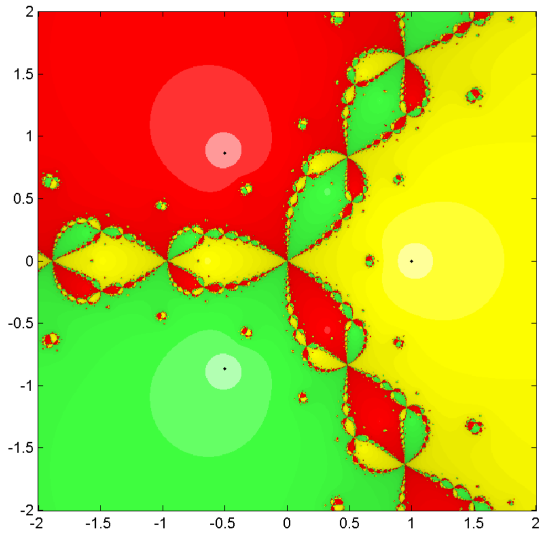

Figure 2. Dynamical plane of G_f for $d = 3, m = 3$. The inclusions in one basin contain points of the other basins.

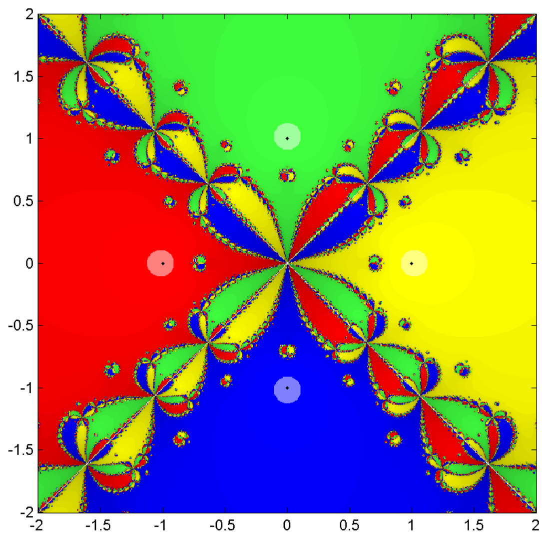

Figure 3. Dynamical plane of G_f for $d = 4, m = 4$.

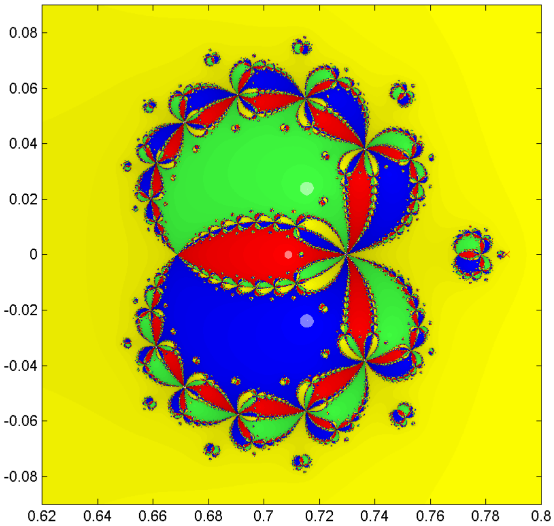

Figure 4. Detail of the dynamical plane of G_f for $d = 4, m = 4$ near the strange fixed point $z = 0.78$, marked with a red x.