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Bounds on the k-restricted arc connectivity of some bipartite tournaments

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A B S T R A C T

For $k \geq 2$, a strongly connected digraph D is called λ'_k-connected if it contains a set of arcs W such that $D - W$ contains at least k non-trivial strong components. The *k-restricted arc connectivity* of a digraph D was defined by Volkmann as $\lambda'_k(D) = \min\{|W| : W$ is a k-restricted arc-cut$\}$. In this paper we bound $\lambda'_k(T)$ for a family of bipartite tournaments T called projective bipartite tournaments. We also introduce a family of "good" bipartite oriented digraphs. For a good bipartite tournament T we prove that if the minimum degree of T is at least $1.5k - 1$ then $k(k-1) \leq \lambda'_k(T) \leq k(N - 2k - 2)$, where N is the order of the tournament. As a consequence, we derive better bounds for circulant bipartite tournaments.

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1. Introduction

Through this work only finite digraphs without loops and multiple arcs are considered. For all definitions not given here we refer the reader to the book of Bang-Jensen and Gutin [9]. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. A vertex u is adjacent to a vertex v if $(u, v) \in A(D)$. The *out-neighborhood* of a vertex u is $N^+(u) = \{v \in V(D) : (u, v) \in A(D)\}$ and the *in-neighborhood* of a vertex u is $N^-(u) = \{v \in V(D) : (v, u) \in A(D)\}$. The *out-degree* is $d^+(v) = |N^+(v)|$ and the *in-degree* $d^-(v) = |N^-(v)|$. We denote by $\delta^+(D)$ the minimum out-degree of the vertices in D, and by $\delta^-(D)$ the minimum

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in-degree of the vertices in D. The minimum degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. Given a vertex subset $X \subset V(D)$, the induced subdigraph of D by X is denoted by $D[X]$. Given two vertex subsets $X, Y \subset V(D)$, we denote by (X, Y) the set of arcs from X to Y.

In a digraph D a vertex v is *reachable* from a vertex u if D has an (u, v)-path. A digraph D is *strongly connected* or *strong* if, for every pair u, v of distinct vertices in D there exists an (u, v)-path and a (v, u)-path. Clearly, a strong digraph D has both $\delta^+(D) \geq 1$ and $\delta^-(D) \geq 1$, that is, $\delta(D) \geq 1$. For a strong digraph D, a set of arcs $W \subseteq A(D)$ is an *arc-cut* if $D - W$ is not strong. A *strong component* of a digraph is a maximal strong induced subdigraph. A digraph D is said to be *k-arc-connected* if D has no arc-cut with less than k arcs. A parameter that can measure the fault tolerance of a network modeled by a digraph D is the classical arc-connectivity $\lambda(D) := \lambda$ of D. The *arc connectivity*λ of a digraph D is the largest integer k such that D is k-arc-connected. If D is a non-strong digraph, we set $\lambda = 0$. Note that $\lambda \geq k$ if and only if $|(X, V(D)\setminus X)| \geq k$ for all proper subsets X of $V(D)$. The arc-connectivity is an important measure for the fault tolerance of a network. However, one might be interested in more refined indices of reliability. Even two digraphs with the same arc-connectivity λ may be considered to have different reliabilities, since the number or type of minimum arc-cuts is different or simply because the existence of some additional structural properties is required. From here arises the notion of restricted arc-connectivity λ' defined by Volkmann [24] as follows. For a strongly connected digraph D the restricted arc-connectivity λ' is defined as the minimum cardinality of an arc-cut over all arc-cuts W satisfying that $D - W$ contains a non trivial strong component D_1 such that $D - V(D_1)$ has an arc. Some results for λ' can be seen in [4,5,13,24,25].

Let $k \geq 2$ be an integer. In the same paper [24] Volkmann also introduced the k-restricted arc-connectivity of a digraph D, λ'_k, as follows. An arc set W of D is a *k-restricted arc-cut* if $D - W$ contains at least k non trivial strong components. The *k-restricted arc connectivity* of D is

$$\lambda'_k(D) = \min\{|W| : W \text{ is a } k\text{-restricted arc-cut}\}.$$

A strong digraph D is said to be *λ'_k-connected* if $\lambda'_k(D)$ exists. k-restricted edge connectivity has been used by many author in graphs, sometimes it is also called extra-connectivity [3,15]. This concept was also introduced for (undirected) graphs independently by Chartrand et al. [12], Sampathkumar [21] and Oellerman [20] as k-connectivity. Recently this parameter has been studied under the name of k-component edge connectivity [22].

Volkmann [24] gives a characterization of the λ'_k-connected digraphs.

Proposition 1.1. [24] *Let $k \geq 2$ be an integer. A strongly connected digraph D is λ'_k-connected if and only if D contains at least k pairwise vertex disjoint cycles.*

Meierling et al. [19] characterize the λ'_2-connected local tournaments and tournaments. They proved that the recognition problem of deciding if a strongly connected local tournament or tournament with n vertices and m arcs is λ'_2-connected can be solved in polynomial time. Whereas the problem of deciding if $\lambda'_k(D)$ exists for a strong digraph D when $k \geq 3$ is **NP**-complete.

Furthermore, Proposition 1.1 states that the number of disjoint cycles in a strong digraph is equal to the maximum k for which the digraph is λ'_k-connected. Therefore, it is important to know the maximum number of disjoint cycles in a digraph. Bermond and Thomassen [11] established the following conjecture, which relates the number of disjoint cycles in a digraph with the minimum out-degree.

Conjecture 1.1. [11] *Every digraph D with $\delta^+(D) \geq 2k - 1$ has k disjoint cycles.*

This conjecture has been proved for general digraphs by Thomassen [23] when $k = 2$, and by Lichiardopol et al. [18] when $k = 3$. In 2010, Bessy et al. [10] proved Conjecture 1.1 for regular tournaments. In 2014, Bang-Jensen et al. [10] proved it for tournaments. Thomassen [23] also established the existence of a finite integer $f(k)$ such that every digraph of minimum out-degree at least $f(k)$ contains k disjoint cycles. Alon [1] proved in 1996 that for every integer k, the value $64k$ is suitable for $f(k)$.

A bipartite tournament is an oriented complete bipartite graph. Hence, the girth of any non acyclic bipartite tournament is four. Very recently, Bai et al. [2], proved Conjecture 1.1 for bipartite tournaments as a consequence of another result related to the numbers of vertex disjoint cycles of a given length in bipartite tournaments with minimum out-degree at least $qr - 1$, for $q \geq 2$ and $r \geq 1$ two integers. In [6] it was proved that every bipartite tournament with minimum out-degree at least $2k - 2$ and minimum in-degree at least one contains k disjoint 4-cycles whenever $k \geq 3$. Moreover, it was shown that every bipartite tournament with both minimum out-degree and minimum in-degree at least $1.5k - 1$ contains at least k disjoint cycles. As an immediate consequence of Proposition 1.1 and this last result we can write the following result.

Corollary 1.1. *Let $k \geq 2$ be an integer. A strongly connected bipartite tournament with minimum degree $\delta \geq 1.5k - 1$ is λ'_k-connected.*

In this paper we give bounds on the k-restricted arc-connectivity in some families of bipartite tournaments. This paper is organized as follows. In the next section we give an upper bound on λ'_k of the projective bipartite tournaments introduced in [7]. In the last section we introduce a family of oriented bipartite digraphs called *good*. The main theorem concerns with good bipartite tournaments. For this family we prove that if the minimum degree is at least $1.5k - 1$, then $k(k-1) \leq \lambda'_k \leq k(N - 2k - 2)$, where N is the order of the tournament. We also prove that complete p-cycles and certain circulant bipartite tournaments are good and removing the hypothesis on the minimum degree we are able to obtain the same lower bound.

2. Projective bipartite tournament

In [7] a family of bipartite tournaments based on projective planes was introduced. A *projective plane* $(P, \mathcal{L})$ consists of a finite set P of elements called *points*, and a finite family $\mathcal{L}$ of subsets of P called *lines* which satisfy the following conditions:

(i) Any two lines intersect at a single point.
(ii) Any two points belongs to a single line.
(iii) There are four points of which no three belong to the same line.

It can be shown that for every projective plane, there is an integer $n \geq 2$ such that every line has exactly $n+1$ points and every point is incident with exactly $n+1$ lines. Hence, the projective plane $(P, \mathcal{L})$ is said to have order n. Moreover, observe that $|P| = |\mathcal{L}| = n^2 + n + 1$.

Definition 2.1. [7] Let $\Pi = (P, \mathcal{L})$ be a projective plane of order k. The projective bipartite tournament $D_k(\Pi)$ of order k with partite sets P and $\mathcal{L}$ is defined as follows: For all $p \in P$ and for all $L \in \mathcal{L}$,

$$p \in N^+(L) \text{ iff } p \text{ belongs to } L; \; L \in N^+(p) \text{ iff } p \text{ does not belong to } L.$$

Remark 2.1. Let $D_k(\Pi)$ be a projective bipartite tournament of order $k \geq 2$. Then $D_k(\Pi)$ has $n = 2(k^2 + k + 1)$ vertices, every vertex $p \in P$ has $d^+(p) = k+1$, $d^-(p) = k^2$, and every $L \in \mathcal{L}$ has $d^+(L) = k^2$, $d^-(L) = k+1$. Moreover, the diameter $Diam(D_k(\Pi)) = 3$ which implies that the edge connectivity is maximum, i.e., $\lambda(D_k(\Pi)) = \delta(D_k(\Pi)) = k+1$, see [14].

Based on Corollary 1.1 and the above remark, we can write the following result.

Corollary 2.1. *A projective bipartite tournament $D_k(\Pi)$ of order $k \geq 2$ is λ'_t-connected with $t \leq \lfloor 2(k+2)/3 \rfloor$.*

In the following theorem we improve the above corollary and we find an upper bound on the t-restricted-arc-connectivity for projective bipartite tournaments.

Theorem 2.1. *If $D_k(\Pi)$ is the projective bipartite tournament of order $k \geq 2$ having n vertices, then $D_k(\Pi)$ is $\lambda'_{(n-2)/4}$-connected, and*

$$\lambda'_{(n-2)/4}(D_k(\Pi)) \leq (3n-10)(n-2)/16.$$

Proof. Let $D_k(\Pi)$ be the projective bipartite tournament of order k. By Remark 2.1, $D_k(\Pi)$ is strong. In order to show that $D_k(\Pi)$ is λ'_α-connected, by Proposition 1.1, it is sufficient to prove that $D_k(\Pi)$ has $\alpha = \frac{k^2+k}{2} = (n-2)/4$ disjoint cycles of length four.

Observe that two points $p_1, p_2 \in \mathcal{P}$ and two lines $l_1, l_2 \in \mathcal{L}$ induce a 4-cycle (p_1, l_1, p_2, l_2) in $D_k(\Pi)$ if $p_1 \in l_1$, $p_2 \in l_2$, $p_1 \notin l_2$ and $p_2 \notin l_1$.

Let $p \in \mathcal{P}$ and $l \in \mathcal{L}$ be such that $p \notin l$. Let $p_1, p_2, \ldots, p_{k+1}$ be the points of l and let l_i be the line through p and p_i for $i = 1, 2, \ldots, k+1$. Let p_i^j, $j = 1, 2, \ldots, k$, be the k distinct points in l_i others than p, where $p_i = p_i^k$ for all $1 \leq i \leq k+1$. Also denote by $[a, b]$ the line through the points a and b,

Case 1. $k+1$ is odd.

Since $p \notin l$, $(p_{2i-1}^k, l_{2i-1}, p_{2i}^k, l_{2i})$ for $i = 1, 2, \ldots, k/2$, are $k/2$ disjoint 4-cycles in $D_k(\Pi)$.

Consider the line l_1 and note that $p_{k+1}^k \notin l_1$, and put $p = p_1^0$. Then

$$(p_1^{2i}, [p_1^{2i}, p_{k+1}^k], p_1^{2i+1}, [p_1^{2i+1}, p_{k+1}^k]) \text{ for } i = 0, 1, \ldots, k/2-1,$$

are $k/2$ disjoint 4-cycles in $D_k(\Pi)$ and also disjoint with the $k/2$ above. Similarly, note that $p_1^k \notin l_{k+1}$. Then

$$(p_{k+1}^{2i-1}, [p_{k+1}^{2i-1}, p_1^k], p_{k+1}^{2i}, [p_{k+1}^{2i}, p_1^k]) \text{ for } i = 1, 2, \ldots, k/2,$$

are $k/2$ disjoint 4-cycles in $D_k(\Pi)$ and also disjoint with the k above. Suppose $k \geq 4$. In this case we can take $p_i^{k-1} \in l_i$ with $i = 2, \ldots, k-1$, such that they are on the same line b and $p_k^{k-1} \notin b$. Hence

$$(p_{2i}^{k-1}, [p_{2i}^{k-1}, p_k^{k-1}], p_{2i+1}^{k-1}, [p_{2i+1}^{k-1}, p_k^{k-1}]) \text{ for } i = 1, 2, \ldots, k/2-1,$$

are $k/2-1$ disjoint 4-cycles in $D_k(\Pi)$ and also disjoint with the $3k/2$ above.

Finally, observe that $p_1^j \notin l_{j+1}$, $j = 1, \ldots, k-1$. Thus,

$$(p_{j+1}^{2i-1}, [p_{j+1}^{2i-1}, p_1^j], p_{j+1}^{2i}, [p_{j+1}^{2i}, p_1^j]) \text{ for } i = 1, 2, \ldots, k/2-1,$$

are $(k/2-1)(k-1)$ disjoint 4-cycles in $D_k(\Pi)$ and also disjoint with the $2k-1$ above. Therefore, the number of disjoint 4-cycles in $D_k(\Pi)$ is at least

$$(k/2-1)(k-1) + 2k - 1 = \frac{k}{2} + \frac{k^2}{2} = \alpha.$$

Case 2. $k+1$ is even.

As in the above case, since $p \notin l$, $(p^k_{2i-1}, l_{2i-1}, p^k_{2i}, l_{2i})$ for $i = 1, 2, \dots, (k+1)/2$, are $(k+1)/2$ disjoint 4-cycles in $D_k(\Pi)$. Consider the line l_1 and note that $p^k_{k+1} \notin l_1$. Then

$$(p_1^{2i}, [p_1^{2i}, p^k_{k+1}], p_1^{2i+1}, [p_1^{2i+1}, p^k_{k+1}]) \text{ for } i = 1, 2, \dots, (k-1)/2,$$

are $(k-1)/2$ disjoint 4-cycles in $D_k(\Pi)$ and also disjoint with the $(k+1)/2$ above.

Finally, observe that $p_1^j \notin l_{j+1}$, $j = 1, \dots, k$. Thus,

$$(p^{2i-1}_{j+1}, [p^{2i-1}_{j+1}, p^j_1], p^{2i}_{j+1}, [p^{2i}_{j+1}, p^j_1]) \text{ for } i = 1, 2, \dots, (k-1)/2,$$

are $k(k-1)/2$ disjoint 4-cycles in $D_k(\Pi)$ and also disjoint with the k above. Therefore, the number of disjoint 4-cycles in $D_k(\Pi)$ is at least

$$k\frac{k-1}{2} + k = \frac{k}{2} + \frac{k^2}{2} = \alpha.$$

In order to prove the upper bound on λ'_k, we count the number of arcs out-coming or in-coming from a 4-cycle in $D_k(\Pi)$. Let $C_0 = (p, l, p', l')$ be a 4-cycle. Since $d^+(p) = d^+(p') = d^-(l) = d^-(l') = k+1$ and $d^-(p) = d^-(p') = d^+(l) = d^+(l') = k^2$, it follows that the minimum number of arcs needed to disconnect C_0 from $T - V(C_0)$ is at least $2(k^2 + k - 1)$. Let $D_1 = D_k(\Pi) - V(C_0)$, and let C_1 be a 4-cycle in D_1. The minimum number of arcs needed to disconnect C_1 from $D_1 - V(C_1)$ is at least $2(k^2 + k - 1) - 2$, because $|V(C_0) \cap N^-(C_1)| \geq 2$ or $|V(C_0) \cap N^+(C_1)| \geq 2$ (note that if $|V(C_0) \cap N^-(C_1)| \leq 1$, then $|V(C_0) \cap N^+(C_1)| \geq 2$, because $D_k(\Pi)$ is a bipartite tournament). Let $D_2 = D_1 - V(C_1)$, and let C_2 be a 4-cycle in D_2. The minimum number of arcs needed to disconnect C_2 is at least $2(k^2 + k - 1) - 4$, because either $|(V(C_0) \cup V(C_1)) \cap N^-(C_2)| \geq 4$ or $|(V(C_0) \cup V(C_1)) \cap N^+(C_2)| \geq 4$. If $D_{\alpha-1}$ is the digraph obtained after removing $\alpha - 1$ disjoint 4-cycles, then the minimum number of arcs needed to disconnect a 4-cycle C_α is at least $2(k^2 + k - 1) - 2(\alpha - 1)$, because either $|\cup_{i=0}^{\alpha-2} V(C_i) \cap N^-(C_{\alpha-1})| \geq 2(\alpha - 1)$ or $|\cup_{i=0}^{\alpha-2} V(C_i) \cap N^+(C_{\alpha-1})| \geq 2(\alpha - 1)$. Hence, the minimum order to disconnect α disjoint 4-cycles is

$$\begin{aligned}\sum_{i=1}^{\alpha}(2(k^2 + k - 1) - 2(i-1)) &= 2\alpha(k^2 + k - 1) - \alpha(\alpha - 1)\\ &= \alpha\frac{3k^2 + 3k - 2}{2}\\ &= 3\alpha^2 - \alpha.\end{aligned}$$

Therefore, the theorem holds. □

3. Good oriented bipartite digraphs

Let D be an oriented bipartite digraph with $\delta^+(D) \geq 1$. Let f: $V(D) \to V(D)$ be a function such that $f(x) \in N^+(x)$. Let us denote by $x_f^+ = N^+(x) \cup N^+(f(x))$, and $x_f^- = N^-(x) \cup N^-(f(x))$. Note that $x \in x_f^-$, $f(x) \in x_f^+$ and $x_f^+ \cap x_f^- = \emptyset$ because D is oriented and bipartite.

Definition 3.1. Let D be an oriented bipartite digraph with $\delta^+(D) \geq 1$ and let f: $V(D) \to V(D)$ be a function such that $f(x) \in N^+(x)$. Then D is said to be f-good if the following assertions hold:

1. Let $u, v \in x_f^\epsilon$, with $\epsilon \in \{-, +\}$. If $v \in u_f^+$, then $u_f^+ \cap v_f^- \subset x_f^\epsilon$.
2. Let $u, v, w \in x_f^\epsilon$, with $\epsilon \in \{-, +\}$. If $v \in u_f^+ \cap w_f^-$, then $u_f^- \cap w_f^- \subset v_f^-$ and $u_f^+ \cap w_f^+ \subset v_f^+$.

In general, we say that D is good if D is f-good for some f.

Next we present two distinct families of bipartite oriented digraphs which are good.

Let D be a digraph such that $V(D)$ can be partitioned into $p \geq 2$ parts V_α, $\alpha = 1, 2, \dots, p$, in such a way that the vertices in the partite set V_α are only adjacent to vertices of $V_{\alpha+1}$, where the sum is in $\mathbb{Z}_p$. These digraphs are known as p-cycles, see [17]. In [4] some sufficient conditions for guaranteeing optimal restricted arc-connectivity λ' of p-cycles are proved. Clearly, the girth of a p-cycle is at least p and when p is even D is bipartite. Moreover, if every vertex of V_α is adjacent to every vertex of $V_{\alpha+1}$, then D is known as a *complete p-cycle*.

Proposition 3.1. *Let $p \geq 4$ be an even number and D a complete p-cycle. Then D is a good oriented bipartite digraph.*

Proof. Let $v_{\alpha,j} \in V_\alpha$ with $j = 1, 2, \dots, |V_\alpha|$. Let us consider the function f: $V(D) \to V(D)$ such that $f(v_{\alpha,j}) = v_{\alpha+1,j}$, where j is taken modulo $|V_{\alpha+1}|$.

Therefore for every $x \in V_\alpha$, we have $x_f^+ = V_{\alpha+1} \cup V_{\alpha+2}$ and $x_f^- = V_{\alpha-1} \cup V_\alpha$. Without loss of generality suppose that $\alpha = 1$ and $x \in V_1$. Let us see that both assertions of Definition 3.1 hold.

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Suppose $u, v \in x_f^+ = V_2 \cup V_3$ (for $\epsilon = -$ the proof is analogous) and $v \in u_f^+$. If $u \in V_3$, then $u_f^+ = V_4 \cup V_5$ yielding that $v \in (V_4 \cup V_5) \cap (V_2 \cup V_3) = \emptyset$, which is impossible. Hence, $u \in V_2$ and $u_f^+ = V_3 \cup V_4$ yielding that $v \in (V_3 \cup V_4) \cap (V_2 \cup V_3) = V_3$, implying that $v_f^- = V_2 \cup V_3$. Hence, $u_f^+ \cap v_f^- = V_3 \subset x_f^+$, and assertion 1 of Definition 3.1 holds.

Next, let $u, v, w \in x_f^+$ and $v \in u_f^+ \cap w_f^-$. Reasoning as above we have $u \in V_2$ and $u_f^+ = V_3 \cup V_4$. If $w \in V_2$, then $w_f^- = V_1 \cup V_2$ yielding that $u_f^+ \cap w_f^- = \emptyset$, which is impossible. Therefore, $w \in V_3$ and $w_f^- = V_2 \cup V_3$ implying that $v \in u_f^+ \cap w_f^- = V_3$. We can check that $u_f^- \cap w_f^- = (V_1 \cup V_2) \cap (V_2 \cup V_3) = V_2 \subset v_f^- = V_2 \cup V_3$; and $u_f^+ \cap w_f^+ = (V_3 \cup V_4) \cap (V_4 \cup V_5) = V_4 \subset v_f^+$. Hence, assertion 2 of Definition 3.1 holds. □

Let $t \geq 0$ be an integer number and $B = \overrightarrow{C}_{4n+2t}(1, 3, \ldots, 2n-1)$ be a circulant bipartite digraph in which $V(B) = \mathbb{Z}_{4n+2t}$ and $A(B) = \{ij : j = i + s \text{ with } s = 1, 3, \ldots, 2n-1\}$. Observe that if $t = 0$, then B is a bipartite tournament.

Proposition 3.2. *The circulant digraph $\overrightarrow{C}_{4n+2t}(1, 3, \ldots, 2n-1)$ is a good oriented bipartite digraph.*

Proof. Let $B = \overrightarrow{C}_{4n+2t}(1, 3, \ldots, 2n-1)$. Let us consider the function f: $V(B) \to V(B)$ such that $f(x) = x + 1$ modulo $4n + 2t$. For simplicity we denote $x_f^+ = x^+$ and $x_f^- = x^-$. Moreover, since B is a vertex transitive digraph, we may assume that $x = 0$ for proving both assertions 1 and 2 of Definition 3.1. We also assume that $\epsilon = -$ and the case $\epsilon = +$ can be done in a similar way.

Let $u, v \in 0^- = N^-(0) \cup N^-(1) = \{0, 4n + 2t - 1, \ldots, 2n + 2t + 1\}$. Since $v \in u^+$, $u \neq 0$ because $0^- \cap 0^+ = \emptyset$ and $v \neq u$. Hence, $u = 2n + 2t + j$ with $1 \leq j \leq 2n - 1$ and then

$$u^+ = N^+(u) \cup N^+(u+1) = \{j, j-1, \ldots, 0, 4n + 2t - 1, \ldots, 2n + 2t + j + 1\}.$$

Since $v \in u^+ \cap 0^-$, $v = 2n + 2t + h$ with $j + 1 \leq h \leq 2n$, it follows that

$$v^- = N^-(v) \cup N^-(v+1) = \{2n + 2t + h, 2n + 2t + h - 1, \ldots, 2n + 2t, \ldots, 2t + h + 1\}.$$

Let $i \in u^+ \cap v^-$, then $2n + 2t + j + 1 \leq i \leq 2n + 2t + h$ yielding that $i \in 0^-$ and assertion 1 holds.

Let $u, v, w \in 0^- = \{0, 4n + 2t - 1, \ldots, 2n + 2t + 1\}$, then $w = 2n + 2t + r$ with $0 \leq r \leq 2n$, and u as before. Since $v \in u^+ \cap w^-$, it follows that $v = 2n + 2t + h \in w^-$, yielding that $w = 2n + 2t + r$ with $h < r \leq 2n$ ($h < r$ because $w \neq v$). Therefore we have $1 \leq j < h < r \leq 2n$. Thus, if $x \in u^- \cap w^-$, then $x \in \{2n + 2t + j, 2n + 2t + j - 1, \ldots, r + 2t + 1\} \subset v^-$ giving $u^- \cap w^- \subset v^-$. If $x \in u^+ \cap w^+$, then $x \in \{2n + 2t + r + 1, \ldots, 0, \ldots, 2t + j + 1\} \subset v^+$, implying $u^+ \cap w^+ \in v^+$. Thus assertion 2 of Definition 3.1 also holds. □

The following result is a direct consequence for paths of length two from Definition 3.1.

Corollary 3.1. *Let D be a f-good oriented bipartite digraph and $D[i_f^\epsilon]$ with $\epsilon \in \{-, +\}$ the induced subdigraph in D by the set i_f^ϵ. Then*

1. *If (u, v, w) is a path in $D[i_f^\epsilon]$, then $u_f^- \cap w_f^- \subset v_f^-$ and $u_f^+ \cap w_f^+ \subset v_f^+$.*
2. *If D is a bipartite tournament and (u, v, w) is a path in $D[i_f^\epsilon]$, then $w \in u_f^+$.*

Proof. 1. If (u, v, w) is a path, then $v \in N^+(u) \cap N^-(w)$, and therefore $v \in u_f^+ \cap w_f^-$. Since $u, v, w \in i_f^\epsilon$ it follows the result by assertion 2 of Definition 3.1.

2. If (u, v, w) is a path in $D[i_f^\epsilon]$, then by the above point we have $u_f^- \cap w_f^- \subset v_f^-$. If $w \in u_f^-$ then $w \in u_f^- \cap w_f^- \subset v_f^-$, which is a contradiction because $w \in N^+(v) \subset v_f^+$ and $v_f^- \cap v_f^+ = \emptyset$. Hence, $w \in u_f^+$. □

3.1. k-restricted arc connectivity of good bipartite tournaments

In this subsection we bound the λ'_k-connectivity of good bipartite tournaments.

Lemma 3.1. *Let T be a f-good bipartite tournament. Let $i \in V(T)$ and (a, b, c, d) be a C_4 in $T - i$ and suppose that $b, d \in N^-(i)$. Then $|\{a, c\} \cap i_f^-| = 1$.*

Proof. For simplicity we denote $x_f^+ = x^+$ and $x_f^- = x^-$ for all $x \in V(T)$. Suppose $d \in b^+$. Since $b, d \in N^-(i) \subset i^-$, by item 1 of Definition 3.1, it follows that $b^+ \cap d^- \subset i^-$, implying that $c \in i^-$. Conversely, if $c \in i^-$, then (b, c, d) is a path in $T[i^-]$ yielding that $d \in b^+$ by item 2 of Corollary 3.1.

If $d \in b^-$ then (d, a, b) is a path in $T[b^-]$ yielding that $b \in d^+$ by item 2 of Corollary 3.1. We have $d^+ \cap b^- \subset i^-$ by item 1 of Definition 3.1, yielding that $a \in i^-$. And reciprocally, suppose $a \in i^-$. Since $b, d \in N^-(i)$ it follows that (a, b, i) is a path in $T[i^-]$ and by item 2 of Corollary 3.1, we have $a^- \cap i^- \subset b^-$ yielding that $d \in b^-$.

Since T is a tournament it follows that either $d \in b^+$ or $d \in b^-$ it follows that either $c \in i^-$ or $a \in i^-$ and the lemma holds. □

Lemma 3.2. *Let T be a f-good bipartite tournament. Then, for every pair C_1, C_2 of disjoint 4-cycles,*

$$|(C_1, C_2)| \geq 2.$$

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Proof. Let $C_1 = (a, b, c, d, a)$ and $C_2 = (y, z, w, x, y)$. Let $T = (X, Y)$ and suppose that $a, c, w, y \in X$ and b, d, x, $z \in Y$. Let us suppose that $|(C_1, C_2)| \le 1$. Without loss of generality, we may assume that $\{xa, za, xc, zc, yd, wd\} \subseteq (C_2, C_1)$.

For simplicity we denote $x_f^+ = x^+$ and $x_f^- = x^-$ for all $x \in V(T)$. Then $x, z, d \in N^-(a) \subset a^-$. By Lemma 3.1, we have $|\{y, w\} \cap a^-| = 1$. Without loss of generality assume that $y \in a^-$ and $w \in a^+$. Let us show that $d \in z^-$. Suppose $d \in z^+$, since $z, d \in a^-$, by item 1 of Definition 3.1, it follows that $z^+ \cap d^- \subset a^-$, implying that $w \in a^-$ because $w \in z^+ \cap d^-$ and $wd \in A(T)$. Since this is a contradiction with our assumption $w \in a^+$, we have $d \in z^-$. Moreover (x, y, d) is a path in $D[a^-]$ because $yd \in A(T)$. By item 2 of Corollary 3.1, we get $d \in x^+$. Hence, $x, z, d \in a^-$ and $d \in x^+ \cap z^-$. By item 2 of Definition 3.1, it follows that $x^+ \cap z^+ \subset d^+$, yielding that $c \in d^+$ since $xc, zc \in A(T)$. This is a contradiction because $c \in d^-$. Hence, $|(C_1, C_2)| \ge 2$. □

Note that $D_k(\Pi)$ is not a good bipartite tournament for $k = 2$. In this case it is possible to find two disjoint C_4 such that there is only one arc from one to another and by the above Lemma 3.2 we get that $D_2(\Pi)$ is not a good bipartite tournament.

As a consequence of the above results we obtain the following theorem.

Theorem 3.1. *Let $k \ge 2$ be an integer. Let T be a λ'_k-connected good bipartite tournament with N vertices. Then*

$$k(k-1) \le \lambda'_k(T) \le k(N - 2k - 2).$$

Proof. Since T is λ'_k-connected, it has at least k-vertex disjoint C_4 by Proposition 1.1. Hence, the lower bound on $\lambda_k(T)$ follows by Lemma 3.2. To obtain the upper bound observe that the number of arcs from a cycle C to $T - V(C)$ plus the number of arcs from $T - V(C)$ to C is at most $2(N-4)$. Then one of the two arc sets has cardinality at most $N - 4$. Let $C_1, \ldots, C_k$ be k vertex disjoint cycles contained in T. Thus, the maximum number of arcs that we need to remove from T to disconnect these k cycles is

$$(N-4) + (N-8) + \cdots + (N-4k) = kN - 2k(k+1) = k(N - 2k - 2),$$

and the result follows. □

Corollary 3.2. *Let $k \ge 2$ be an integer. Let T be a good bipartite tournament with N vertices and $\delta(T) \ge 1.5k - 1$. Then*

$$k(k-1) \le \lambda'_k(T) \le k(N - 2k - 2).$$

Proof. Since $\delta(T) \ge 1.5k - 1$, it follows that T is λ'_k-connected by Corollary 1.1. The result is a direct consequence of Theorem 3.1. □

For circulant bipartite tournaments $\overrightarrow{C}_{4n}(1, 3, \ldots, 2n-1)$ we have the following known result.

Theorem 3.2. [16] *If $n \ge 2$, then for every $i \in V(\overrightarrow{C}_{4n}(1, 3, \ldots, 2n-1))$, $\overrightarrow{C}_{4n}(1, 3, \ldots, 2n-1) - \{i, i+1, i+2n, i+2n+1\} \cong \overrightarrow{C}_{4(n-1)}(1, 3, \ldots, 2(n-1)-1)$.*

From the above theorem it follows that $\overrightarrow{C}_{4n}(1, 3, \ldots, 2n-1)$ has n disjoint 4-cycles. Therefore, by Theorem 3.2 and Proposition 3.2 we can write the following result.

Corollary 3.3. *Let k, n be integers such that $2 \le k \le n$. Let $T = \overrightarrow{C}_{4n}(1, 3, \ldots, 2n-1)$ be a circulant bipartite tournament. Then T is λ'_k-connected and*

$$k(k-1) \le \lambda'_k(T) \le 2(2n-k)(k-1).$$

Analogously, we can write the following result for 4-cycles.

Corollary 3.4. *Let T be a complete 4-cycle with N vertices and $|V_\alpha| \ge k$ for each $\alpha = 1, 2, 3, 4$. Then T is λ'_k connected and*

$$k(k-1) \le \lambda'_k(T) \le k(N - 2k - 2).$$

Uncited reference

[8].

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