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Joint moments of the total discounted gains and losses in the renewal risk model with two-sided jumps

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Abstract

This paper considers a renewal insurance risk model with two-sided jumps (e.g. Labbé et al. (2011)), where downward and upward jumps typically represent claim amounts and random gains respectively. A generalization of the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) is proposed and analyzed for sample paths leading to ruin. In particular, we shall incorporate the joint moments of the total discounted costs associated with claims and gains until ruin into the Gerber-Shiu function. Because ruin may not occur, the joint moments of the total discounted claim costs and gain costs are also studied upon ultimate survival of the process. General recursive integral equations satisfied by these functions are derived, and our analysis relies on the concept of 'moment-based discounted densities' introduced by Cheung (2013). Some explicit solutions are obtained in two examples under different cost functions when the distribution of each claim is exponential or a combination of exponentials (while keeping the distributions of the gains and the inter-arrival times between successive jumps arbitrary). The first example looks at the joint moments of the total discounted amounts of claims and gains whereas the second focuses on the joint moments of the numbers of downward and upward jumps until ruin. Numerical examples including the calculations of covariances between the afore-mentioned quantities are given at the end along with some interpretations.

Keywords: Renewal risk model; Two-sided jumps; Joint moments; Total discounted claims/gains; Number of downward/upward jumps.

1 Introduction

1.1 The model, quantities of interest and literature review

In this paper, the renewal risk process with two-sided jumps is used to model the evolution of an insurance company's surplus over time (e.g. Labbé et al. (2011)). Specifically, the surplus level of the insurance company at time t is given by

$$U(t) = u + ct + \sum_{i=1}^{N(t)} Y_i, \qquad t \geq 0, \tag{1.1}$$

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where $u = U(0) \geq 0$ is the initial surplus, $c \geq 0$ is the (net) premium income per unit time that is assumed to be a constant, $\{N(t)\}_{t\geq 0}$ is a renewal process that counts the number of jumps, and $\{Y_i\}_{i=1}^{\infty}$ forms an independent and identically distributed (i.i.d.) sequence representing the jump sizes that may be positive or negative. The Y_i's are assumed to be continuous random variables with common density $p(\cdot)$. In addition, the counting process $\{N(t)\}_{t\geq 0}$ is characterized by the sequence of arrival epochs $\{T_i\}_{i=1}^{\infty}$ such that $N(t) = \sup\{i \in \mathbb{N} : T_i \leq t\}$ (where $\mathbb{N}$ is the set of non-negative integers). The corresponding inter-arrival times $\{V_i = T_i - T_{i-1}\}_{i=1}^{\infty}$ (with the definition $T_0 = 0$) form an i.i.d. sequence of positive continuous random variables with common density $k(\cdot)$. Furthermore, the sequences $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are assumed to be mutually independent. The time of ruin of the surplus process (1.1) is defined as $\tau = \inf\{t \geq 0 : U(t) < 0\}$ with the convention that $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$.

Because each jump size Y_i in the insurance risk process (1.1) may take on positive or negative values, in our analysis it will be more convenient to separate the contributions of upward and downward jumps. To this end, we define $N_+(t) = \sum_{i=1}^{N(t)} 1\{Y_i > 0\}$ to be the number of upward jumps until time t (where $1\{A\}$ is the indicator function of an event A), which is related to the arrival epochs $\{T_{+,i}\}_{i=1}^{\infty}$ of the upward jumps via $N_+(t) = \sup\{i \in \mathbb{N} : T_{+,i} \leq t\}$. Similarly, let $N_-(t) = \sum_{i=1}^{N(t)} 1\{Y_i < 0\} = \sup\{i \in \mathbb{N} : T_{-,i} \leq t\}$ with $\{T_{-,i}\}_{i=1}^{\infty}$ being the arrival epochs of the downward jumps. For each $i \in \mathbb{N}^+$ (where $\mathbb{N}^+$ is the set of positive integers), there is a unique $j \in \mathbb{N}^+$ such that $T_{+,i} = T_j$ for which we let $Y_{+,i} = Y_j$ be the size of the ith upward jump. Analogously, we define $Y_{-,i} = -Y_j$ to be the magnitude of the ith downward jump if $T_{-,i} = T_j$. Hence, the surplus process (1.1) can be rewritten as

$$U(t) = u + ct - \sum_{i=1}^{N_-(t)} Y_{-,i} + \sum_{i=1}^{N_+(t)} Y_{+,i}, \qquad t \geq 0. \tag{1.2}$$

It is assumed that $\Pr\{Y_1 > 0\} = q_+ \geq 0$ and $\Pr\{Y_1 < 0\} = q_- = 1 - q_+ > 0$. (We exclude the case $q_- = 0$ which results in monotonically increasing sample paths.) Then for $y > 0$ the common densities of the sequences $\{Y_{+,i}\}_{i=1}^{\infty}$ and $\{Y_{-,i}\}_{i=1}^{\infty}$ are given by $p_+(y) = (d/dy)\Pr\{Y_1 \leq y | Y_1 > 0\}$ and $p_-(y) = (d/dy)\Pr\{-Y_1 \leq y | Y_1 < 0\}$ respectively, and one can write

$$p(y) = q_+ p_+(y) 1\{y > 0\} + q_- p_-(-y) 1\{y < 0\}.$$

Furthermore, the positive security loading condition $cE[V_1] + E[Y_1] = cE[V_1] + q_+ E[Y_{+,1}] - q_- E[Y_{-,1}] > 0$ is assumed to hold, and this ensures that the ruin probability $\psi(u) = \Pr\{\tau < \infty | U(0) = u\}$ is strictly less than 1 for all $u \geq 0$ (e.g. Asmussen and Albrecher (2010, Chapter III, Corollary 3.2(a))). Note that the random sums $\sum_{i=1}^{N_-(t)} Y_{-,i}$ and $\sum_{i=1}^{N_+(t)} Y_{+,i}$ in (1.2) are independent if $\{N(t)\}_{t\geq 0}$ is a Poisson process, but this is generally not true when $\{N(t)\}_{t\geq 0}$ is a renewal process. In addition, $\{N_+(t)\}_{t\geq 0}$ (resp. $\{N_-(t)\}_{t\geq 0}$) is a renewal process with i.i.d. inter-arrival times having common density $\sum_{i=1}^{\infty} q_-^{i-1} q_+ k^{*i}(\cdot)$ (resp. $\sum_{i=1}^{\infty} q_+^{i-1} q_- k^{*i}(\cdot)$), where $k^{*i}(\cdot)$ is the i-fold convolution density of $k(\cdot)$.

In the risk process (1.1) or (1.2), an upward jump is regarded as a stochastic income and a downward jump is the result of a loss. While it is clear that losses or insurance claims usually arise from property and casualty insurance business, random income arises for life annuity or pension funds in which the insurer pays annuities to its policyholders and earns part of the reserves when a policyholder dies (e.g. Seal (1969, p.116)). Therefore, the model in the present paper can be suitable for insurance companies with business in both property and casualty insurance and life annuities. Apart from the above interpretation, corrections of previous overstatement of losses can also be another source of random income. For the rest of the paper, we shall use the terminologies 'loss', 'claim' and 'downward jump' interchangeably, and the same applies to the words ‘gain’ and ‘upward jump’. There have been a number of studies concerning risk models with two-sided jumps in recent years. Much of the pertinent literature has focused on the analysis of the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)), which is defined as

$$\phi_{\delta_1}(u) = E[e^{-\delta_1 \tau} w(U(\tau^-), |U(\tau)|) 1\{\tau < \infty\} | U(0) = u], \qquad u \geq 0. \tag{1.3}$$

Here $\delta_1 \geq 0$ can be regarded as a force of interest or the Laplace transform argument of τ, and $w(\cdot, \cdot)$ is the so-called ‘penalty’ as a non-negative function of the surplus immediately before ruin $U(\tau^-)$ and the deficit at ruin $|U(\tau)|$. Under the compound Poisson model without deterministic premium income (i.e. $c = 0$), Labbé and Sendova (2009) and Albrecher et al. (2010) derived the solution of the Gerber-Shiu function (and its special case where $w(\cdot, \cdot)$ only depends on $|U(\tau)|$) under specific distributional assumptions on either downward or upward jumps. Cai et al. (2009, Section 8) also contains some explicit results under two-sided exponential jumps as well as an application of a Gerber-Shiu function to price perpetual American put options. For the risk model (1.1) with arbitrary inter-arrival times, Cheung (2011) and Labbé et al. (2011) respectively considered the cases $c < 0$ and $c \geq 0$, and some structural properties of the Gerber-Shiu function in connection with defective renewal equations are given. Interested readers are referred to Perry et al. (2002), Kou and Wang (2003), Asmussen et al. (2004), Breuer (2008) and Cai (2009) for the study of first passage times and one-sided or two-sided exit problems in related models with two-sided jumps.

While the Gerber-Shiu function (1.3) serves as a powerful tool that unifies the study of τ, $U(\tau^-)$ and $|U(\tau)|$, it is instructive to note that the random variables $U(\tau^-)$ and $|U(\tau)|$ are defined at the ruin time τ. This led various researchers to analyze generalizations of the Gerber-Shiu function (mainly in risk models without upward jumps). In particular, additional random variables defined before the time of ruin are incorporated into the penalty function (or the joint law underlying the Gerber-Shiu function), such as the minimum surplus level before ruin (e.g. Doney and Kyprianou (2006), Biffis and Morales (2010), and Cheung et al. (2010)), the maximum surplus before ruin (e.g. Kyprianou and Zhou (2009), and Cheung and Landriault (2010)), the surplus level immediately after the second last claim before ruin (see e.g. Cheung et al. (2010), Woo (2010, 2012), and Zhang and Yang (2010)). On the other hand, Landriault et al. (2011) and Frostig et al. (2012) included the number of claims until ruin in the Gerber-Shiu function in the form of a generating function. An alternative extension was also proposed by Cai et al. (2009), who looked at the expected total discounted operating costs based on an integral of the entire sample path until ruin. It is shown that the traditional Gerber-Shiu function can be retrieved from such a function as a special case in a non-trivial manner in certain models (see also Feng (2009a,b)). In another direction of generalization, Cheung (2013) and Cheung and Woo (2016) added a moment-based component concerning the total discounted claim costs until ruin into the Gerber-Shiu function. Under barrier and threshold dividend strategies respectively, Cheung et al. (2015) and Cheung and Liu (2016) further incorporated the higher moments of the total discounted dividends until ruin, thereby allowing for the computation of the covariance of the insurance company’s payments to its policyholders (claims) and shareholders (dividends). Motivated by the last four works and the fact that the present model (1.2) contains upward jumps, we are interested in the generalized Gerber-Shiu function

$$\phi_{\delta_1,\delta_2,\delta_3,n,m}(u) = E[e^{-\delta_1 \tau} Z^n_{-,\delta_2}(\tau) Z^m_{+,\delta_3}(\tau) w(U(\tau^-), |U(\tau)|) 1\{\tau < \infty\} | U(0) = u], \qquad u \geq 0, \tag{1.4}$$

where $Z_{-,\delta_2}(\tau) = \sum_{i=1}^{N_-(\tau)} e^{-\delta_2 T_{-,i}} f_-(Y_{-,i})$ and $Z_{+,\delta_3}(\tau) = \sum_{i=1}^{N_+(\tau)} e^{-\delta_3 T_{+,i}} f_+(Y_{+,i})$ are respectively the total discounted claim costs and the total discounted gain costs (which are both path-dependent random variables), and $n, m \in \mathbb{N}$ are the orders of moments of these two variables. Here $f_-(\cdot)$ and $f_+(\cdot)$ are non-negative functions on $(0, \infty)$ which determine the ‘costs’ associated with downward and upward jumps respectively, and $\delta_2, \delta_3 \geq 0$ are interest rates used for discounting these 'costs'. Concerning the choice of cost function $f_-(\cdot)$, if it is assumed that $f_-(x) = x$ then $Z_{-,\delta_2}(\tau) = \sum_{i=1}^{N_-(\tau)} e^{-\delta_2 T_{-,i}} Y_{-,i}$ represents the total discounted claim amounts until ruin. On the other hand, if $f_-(\cdot) \equiv 1$ and $\delta_2 = 0$ then $Z_{-,\delta_2}(\tau) = N_-(\tau)$ becomes the number of claims until ruin. Similar comments are applicable to $f_+(\cdot)$. Note that if $n = m = 0$ then $\phi_{\delta_1,\delta_2,\delta_3,n,m}(u)$ in (1.4) reduces to the classical Gerber-Shiu function $\phi_{\delta_1}(u)$. If one lets $\delta_1 = 0$ and $w(\cdot,\cdot) \equiv 1$, then $\phi_{\delta_1,\delta_2,\delta_3,n,m}(u)$ corresponds to the (defective) joint moment of $Z_{-,\delta_2}(\tau)$ and $Z_{+,\delta_3}(\tau)$ from which covariance (conditional on ruin occurrence) can be calculated. Assuming $w(\cdot,\cdot) \equiv 1$, the joint moments involving the ruin time are also obtainable by successive differentiation of $\phi_{\delta_1,\delta_2,\delta_3,n,m}(u)$ with respect to δ_1 at $\delta_1 = 0$. For example, we have $(-1)^r (d^r/d\delta_1^r)\phi_{\delta_1,\delta_2,\delta_3,n,0}(u) = E[\tau^r Z_{-,\delta_2}^n(\tau) 1\{\tau < \infty\} | U(0) = u]$. The study of the covariance of the time of ruin and the total discounted claims until ruin in a dependent Sparre Andersen risk model can be found in Cheung and Woo (2016). We also remark that, in a Markov-modulated risk model without upward jumps, the covariance between the total (non-discounted) claims in different environmental states and that between the numbers of claims were computed by Li et al. (2016, Section 7).

The generalized Gerber-Shiu function (1.4) is confined to the sample paths for which ruin occurs. In the case where ruin does not occur (thanks to the positive loading assumption), the surplus prior to ruin $U(\tau^-)$ and the deficit at ruin $|U(\tau)|$ are undefined but the total discounted claim costs $Z_{-,\delta_2}(\tau)$ and the total discounted gain costs $Z_{+,\delta_3}(\tau)$ can still be analyzed. This leads us to study the joint moment, for $n, m \in \mathbb{N}$, given by

$$\varphi_{\delta_2,\delta_3,n,m}(u) = E[Z_{-,\delta_2}^n(\tau) Z_{+,\delta_3}^m(\tau) 1\{\tau = \infty\} | U(0) = u], \qquad u \geq 0. \tag{1.5}$$

It is always assumed that $\delta_2 > 0$ and $\delta_3 > 0$ as far as $\varphi_{\delta_2,\delta_3,n,m}(u)$ is concerned. Clearly, the (n, m)th joint moment of $(Z_{-,\delta_2}(\tau), Z_{+,\delta_3}(\tau))$ which takes into account all sample paths is given by

$$E[Z_{-,\delta_2}^n(\tau) Z_{+,\delta_3}^m(\tau) | U(0) = u] = \phi_{\delta_1,\delta_2,\delta_3,n,m}(u)|_{\delta_1=0,w\equiv1} + \varphi_{\delta_2,\delta_3,n,m}(u). \tag{1.6}$$

1.2 Note on notation and organization of paper

For the remainder of the paper, whenever there is no confusion, the generalized Gerber-Shiu function $\phi_{\delta_1,\delta_2,\delta_3,n,m}(\cdot)$ in (1.4) will be abbreviated as $\phi_{\delta_{123},n,m}(\cdot)$, which is further simplified to $\phi_{\delta_{12},n}(u)$ when $m = 0$. Similarly, concerning $\varphi_{\delta_2,\delta_3,n,m}(\cdot)$ defined in (1.5), for simplicity we shall write $\varphi_{\delta_{23},n,m}(\cdot)$ instead.

The rest of the paper is organized as follows. In Section 2, for each of $\phi_{\delta_{123},n,m}(u)$ and $\varphi_{\delta_{23},n,m}(u)$, two different integral equations will be derived based on the first jump event and the first time the surplus process $\{U(t)\}_{t\geq0}$ drops below its initial level. While the first equation is standard, to obtain the second one we adopt an analogous approach to Cheung (2013) by utilizing an extension of the discounted density involving the moments of $Z_{-,\delta_2}(\tau)$ and $Z_{+,\delta_3}(\tau)$. Sections 3 and 4 are concerned with the application of both integral equations to obtain some explicit solutions to $\phi_{\delta_{123},n,m}(u)$ and $\varphi_{\delta_{23},n,m}(u)$ by making distributional assumption only on the downward jumps. In particular, Section 3 looks at the joint moments of the total discounted gains and losses when $p_-(\cdot)$ is an exponential density, whereas the joint moments of the numbers of upward and downward jumps are considered in Section 4 when $p_-(\cdot)$ follows a combination of exponentials. Numerical examples involving the covariance measures in relation to the above quantities of interest are given in Section 5 along with some interpretations.

2 General results

2.1 Integral equations via the first jump event

We begin with the classical approach to derive an integral equation for the generalized Gerber-Shiu function (1.4) by conditioning on the time V_1 and the amount Y_1 of the first jump. If the first jump is a loss (i.e. $Y_1 = -Y_{-,1}$), then there are two possible cases. If $Y_{-,1} \leq u + cV_1$, then the process restarts with the surplus level $u + cV_1 - Y_{-,1}$. But if $Y_{-,1} > u + cV_1$ then ruin occurs at time $\tau = V_1$, where $U(\tau^-) = u + c\tau$; $|U(\tau)| = Y_{-,1} - u - c\tau$; $Z_{-,\delta_2}(\tau) = e^{-\delta_2 \tau} f_-(Y_{-,1})$; and $Z_{+,\delta_3}(\tau) = 0$. On the other hand, if the first jump is a gain (i.e. $Y_1 = Y_{+,1}$) then the process restarts at the level $u + cV_1 + Y_{+,1}$. Because $Z_{+,\delta_3}(\tau) = 0$ when the first jump causes ruin, we need to separate the cases $m = 0$ and $m \in \mathbb{N}^+$. Combining the above observations followed by appropriate binomial expansions leads to, for $n \in \mathbb{N}$ and $m = 0$,

$$\phi_{\delta_{12},n}(u) = \int_0^\infty e^{-(\delta_1 + n\delta_2)t} \Bigg(q_- \sum_{i=0}^n \binom{n}{i} \int_0^{u+ct} f_-^{n-i}(y) \phi_{\delta_{12},i}(u + ct - y) p_-(y)\, dy \\ + q_- \int_{u+ct}^\infty f_-^n(y) w(u + ct, y - u - ct) p_-(y)\, dy + q_+ \int_0^\infty \phi_{\delta_{12},n}(u + ct + y) p_+(y)\, dy \Bigg) k(t)\, dt, \tag{2.1}$$

and for $n \in \mathbb{N}$ and $m \in \mathbb{N}^+$,

$$\phi_{\delta_{123},n,m}(u) = \int_0^\infty e^{-(\delta_1 + n\delta_2 + m\delta_3)t} \Bigg(q_- \sum_{i=0}^n \binom{n}{i} \int_0^{u+ct} f_-^{n-i}(y) \phi_{\delta_{123},i,m}(u + ct - y) p_-(y)\, dy \\ + q_+ \sum_{j=0}^m \binom{m}{j} \int_0^\infty f_+^{m-j}(y) \phi_{\delta_{123},n,j}(u + ct + y) p_+(y)\, dy \Bigg) k(t)\, dt. \tag{2.2}$$

The integral expression for the joint moment (1.5) upon survival is obtainable using the same approach. However, one should be aware that the case where the first jump causes ruin has zero contribution to (1.5). We arrive at, for $n, m \in \mathbb{N}$,

$$\varphi_{\delta_{23},n,m}(u) = \int_0^\infty e^{-(n\delta_2 + m\delta_3)t} \Bigg(q_- \sum_{i=0}^n \binom{n}{i} \int_0^{u+ct} f_-^{n-i}(y) \varphi_{\delta_{23},i,m}(u + ct - y) p_-(y)\, dy \\ + q_+ \sum_{j=0}^m \binom{m}{j} \int_0^\infty f_+^{m-j}(y) \varphi_{\delta_{23},n,j}(u + ct + y) p_+(y)\, dy \Bigg) k(t)\, dt. \tag{2.3}$$

Remark 1 While the integral equations (2.2) (when $\delta_1 = 0$) and (2.3) satisfied by $\phi_{\delta_{123},n,m}(\cdot)|_{\delta_1=0}$ for ruin and $\varphi_{\delta_{23},n,m}(\cdot)$ for survival look identical when $n \in \mathbb{N}$ and $m \in \mathbb{N}^+$, the quantities $\phi_{\delta_{123},n,m}(\cdot)|_{\delta_1=0}$ and $\varphi_{\delta_{23},n,m}(\cdot)$ have different solutions in general. The reason is that these integral equations are recursive in n and m, and when $m = 0$ the integral equation (2.1) satisfied by $\phi_{\delta_{123},n,0}(\cdot)|_{\delta_1=0} = \phi_{\delta_{12},n}(\cdot)|_{\delta_1=0}$ is different from (2.3). In particular, the starting point for evaluating $\phi_{\delta_{123},n,m}(\cdot)|_{\delta_1=0}$ is $\phi_{\delta_{123},0,0}(\cdot)|_{\delta_1=0} = \phi_0(\cdot)$ which corresponds to the classical Gerber-Shiu function without discounting, while that for $\varphi_{\delta_{23},n,m}(\cdot)$ is $\varphi_{\delta_{23},0,0}(\cdot) = 1 - \psi(\cdot)$ which is the survival probability. □

For risk models under specific inter-arrival time distributions (e.g. exponential or Erlang(n)), solutions to Gerber-Shiu type functions can typically be found by transforming the integral equation obtainable via the first jump event to a defective renewal equation (see e.g. Gerber and Shiu (1998), Dickson and Hipp (2011), and Li and Garrido (2004)). This is possible because the presence of exponential terms in $k(\cdot)$ typically allows us to differentiate conveniently and/or utilize Dickson-Hipp operators (see (3.10)). However, such a transformation does not appear to be feasible when inter-arrival times of the jumps are left arbitrary. As a result, it is not clear how full solutions to $\phi_{\delta_{123},n,m}(\cdot)$ and $\varphi_{\delta_{23},n,m}(\cdot)$ can be derived from (2.1)-(2.3) only. Therefore, we shall use the probabilistic argument of conditioning on the first drop of $\{U(t)\}_{t\geq 0}$ below the initial surplus (e.g. Willmot (2007)) to arrive at integral equations for $\phi_{\delta_{123},n,m}(\cdot)$ and $\varphi_{\delta_{23},n,m}(\cdot)$, which will yield additional information.

2.2 Integral equations via the first drop

When considering the event of the first drop of the surplus process below its initial level u, one needs to keep track of the discounted jump costs until such a drop. To this end, 'moment-based discounted densities' play an important role, and the concept is introduced as follows (see Cheung (2013) and Cheung and Woo (2016)). For $\tau < \infty$, we first let $g_{\delta_2,\delta_3}(t, x, y, z_2, z_3|u)$ be the (defective) joint density of $(\tau, U(\tau^-), |U(\tau)|, Z_{-,\delta_2}(\tau), Z_{+,\delta_3}(\tau))$ at (t, x, y, z_2, z_3) and $g^*_{\delta_2,\delta_3}(t, x, z_2, z_3|u)$ be the (defective) joint density of $(\tau, U(\tau^-), \sum_{i=1}^{N_-(\tau)-1} e^{-\delta_2 T_{-,i}} f_-(Y_{-,i}), Z_{+,\delta_3}(\tau))$ at (t, x, z_2, z_3) (see Remark 3). Then, for $t, y > 0$; $z_2 > e^{-\delta_2 t} f_-(x+y)$ and $x, z_3 \geq 0$, one can argue that

$$g_{\delta_2,\delta_3}(t, x, y, z_2, z_3|u) = g^*_{\delta_2,\delta_3}(t, x, z_2 - e^{-\delta_2 t} f_-(x+y), z_3|u) p_{-,x}(y), \tag{2.4}$$

where $p_{-,x}(y) = p_-(x+y)/\overline{P}_-(x)$ and $\overline{P}_-(x) = \int_x^\infty p_-(y)\, dy$ are the residual lifetime density and the survival function corresponding to $p_-(\cdot)$. To explain the above probabilistic identity, it is noted that, for $(U(\tau^-), |U(\tau)|)$ to be equal to (x, y), the final jump that causes ruin must be a downward jump of size $x + y$ whose associated cost is $f_-(x + y)$. Then, in order for $(\tau, Z_{-,\delta_2}(\tau))$ to be at (t, z_2), the quantity $\sum_{i=1}^{N_-(\tau)-1} e^{-\delta_2 T_{-,i}} f_-(Y_{-,i})$ should be $z_2 - e^{-\delta_2 t} f_-(x + y)$. At the moment just prior to ruin, a downward jump of size $x+y$ occurs immediately to bring the surplus from x to $-y$, but this is conditional on the fact that the jump must be larger than x for ruin to occur. This explains the term $p_{-,x}(y)$ on the right-hand side of (2.4). With these definitions, the Gerber-Shiu function can be represented, for $n, m \in \mathbb{N}$, as

$$\begin{aligned}\phi_{\delta_{123},n,m}(u) &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_{e^{-\delta_2 t} f_-(x+y)}^\infty e^{-\delta_1 t} z_2^n z_3^m w(x, y) g_{\delta_2,\delta_3}(t, x, y, z_2, z_3|u)\, dz_2\, dz_3\, dt\, dx\, dy \\ &= \int_0^\infty \int_0^\infty w(x, y) h^*_{\delta_1,\delta_2,\delta_3,n,m}(x, y|u)\, dx\, dy, \end{aligned}\tag{2.5}$$

where

$$\begin{aligned} h^*_{\delta_1,\delta_2,\delta_3,n,m}(x, y|u) &= \int_0^\infty \int_0^\infty \int_{e^{-\delta_2 t} f_-(x+y)}^\infty e^{-\delta_1 t} z_2^n z_3^m g_{\delta_2,\delta_3}(t, x, y, z_2, z_3|u)\, dz_2\, dz_3\, dt &(2.6)\\ &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta_1 t} (z_2 + e^{-\delta_2 t} f_-(x+y))^n z_3^m g^*_{\delta_2,\delta_3}(t, x, z_2, z_3|u) p_{-,x}(y)\, dz_2\, dz_3\, dt &(2.7)\end{aligned}$$

is the moment-based discounted joint density of $(U(\tau^-), |U(\tau)|)$ at (x, y). Clearly, if $n = m = 0$ then $h^*_{\delta_1,\delta_2,\delta_3,n,m}(x, y|u)$ becomes the usual discounted joint density of $(U(\tau^-), |U(\tau)|)$. Then

$$h_{\delta_1,\delta_2,\delta_3,n,m}(y|u) = \int_0^\infty h^*_{\delta_1,\delta_2,\delta_3,n,m}(x, y|u)\, dx \tag{2.8}$$

is the moment-based discounted density of the deficit $|U(\tau)|$.

Because the first time when $\{U(t)|U(0)=u\}_{t\geq 0}$ falls below u is equivalent to the ruin time of $\{U(t)|U(0)=0\}_{t\geq 0}$ thanks to the spatial homogeneity of $\{U(t)\}_{t\geq 0}$, it is sufficient to apply the above densities under zero initial surplus. For $\phi_{\delta_{123},n,m}(u)$, the presence of the indicator $1\{\tau<\infty\}$ in the definition (1.4) means that a drop below the initial level must occur. If the amount of the drop is $y\leq u$ then the process restarts at level $u-y$; but if $y>u$ then ruin occurs upon the first drop. Therefore, we arrive at, for $n,m\in\mathbb{N}$,

$$\begin{aligned}\phi_{\delta_{123},n,m}(u)=&\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\int_0^u\int_0^\infty\int_0^\infty\int_0^\infty\int_{e^{-\delta_2 t}f_-(x+y)}^\infty e^{-(\delta_1+i\delta_2+j\delta_3)t}z_2^{n-i}z_3^{m-j}\phi_{\delta_{123},i,j}(u-y)\\&\times g_{\delta_2,\delta_3}(t,x,y,z_2,z_3|0)\,dz_2\,dz_3\,dt\,dx\,dy\\&+\int_u^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{e^{-\delta_2 t}f_-(x+y)}^\infty e^{-\delta_1 t}z_2^n z_3^m w(x+u,y-u)\\&\times g_{\delta_2,\delta_3}(t,x,y,z_2,z_3|0)\,dz_2\,dz_3\,dt\,dx\,dy\\=&\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\int_0^u h_{\delta_1+i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j}(y|0)\phi_{\delta_{123},i,j}(u-y)\,dy\\&+\int_u^\infty\int_0^\infty h^*_{\delta_1,\delta_2,\delta_3,n,m}(x,y|0)w(x+u,y-u)\,dx\,dy,\end{aligned}\tag{2.9}$$

where (2.6) and (2.8) have been applied. This is a renewal equation satisfied by $\phi_{\delta_{123},n,m}(\cdot)$. To see it, we rewrite (2.9) by separating the term $(i,j)=(n,m)$ in the summation, leading to

$$\phi_{\delta_{123},n,m}(u)=\int_0^u h_{\delta_1+n\delta_2+m\delta_3,\delta_2,\delta_3,0,0}(y|0)\phi_{\delta_{123},n,m}(u-y)\,dy+\alpha_{\delta_{123},n,m}(u),\tag{2.10}$$

where

$$\begin{aligned}\alpha_{\delta_{123},n,m}(u)=&\sum_{(i,j)\in\Gamma(n,m)}\binom{n}{i}\binom{m}{j}\int_0^u h_{\delta_1+i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j}(y|0)\phi_{\delta_{123},i,j}(u-y)\,dy\\&+\int_u^\infty\int_0^\infty h^*_{\delta_1,\delta_2,\delta_3,n,m}(x,y|0)w(x+u,y-u)\,dx\,dy.\end{aligned}\tag{2.11}$$

The summation above is taken over the set $\Gamma(n,m)=\{(i,j):0\leq i\leq n;0\leq j\leq m;(i,j)\neq(n,m)\}$. From (2.5), (2.8) and the definition (1.4), we note that

$$\int_0^\infty h_{\delta_1+n\delta_2+m\delta_3,\delta_2,\delta_3,0,0}(y|0)\,dy=E[e^{-(\delta_1+n\delta_2+m\delta_3)\tau}1\{\tau<\infty\}|U(0)=0]<1,$$

where the last inequality holds when $\delta_1+n\delta_2+m\delta_3>0$ or the positive security loading condition holds. Hence, one asserts that the renewal equation (2.10) is defective.

Remark 2 The defective renewal equation (2.10) for $\phi_{\delta_{123},n,m}(\cdot)$ is recursive in $n,m\in\mathbb{N}$ since the non-homogeneous term $\alpha_{\delta_{123},n,m}(u)$ defined in (2.11) depends on the 'lower-order' Gerber-Shiu functions, and the starting point is the classical Gerber-Shiu function $\phi_{\delta_{123},0,0}(\cdot)=\phi_{\delta_1}(\cdot)$. However, although the solution of a defective renewal equation is unique, (2.10) itself does not immediately lead to full solution to $\phi_{\delta_{123},n,m}(\cdot)$ because the densities $h^*_{\delta_1,\delta_2,\delta_3,n,m}(x,y|0)$ and $h_{\delta_1+i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j}(y|0)$ therein are not completely known. Instead, (2.10) will be used for determining the solution form of $\phi_{\delta_{123},n,m}(\cdot)$ upon further assumption on the downward jump density $p_-(\cdot)$, the penalty function $w(\cdot,\cdot)$ and the cost function $f_-(\cdot)$ (see (3.8) and (4.7)). $\square$

Remark 3 Indeed, the random variables $Z_{-,\delta_2}(\tau)$ and $Z_{+,\delta_3}(\tau)$ may not be always continuous. For example, if $f_-(\cdot) \equiv 1$ and $\delta_2 = 0$ then $Z_{-,\delta_2}(\tau) = N_-(\tau)$ is the number of downward jumps until ruin which is a fully discrete random variable. Moreover, if no upward jumps happen before ruin then $Z_{+,\delta_3}(\tau) = 0$ and therefore $Z_{+,\delta_3}(\tau)$ is possibly a mixed random variable with a point mass at zero plus a density part. Similar comments apply to the random variable $\sum_{i=1}^{N_-(\tau)-1} e^{-\delta_2 T_{-,i}} f_-(Y_{-,i})$. But for the ease of presentation, we have assumed that all variables in the quintuple $(\tau, U(\tau^-), |U(\tau)|, Z_{-,\delta_2}(\tau), Z_{+,\delta_3}(\tau))$ and the quadruple $(\tau, U(\tau^-), \sum_{i=1}^{N_-(\tau)-1} e^{-\delta_2 T_{-,i}} f_-(Y_{-,i}), Z_{+,\delta_3}(\tau))$ are fully continuous, and integrations over the 'joint density' (2.4) have been used. If we are to take into account any discrete component, then the integrals with respect to z_2 and z_3 in the derivations should be replaced by summations or a mix of integrations and summations. Nonetheless, the variable $|U(\tau)|$ (corresponding to the argument y) is always continuous, and it is the form in y of the density $h_{\delta_1,\delta_2,\delta_3,n,m}(y|0)$ defined via (2.8) that is important for our analysis (see (3.2) and (4.2)). Hence, the existence of discrete components does not affect the form of $h_{\delta_1,\delta_2,\delta_3,n,m}(y|0)$ and all the main results in the paper still hold true. $\square$

To derive a defective renewal equation for $\varphi_{\delta_{23},n,m}(\cdot)$ via the first drop, it is instructive to note that the surplus process may actually stay above the initial level forever as we require the process to survive according to the indicator $1\{\tau = \infty\}$ in the definition (1.5). The contribution of such a scenario to $\varphi_{\delta_{23},n,m}(u)$ is simply $\varphi_{\delta_{23},n,m}(0)$. For the case where a drop below the initial level occurs, the same arguments used to obtain the first term in (2.9) are applicable. Consolidating these observations, we arrive at, for $n, m \in \mathbb{N}$,

$$\begin{aligned}\varphi_{\delta_{23},n,m}(u) &= \varphi_{\delta_{23},n,m}(0) + \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\int_0^u\int_0^\infty\int_0^\infty\int_0^\infty\int_{e^{-\delta_2 t}f_-(x+y)}^\infty e^{-(i\delta_2+j\delta_3)t} z_2^{n-i} z_3^{m-j}\varphi_{\delta_{23},i,j}(u-y) \\ &\qquad \times g_{\delta_2,\delta_3}(t,x,y,z_2,z_3|0)\,dz_2\,dz_3\,dt\,dx\,dy \\ &= \varphi_{\delta_{23},n,m}(0) + \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\int_0^u h_{i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j}(y|0)\varphi_{\delta_{23},i,j}(u-y)\,dy, \end{aligned} \tag{2.12}$$

which is clearly a defective renewal equation resembling (2.10).

3 Joint moments of total discounted gains and losses

In this section, we are mainly interested in the joint moments of the total discounted upward and downward jumps. It is assumed that $\delta_2, \delta_3 > 0$, and the penalty function $w(x, y) = w_1(y)$ only depends on the deficit argument but not the surplus prior to ruin. Let $f_-(x) = x$ so that $Z_{-,\delta_2}(\tau) = \sum_{i=1}^{N_-(\tau)} e^{-\delta_2 T_{-,i}} Y_{-,i}$ is the total discounted losses until ruin. But we do not make any assumptions on $f_+(\cdot)$ since the derivations are not more complicated than the case where one assumes that $f_+(x) = x$ from the outset.

3.1 Note on the discounted density $h_{\delta_1,\delta_2,\delta_3,n,m}(y|u)$

In the spirit of Willmot (2007), we first study some properties of the moment-based discounted density $h_{\delta_1,\delta_2,\delta_3,n,m}(y|u)$ when the loss density $p_-(\cdot)$ satisfies

$$p_-(x+y) = \sum_{k=1}^{r} \eta_k(x)\gamma_k(y), \tag{3.1}$$

for some functions $\eta_k(\cdot)$'s and $\gamma_k(\cdot)$'s. The factorization is known to be satisfied by a variety of distributions, such as the classes of combinations of exponentials (see (4.4)) and Erlang mixtures for which $\gamma_k(\cdot)$ would take on exponential and Erlang forms respectively. Under the assumption (3.1), it is clear that the residual lifetime density $p_{-,x}(y) = p_-(x+y)/\overline{P}_-(x)$ satisfies

$$p_{-,x}(y) = \sum_{k=1}^{r} \eta_k^*(x)\gamma_k(y),$$

where $\eta_k^*(x) = \eta_k(x)/\overline{P}_-(x)$. Substitution of (2.7) and the above expression into (2.8) yields

$$\begin{aligned}
&h_{\delta_1,\delta_2,\delta_3,n,m}(y|u)\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta_1 t}(z_2 + e^{-\delta_2 t}(x+y))^n z_3^m g^*_{\delta_2,\delta_3}(t,x,z_2,z_3|u) \sum_{k=1}^{r} \eta_k^*(x)\gamma_k(y)\, dz_2\, dz_3\, dt\, dx\\
&= \sum_{l=0}^{n}\sum_{k=1}^{r}\binom{n}{l} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta_1 t}(z_2 + e^{-\delta_2 t}x)^{n-l}(e^{-\delta_2 t}y)^l z_3^m g^*_{\delta_2,\delta_3}(t,x,z_2,z_3|u)\eta_k^*(x)\gamma_k(y)\, dz_2\, dz_3\, dt\, dx\\
&= \sum_{l=0}^{n}\sum_{k=1}^{r} D_{\delta_1,\delta_2,\delta_3,n,m,l,k}(u) y^l \gamma_k(y), \qquad (3.2)
\end{aligned}$$

where $D_{\delta_1,\delta_2,\delta_3,n,m,l,k}(u)$'s are constants that do not depend on y (and their obvious definitions are omitted as they are not required in the analysis). The above result indicates that, for example, if $p_-(\cdot)$ is a combination of exponentials then the density $h_{\delta_1,\delta_2,\delta_3,n,m}(\cdot|u)$ will be a combination of Erlangs. With (2.7) and the assumptions in this section, the second term on the right-hand side of (2.9) can be evaluated as

$$\begin{aligned}
&\int_u^\infty \int_0^\infty h^*_{\delta_1,\delta_2,\delta_3,n,m}(x,y|0) w_1(y-u)\, dx\, dy = \int_0^\infty \int_0^\infty h^*_{\delta_1,\delta_2,\delta_3,n,m}(x,y+u|0) w_1(y)\, dx\, dy\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta_1 t}(z_2 + e^{-\delta_2 t}(x+y+u))^n z_3^m g^*_{\delta_2,\delta_3}(t,x,z_2,z_3|0) \frac{p_-(x+y+u)}{\overline{P}_-(x)} w_1(y)\, dz_2\, dz_3\, dt\, dx\, dy\\
&= \sum_{l=0}^{n}\sum_{k=1}^{r}\binom{n}{l} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta_1 t}(z_2 + e^{-\delta_2 t}(x+y))^{n-l}(e^{-\delta_1 t}u)^l z_3^m g^*_{\delta_2,\delta_3}(t,x,z_2,z_3|0)\\
&\qquad\qquad \times \frac{\eta_k(x+y)}{\overline{P}_-(x)}\gamma_k(u) w_1(y)\, dz_2\, dz_3\, dt\, dx\, dy\\
&= \sum_{l=0}^{n}\sum_{k=1}^{r} E_{\delta_1,\delta_2,\delta_3,n,m,l,k} u^l \gamma_k(u), \qquad (3.3)
\end{aligned}$$

for some constants $E_{\delta_1,\delta_2,\delta_3,n,m,l,k}$'s with obvious definitions.

3.2 Detailed analysis for exponential claims

In the remainder of this section, we further assume exponential losses so that $p_-(y) = \beta e^{-\beta y}$. Then, the results in Section 3.1 are applicable with $r = 1$; $\eta_1(x) = \beta e^{-\beta x}$ and $\gamma_1(y) = e^{-\beta y}$. For every pair of (n,m) such that $n, m \in \mathbb{N}$, let $\kappa_{n,m} \in (0,\beta)$ be the unique positive root of the Lundberg's equation (in ξ)

$$\left(q_- \frac{\beta}{\beta - \xi} + q_+ \widetilde{p}_+(\xi)\right) \widetilde{k}(\delta_1 + n\delta_2 + m\delta_3 + c\xi) = 1. \qquad (3.4)$$

See e.g. Cheung (2011, p.16) and Labbé et al. (2011, Proposition 5.1). We suppress dependence of $\kappa_{n,m}$ on δ_1, δ_2 and δ_3 for simplicity. In particular, $\kappa_{n,m}$ will be denoted as $\kappa^*_{n,m}$ when $\delta_1 = 0$. For later use, we note from (1.4), (2.5) and (2.8) that the Laplace transform of the discounted density $h_{\delta_1,\delta_2,\delta_3,0,0}(\cdot|0)$ is given by

$$\widetilde{h}_{\delta_1,\delta_2,\delta_3,0,0}(s|0) = E[e^{-\delta_1\tau - s|U(\tau)|}1\{\tau < \infty\}|U(0) = 0] = \frac{\beta - \kappa_{0,0}}{\beta + s}, \tag{3.5}$$

where the last equality follows from Labbé et al. (2011, Theorem 5.1). See Remark 4 below. (In this paper, the Laplace transform of a function $a(\cdot)$ is denoted by $\widetilde{a}(s) = \int_0^\infty e^{-sx}a(x)\,dx$.)

Remark 4 When claims are exponentially distributed, Labbé et al. (2011, Theorem 5.1) provided an expression for the classical Gerber-Shiu function (1.3) under the penalty function $w(x, y) = e^{-zx}w_1(y)$. Therefore, $E[e^{-\delta_1\tau-s|U(\tau)|}1\{\tau < \infty\}|U(0) = 0]$ can be obtained from their Equation (5.1) by letting $z = 0$ and $w_1(y) = e^{-sy}$ with the initial surplus $u = 0$, and this equals $b_{\delta,0}\alpha\phi_\delta$ according to their notation. By comparing our notation with theirs, it can be seen that our inter-arrival time density $k(\cdot)$, upward jump density $p_+(\cdot)$, exponential downward jump parameter β, upward jump probability q_+, downward jump probability q_- and force of interest δ_1 are denoted by $f_V(\cdot)$, $g(\cdot)$, α, q, p and δ in their paper. With the above notation, comparison of our (3.4) at $n = m = 0$ with Equation (5.3) of Labbé et al. (2011) first reveals that our Lundberg's root $\kappa_{0,0}$ is equivalent to their $\alpha(1 - \phi_\delta)$. As their $Z_1|Z_1 > 0$ is exponential with parameter α, their Equation (5.2) reduces to $b_{\delta,0} = 1/(\alpha + s)$ and hence $E[e^{-\delta_1\tau-s|U(\tau)|}1\{\tau < \infty\}|U(0) = 0] = \alpha\phi_\delta/(\alpha + s)$, which is $(\beta - \kappa_{0,0})/(\beta + s)$ in our notation. □

To determine the solution form of $\phi_{\delta_{123},n,m}(\cdot)$, we proceed by taking Laplace transforms on both sides of (2.9) and separating the term $(i, j) = (n, m)$ in the summation. This leads us to

$$\begin{aligned}\widetilde{\phi}_{\delta_{123},n,m}(s) = \widetilde{h}_{\delta_1+n\delta_2+m\delta_3,\delta_2,\delta_3,0,0}(s|0)\widetilde{\phi}_{\delta_{123},n,m}(s) + \sum_{(i,j)\in\Gamma(n,m)} \binom{n}{i}\binom{m}{j}\widetilde{h}_{\delta_1+i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j}(s|0)\widetilde{\phi}_{\delta_{123},i,j}(s)\\ + \int_0^\infty e^{-su}\int_u^\infty\int_0^\infty h^*_{\delta_1,\delta_2,\delta_3,n,m}(x, y|0)w_1(y - u)\,dx\,dy\,du.\end{aligned} \tag{3.6}$$

Utilizing (3.5) (with δ_1 replaced by $\delta_1+n\delta_2+m\delta_3$), (3.2) and (3.3), rearrangements of the above equation yield

$$\widetilde{\phi}_{\delta_{123},n,m}(s) = \frac{1}{\kappa_{n,m} + s}\left(\sum_{(i,j)\in\Gamma(n,m)}\binom{n}{i}\binom{m}{j}\sum_{l=0}^{n-i}\frac{D_{\delta_1+i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j,l,1}(0)l!}{(\beta + s)^l}\widetilde{\phi}_{\delta_{123},i,j}(s)+\sum_{l=0}^{n}\frac{E_{\delta_1,\delta_2,\delta_3,n,m,l,1}l!}{(\beta + s)^l}\right). \tag{3.7}$$

Then, recursively in $n, m \in \mathbb{N}$ it is observed that

$$\widetilde{\phi}_{\delta_{123},n,m}(s) = \sum_{i=0}^{n}\sum_{j=0}^{m}\frac{A_{n,m,i,j}}{\kappa_{i,j} + s} + \sum_{i=0}^{n-1}\frac{B_{n,m,i}}{(\beta + s)^{i+1}},$$

for some constants $A_{n,m,i,j}$'s and $B_{n,m,i}$'s, and hence inversion of the Laplace transforms gives rise to

$$\phi_{\delta_{123},n,m}(u) = \sum_{i=0}^{n}\sum_{j=0}^{m}A_{n,m,i,j}e^{-\kappa_{i,j}u} + \sum_{i=0}^{n-1}B_{n,m,i}\frac{u^ie^{-\beta u}}{i!}, \qquad u \geq 0. \tag{3.8}$$

Remark 5 If the values of $i\delta_2+j\delta_3$ coincide for some $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$, then the related Lundberg's roots will also be identical. For example, if $\delta_2 = \delta_3$, then $\kappa_{i,j} = \kappa_{i+j,0} = \kappa_{0,i+j}$ for all $i, j \in \mathbb{N}$.

In such cases, the corresponding exponential terms $e^{-\kappa_{i,j}u}$ and hence their coefficients $A_{n,m,i,j}$ can be combined in the solution form (3.8). Nonetheless, the subsequent results are still valid as it is no less general for us to use the solution (3.8) with all exponential terms separated. $\square$

Next, the solution form of $\varphi_{\delta_{23},n,m}(\cdot)$ can be obtained by taking Laplace transforms on (2.12). Following the same arguments that lead to (3.7), it is found that

$$\widetilde{\varphi}_{\delta_{23},n,m}(s) = \frac{1}{\kappa^*_{n,m}+s}\left(\sum_{(i,j)\in\Gamma(n,m)}\binom{n}{i}\binom{m}{j}\sum_{l=0}^{n-i}\frac{D_{i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j,l,1}(0)l!}{(\beta+s)^l}\widetilde{\varphi}_{\delta_{23},i,j}(s)+\frac{(\beta+s)\varphi_{\delta_{23},n,m}(0)}{s}\right),$$

from which one deduces that

$$\widetilde{\varphi}_{\delta_{23},n,m}(s) = \sum_{i=0}^{n}\sum_{j=0}^{m}\frac{A^*_{n,m,i,j}}{\kappa^*_{i,j}+s}+\sum_{i=0}^{n-1}\frac{B^*_{n,m,i}}{(\beta+s)^{i+1}}+\frac{C^*_{n,m}}{s}$$

for some constants $A^*_{n,m,i,j}$'s, $B^*_{n,m,i}$'s and $C^*_{n,m}$. Therefore, we arrive at

$$\varphi_{\delta_{23},n,m}(u) = \sum_{i=0}^{n}\sum_{j=0}^{m}A^*_{n,m,i,j}e^{-\kappa^*_{i,j}u}+\sum_{i=0}^{n-1}B^*_{n,m,i}\frac{u^ie^{-\beta u}}{i!}+C^*_{n,m}, \qquad u\geq 0. \tag{3.9}$$

In what follows, we shall determine the unknown coefficients appearing in (3.8) and (3.9). In particular, it will be seen that the constants $B_{n,m,i}$'s and $B^*_{n,m,i}$'s therein are indeed zero. The results are stated in Theorems 1-3, which can be proved by mathematical induction. Theorems 1 and 2 are concerned with the Gerber-Shiu function $\phi_{\delta_{123},n,m}(u)$ when $m=0$ and $m\in\mathbb{N}^+$ whereas Theorem 3 involves the joint moment $\varphi_{\delta_{23},n,m}(u)$ for the event of survival of the process. Only the proof of Theorem 1 is provided as the other two proofs are essentially the same as the first. For ease of presentation, the notion of the Dickson-Hipp operator $\mathcal{T}_s$ (see e.g. Dickson and Hipp (2001) and Li and Garrido (2004, Section 3)) will be used. For any integrable function $a(\cdot)$ on $(0,\infty)$ and complex number s with $\Re(s)\geq 0$, it is defined by

$$\mathcal{T}_s a(y) = \int_y^\infty e^{-s(x-y)}a(x)\,dx, \qquad y\geq 0. \tag{3.10}$$

(Note that $\mathcal{T}_s a(0)=\widetilde{a}(s)$.) Then, the multiple Dickson-Hipp operator $\mathcal{T}_s^i$ (for $i\in\mathbb{N}^+$) is given by

$$\mathcal{T}_s^i a(y) = \int_y^\infty \frac{(x-y)^{i-1}e^{-s(x-y)}}{(i-1)!}a(x)\,dx, \qquad y\geq 0.$$

See Li and Garrido (2004, Section 3, Property 5).

Theorem 1 *Suppose that claim amounts are exponential with density $p_-(y)=\beta e^{-\beta y}$, the claim cost function is $f_-(x)=x$ and the penalty function is $w(x,y)=w_1(y)$. For $n\in\mathbb{N}$, the Gerber-Shiu function $\phi_{\delta_{123},n,0}(u)=\phi_{\delta_{12},n}(u)$ defined in* (1.4) *is given by*

$$\phi_{\delta_{12},n}(u) = \sum_{i=0}^{n}A_{n,0,i,0}e^{-\kappa_{i,0}u}, \qquad u\geq 0, \tag{3.11}$$

where $\kappa_{i,0} \in (0,\beta)$ is defined via the Lundberg's equation (3.4)*. The first n constants are obtained directly as*

$$A_{n,0,i,0} = \frac{n!\beta q_- \sum_{j=i}^{n-1} \frac{A_{j,0,i,0}}{j!(\beta-\kappa_{i,0})^{n-j+1}} \widetilde{k}(\delta_1 + n\delta_2 + c\kappa_{i,0})}{1 - \frac{\beta q_-}{\beta-\kappa_{i,0}} \widetilde{k}(\delta_1 + n\delta_2 + c\kappa_{i,0}) - q_+ \widetilde{p}_+(\kappa_{i,0}) \widetilde{k}(\delta_1 + n\delta_2 + c\kappa_{i,0})}, \qquad i = 0, 1, \ldots, n-1, \quad (3.12)$$

recursively in terms of the constants $\{A_{j,0,i,0}\}_{j=i}^{n-1}$ pertaining to the lower-order Gerber-Shiu functions. Then $A_{n,0,n,0}$ can be calculated from

$$\sum_{i=0}^{n} \sum_{j=0}^{i} \frac{A_{i,0,j,0}}{i!(\beta-\kappa_{j,0})^{n-i+1}} = \mathcal{T}_\beta^{n+1} w_1(0) \qquad (3.13)$$

as the only unknown. The recursion starts itself when $n = 0$ with the constant $A_{0,0,0,0}$ computed by (3.13)*.*

Proof. From Labbé et al (2011, Theorem 5.1) (see also Remark 4), Theorem 1 is valid for $n = 0$. Assume that Theorem 1 is true for $n = 0, 1, \ldots, N-1$ for some $N \in \mathbb{N}^+$. We shall substitute $\{\phi_{\delta_{12},i}(u)\}_{i=0}^{N-1}$ from the induction assumption and the solution form of $\phi_{\delta_{12},N}(u)$ from (3.8) into the integral equation (2.1), and show that the statement of Theorem is also true for $n = N$. Since some of the integrals are structurally identical to those in Section 3.1 of Cheung (2013, Section 3.1) certain results therein are directly applicable with a slight change of notation. Using Cheung (2013, Equations (3.9) and (3.7)), the first and second integrals in (2.1) are respectively

$$\begin{aligned}
&\int_0^\infty e^{-(\delta_1+N\delta_2)t} \left(q_- \sum_{j=0}^{N} \binom{N}{j} \int_0^{u+ct} y^{N-j} \phi_{\delta_{12},j}(u+ct-y) p_-(y)\, dy \right) k(t)\, dt \\
&= N!\beta q_- \sum_{i=0}^{N} \left(\sum_{j=i}^{N} \frac{A_{j,0,i,0}}{j!(\beta-\kappa_{i,0})^{N-j+1}} \widetilde{k}(\delta_1 + N\delta_2 + c\kappa_{i,0}) \right) e^{-\kappa_{i,0}u} \\
&\quad - N!\beta q_- \sum_{i=0}^{N} \left(\sum_{j=0}^{N-i} \sum_{l=0}^{j} \sum_{k=i}^{N-j} \frac{A_{j,0,l,0} c^{k-i}}{j!(\beta-\kappa_{l,0})^{N-j+1-k}} \mathcal{T}_{\delta_1+N\delta_2+\beta c}^{k-i+1} k(0) \right) \frac{u^i e^{-\beta u}}{i!} \\
&\quad + \beta q_- \sum_{i=0}^{N} \left(\sum_{k=0 \vee (i-1)}^{N-1} B_{N,0,k} c^{k+1-i} \mathcal{T}_{\delta_1+N\delta_2+\beta c}^{k+2-i} k(0) \right) \frac{u^i e^{-\beta u}}{i!},
\end{aligned} \qquad (3.14)$$

and

$$\begin{aligned}
&\int_0^\infty e^{-(\delta_1+N\delta_2)t} \left(q_- \int_{u+ct}^\infty y^N w_1(y-u-ct) p_-(y)\, dy \right) k(t)\, dt \\
&= N!\beta q_- \sum_{i=0}^{N} \left(\sum_{j=i}^{N} c^{j-i} \mathcal{T}_\beta^{N-j+1} w_1(0) \right) \mathcal{T}_{\delta_1+N\delta_2+\beta c}^{j-i+1} k(0) \frac{u^i e^{-\beta u}}{i!}.
\end{aligned} \qquad (3.15)$$

In (3.14), we have used the notation $a \vee b = \max(a,b)$. To evaluate the third integral in (2.1), we note that

$$\begin{aligned}
&\int_0^\infty \phi_{\delta_{12},N}(u+ct+y) p_+(y)\, dy \\
&= \int_0^\infty \left(\sum_{i=0}^{N} A_{N,0,i,0} e^{-\kappa_{i,0}(u+ct+y)} + \sum_{i=0}^{N-1} B_{N,0,i} \frac{(u+ct+y)^i e^{-\beta(u+ct+y)}}{i!} \right) p_+(y)\, dy
\end{aligned}$$

$$= \sum_{i=0}^{N} A_{N,0,i,0}\widetilde{p}_+(\kappa_{i,0})e^{-\kappa_{i,0}ct}e^{-\kappa_{i,0}u} + \sum_{i=0}^{N-1} B_{N,0,i}\sum_{j=0}^{i}\left(\int_0^\infty \frac{y^{i-j}e^{-\beta y}}{(i-j)!}p_+(y)\,dy\right)\frac{(u+ct)^j e^{-\beta(u+ct)}}{j!}$$

$$= \sum_{i=0}^{N} A_{N,0,i,0}\widetilde{p}_+(\kappa_{i,0})e^{-\kappa_{i,0}ct}e^{-\kappa_{i,0}u} + \sum_{i=0}^{N-1} B_{N,0,i}\sum_{j=0}^{i}\mathcal{T}_\beta^{i-j+1}p_+(0)\sum_{k=0}^{j}c^{j-k}\frac{t^{j-k}e^{-\beta ct}}{(j-k)!}\frac{u^k e^{-\beta u}}{k!},$$

and hence

$$\int_0^\infty e^{-(\delta_1+N\delta_2)t}\left(q_+\int_0^\infty \phi_{\delta_{12},N}(u+ct+y)p_+(y)\,dy\right)k(t)\,dt$$

$$= q_+\sum_{i=0}^{N} A_{N,0,i,0}\widetilde{p}_+(\kappa_{i,0})\widetilde{k}(\delta_1+N\delta_2+c\kappa_{i,0})e^{-\kappa_{i,0}u} + q_+\sum_{i=0}^{N-1} B_{N,0,i}\sum_{j=0}^{i}\mathcal{T}_\beta^{i-j+1}p_+(0)\sum_{k=0}^{j}c^{j-k}\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{j-k+1}k(0)\frac{u^k e^{-\beta u}}{k!}$$

$$= q_+\sum_{i=0}^{N} A_{N,0,i,0}\widetilde{p}_+(\kappa_{i,0})\widetilde{k}(\delta_1+N\delta_2+c\kappa_{i,0})e^{-\kappa_{i,0}u} + q_+\sum_{i=0}^{N-1}\sum_{k=i}^{N-1}\sum_{j=i}^{k} B_{N,0,k}c^{j-i}\mathcal{T}_\beta^{k-j+1}p_+(0)\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{j-i+1}k(0)\frac{u^i e^{-\beta u}}{i!}. \tag{3.16}$$

According to (2.1), the sum of (3.14)-(3.16) is equal to (3.8) (with $(n,m)=(N,0)$). By equating the coefficients of $e^{-\kappa_{i,0}u}$, we get

$$A_{N,0,i,0} = N!\beta q_-\sum_{j=i}^{N}\frac{A_{j,0,i,0}}{j!(\beta-\kappa_{i,0})^{N-j+1}}\widetilde{k}(\delta_1+N\delta_2+c\kappa_{i,0}) + q_+A_{N,0,i,0}\widetilde{p}_+(\kappa_{i,0})\widetilde{k}(\delta_1+N\delta_2+c\kappa_{i,0})$$

for $i=0,1,\ldots,N$. When $i=N$, the above equation must be automatically true because of (3.4). When $i=0,1,\ldots,N-1$, by rearrangement one asserts that (3.12) holds true for $n=N$.

Next, from the coefficients of $u^N e^{-\beta u}/N!$, we arrive at

$$0 = N!\beta q_-\mathcal{T}_\beta w_1(0)\mathcal{T}_{\delta_1+N\delta_2+\beta c}k(0) - N!\beta q_-\frac{A_{0,0,0,0}}{\beta-\kappa_{0,0}}\mathcal{T}_{\delta_1+N\delta_2+\beta c}k(0) + \beta q_- B_{N,0,N-1}\mathcal{T}_{\delta_1+N\delta_2+\beta c}k(0),$$

and therefore

$$B_{N,0,N-1} = N!\left(\frac{A_{0,0,0,0}}{\beta-\kappa_{0,0}} - \mathcal{T}_\beta w_1(0)\right) = 0, \tag{3.17}$$

where the last equality follows from validity of Theorem 1 at $n=0$. Meanwhile, for $i=0,1,\ldots,N-1$ the coefficients of $u^i e^{-\beta u}/i!$ imply (after rearrangement) that

$$B_{N,0,i} - \beta q_-\sum_{k=0\vee(i-1)}^{N-1} B_{N,0,k}c^{k+1-i}\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{k+2-i}k(0) - q_+\sum_{k=i}^{N-1}\sum_{j=i}^{k} B_{N,0,k}c^{j-i}\mathcal{T}_\beta^{k-j+1}p_+(0)\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{j-i+1}k(0)$$

$$= N!\beta q_-\sum_{j=i}^{N}c^{j-i}\mathcal{T}_\beta^{N-j+1}w_1(0)\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{j-i+1}k(0) - N!\beta q_-\sum_{j=0}^{N-i}\sum_{l=0}^{j}\sum_{k=i}^{N-j}\frac{A_{j,0,l,0}c^{k-i}}{j!(\beta-\kappa_{l,0})^{N-j+1-k}}\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{k-i+1}k(0)$$

$$= N!\beta q_-\sum_{k=i}^{N}c^{k-i}\mathcal{T}_\beta^{N-k+1}w_1(0)\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{k-i+1}k(0) - N!\beta q_-\sum_{k=i}^{N}\sum_{j=0}^{N-k}\sum_{l=0}^{j}\frac{A_{j,0,l,0}c^{k-i}}{j!(\beta-\kappa_{l,0})^{N-j+1-k}}\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{k-i+1}k(0)$$

$$= N!\beta q_-\sum_{k=i}^{N}c^{k-i}\mathcal{T}_{\delta_1+N\delta_2+\beta c}^{k-i+1}k(0)\left(\mathcal{T}_\beta^{N-k+1}w_1(0) - \sum_{j=0}^{N-k}\sum_{l=0}^{j}\frac{A_{j,0,l,0}}{j!(\beta-\kappa_{l,0})^{N-j+1-k}}\right). \tag{3.18}$$

Now, we aim at using (3.18) to show that $B_{N,0,i}=0$ for $i=0,1,\dots,N-1$. Since it is already known from (3.17) that $B_{1,0,0}=0$ for $N=1$, we can restrict ourselves to the case $N\geq 2$ and then $\{1,2,\dots,N-1\}$ is non-empty. For $i=1,2,\dots,N-1$, the expression inside the big brackets on the right-hand side of (3.18) equals zero according to the induction assumption, and therefore

$$B_{N,0,i}-\beta q_-\sum_{k=i-1}^{N-1}B_{N,0,k}c^{k+1-i}\mathcal{T}^{k+2-i}_{\delta_1+N\delta_2+\beta c}k(0)-q_+\sum_{k=i}^{N-1}\sum_{j=i}^{k}B_{N,0,k}c^{j-i}\mathcal{T}^{k-j+1}_{\beta}p_+(0)\mathcal{T}^{j-i+1}_{\delta_1+N\delta_2+\beta c}k(0)=0. \tag{3.19}$$

Putting $i=N-1$ and noting that $B_{N,0,N-1}=0$ from (3.17), it is immediate that $B_{N,0,N-2}=0$. If $N\geq 3$ then we further insert $i=N-2$ into (3.19) to arrive at $B_{N,0,N-3}=0$. The procedure is repeated until we reach $B_{N,0,0}=0$, and one concludes that $B_{N,0,i}=0$ for $i=0,1,\dots,N-1$. Plugging these into (3.8) (with $(n,m)=(N,0)$), it is confirmed that the solution form (3.11) is valid when $n=N$.

Finally, noting that all $B_{N,0,i}$'s are zero, we substitute $i=0$ into (3.18) to get

$$\begin{aligned}0&=N!\beta q_-\sum_{k=0}^{N}c^k\mathcal{T}^{k+1}_{\delta_1+N\delta_2+\beta c}k(0)\left(\mathcal{T}^{N-k+1}_{\beta}w_1(0)-\sum_{j=0}^{N-k}\sum_{l=0}^{j}\frac{A_{j,0,l,0}}{j!(\beta-\kappa_{l,0})^{N-j+1-k}}\right)\\&=N!\beta q_-\mathcal{T}_{\delta_1+N\delta_2+\beta c}k(0)\left(\mathcal{T}^{N+1}_{\beta}w_1(0)-\sum_{j=0}^{N}\sum_{l=0}^{j}\frac{A_{j,0,l,0}}{j!(\beta-\kappa_{l,0})^{N-j+1}}\right),\end{aligned}$$

where the last line follows from the induction assumption that the terms in the summation equal zero when $k=1,2,\dots,N$. Thus, it is clear that (3.13) holds true when $n=N$. Combining all the above results, Theorem 1 is true for all $n\in\mathbb{N}$ by mathematical induction. ■

Theorem 2 *Suppose that claim amounts are exponential with density $p_-(y)=\beta e^{-\beta y}$, the claim cost function is $f_-(x)=x$ and the penalty function is $w(x,y)=w_1(y)$. For $n\in\mathbb{N}$ and $m\in\mathbb{N}^+$, the Gerber-Shiu function $\phi_{\delta_{123},n,m}(u)$ defined in (1.4) is given by*

$$\phi_{\delta_{123},n,m}(u)=\sum_{i=0}^{n}\sum_{j=0}^{m}A_{n,m,i,j}e^{-\kappa_{i,j}u},\qquad u\geq 0,$$

where $\kappa_{i,j}\in(0,\beta)$ is defined via the Lundberg's equation (3.4). The $(n+1)(m+1)-1$ constants $\{A_{n,m,i,j}:(i,j)\in\Gamma(n,m)\}$ are first solved from the system of $(n+1)(m+1)-1$ linear equations

$$\begin{aligned}A_{n,m,i,l}=\;&n!\beta q_-\sum_{j=i}^{n}\frac{A_{j,m,i,l}}{j!(\beta-\kappa_{i,l})^{n-j+1}}\widetilde{k}(\delta_1+n\delta_2+m\delta_3+c\kappa_{i,l})\\&+q_+\sum_{j=l}^{m}\binom{m}{j}A_{n,j,i,l}\widetilde{\zeta}_{m-j}(\kappa_{i,l})\widetilde{k}(\delta_1+n\delta_2+m\delta_3+c\kappa_{i,l}),\qquad (i,l)\in\Gamma(n,m),\end{aligned}$$

recursively in terms of the constants pertaining to the lower-order Gerber-Shiu functions (and this requires the application of Theorem 1 as a starting point), where $\widetilde{\zeta}_{m-j}(\cdot)$ is the Laplace transform of $\zeta_{m-j}(y)=f_+^{m-j}(y)p_+(y)$. Then $A_{n,m,n,m}$ can be calculated from

$$\sum_{l=0}^{m}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{A_{i,m,j,l}}{i!(\beta-\kappa_{j,l})^{n-i+1}}=0$$

as the only unknown.

Theorem 3 *Suppose that claim amounts are exponential with density $p_-(y) = \beta e^{-\beta y}$ and the claim cost function is $f_-(x) = x$. For $n, m \in \mathbb{N}$, the joint moment defined in* (1.5) *is given by*

$$\varphi_{\delta_{23},n,m}(u) = \sum_{i=0}^{n} \sum_{j=0}^{m} A^*_{n,m,i,j} e^{-\kappa^*_{i,j} u} + C^*_{n,m}, \qquad u \geq 0, \tag{3.20}$$

where $\kappa^*_{i,j} \in (0, \beta)$ is defined via the Lundberg's equation (3.4) *with $\delta_1 = 0$. The $(n+1)(m+1)-1$ constants $\{A^*_{n,m,i,j} : (i,j) \in \Gamma(n,m)\}$ are first solved from the system of $(n+1)(m+1)-1$ linear equations*

$$\begin{aligned} A^*_{n,m,i,l} = {} & n!\beta q_- \sum_{j=i}^{n} \frac{A^*_{j,m,i,l}}{j!(\beta - \kappa^*_{i,l})^{n-j+1}} \widetilde{k}(n\delta_2 + m\delta_3 + c\kappa^*_{i,l}) \\ & + q_+ \sum_{j=l}^{m} \binom{m}{j} A^*_{n,j,i,l} \widetilde{\zeta}_{m-j}(\kappa^*_{i,l}) \widetilde{k}(n\delta_2 + m\delta_3 + c\kappa^*_{i,l}), \qquad (i,l) \in \Gamma(n,m), \end{aligned}$$

recursively in terms of the constants pertaining to the lower-order joint moments. Then $A^*_{n,m,n,m}$ can be calculated from

$$\sum_{l=0}^{m} \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{A^*_{i,m,j,l}}{i!(\beta - \kappa^*_{j,l})^{n-i+1}} + \sum_{i=0}^{n} \frac{C^*_{i,m}}{i!\beta^{n-i+1}} = 0.$$

as the only unknown. For $(n,m) \neq (0,0)$, the constant $C^*_{n,m}$ can be computed recursively via

$$C^*_{n,m} = \frac{\widetilde{k}(n\delta_2 + m\delta_3)}{1 - \widetilde{k}(n\delta_2 + m\delta_3)} \left(q_- \sum_{i=0}^{n-1} \binom{n}{i} \frac{(n-i)!}{\beta^{n-i}} C^*_{i,m} + q_+ \sum_{i=0}^{m-1} \binom{m}{i} E[f_+^{m-i}(Y_{+,1})] C^*_{n,i} \right), \tag{3.21}$$

with starting point $C^*_{0,0} = 1$.

Remark 6 Theorem 1 can be regarded as a generalization of the results in Cheung (2013, Section 3.1) in that the current model allows for the possibility of upward jumps. In addition, Cheung (2013, Section 3.1) showed that (in the absence of upward jumps) the constants $B_{n,0,i}$'s in the solution form (3.8) satisfy a linear system of equations but did not prove that they are indeed all equal to zero. $\square$

Remark 7 For $n, m \in \mathbb{N}$ and $\delta_2, \delta_3 > 0$, define the quantity

$$\theta_{n,m} = E\left[\left(\sum_{i=1}^{\infty} e^{-\delta_2 T_{-,i}} f_-(Y_{-,i}) \right)^n \left(\sum_{i=1}^{\infty} e^{-\delta_3 T_{+,i}} f_+(Y_{+,i}) \right)^m \right]. \tag{3.22}$$

For general cost functions $f_-(\cdot)$ and $f_+(\cdot)$, it follows in almost an identical manner to the proof of Cheung and Liu (2016, Lemma 2) that under the positive loading condition one has $\lim_{u\to\infty} \varphi_{\delta_{23},n,m}(u) = \theta_{n,m}$ as long as the expectation (3.22) is finite (which is true if $E[f_-^n(Y_{-,1})]$ and $E[f_+^m(Y_{+,1})]$ are finite). On the other hand, the solution (3.20) under the assumptions in Theorem 3 implies that $\lim_{u\to\infty} \varphi_{\delta_{23},n,m}(u) = C^*_{n,m}$ since $\kappa^*_{i,j} \in (0, \beta)$, and therefore we must have that $C^*_{n,m} = \theta_{n,m}$. Indeed, by conditioning on whether the first jump Y_1 is a downward or upward jump, it is observed that $\theta_{n,m}$ for $(n,m) \neq (0,0)$ satisfies the recursion

$$\theta_{n,m} = q_- \sum_{i=0}^{n} \binom{n}{i} \widetilde{k}(n\delta_2 + m\delta_3) E[f_-^{n-i}(Y_{-,1})] \theta_{i,m} + q_+ \sum_{i=0}^{m} \binom{m}{i} \widetilde{k}(n\delta_2 + m\delta_3) E[f_+^{m-i}(Y_{+,1})] \theta_{n,i},$$

where the trivial starting point $\theta_{0,0} = 1$ is obtainable by putting $n = m = 0$ in (3.22). Because $E[f_-^{n-i}(Y_{-,1})] = E[Y_{-,1}^{n-i}] = (n-i)!/\beta^{n-i}$ under the cost function $f_-(x) = x$ and the exponential downward jump assumption, rearranging the above equation reveals that $\theta_{n,m}$ satisfies the same recursion (3.21) as $C^*_{n,m}$ does. $\square$

4 Joint moments of numbers of upward and downward jumps

In this section, we shall focus on the numbers of upward and downward jumps $N_+(\tau)$ and $N_-(\tau)$, which can be retrieved from $Z_{+,\delta_2}(\tau)$ and $Z_{-,\delta_3}(\tau)$ respectively by letting $f_+(\cdot) \equiv f_-(\cdot) \equiv 1$ and $\delta_2 = \delta_3 = 0$. Since $\{N_+(t)\}_{t\geq 0}$ and $\{N_-(t)\}_{t\geq 0}$ are renewal processes, $N_+(\tau)$ and $N_-(\tau)$ are both infinite almost surely on the set $\{\tau = \infty\}$. Hence, we only consider the Gerber-Shiu function $\phi_{\delta_{123},n,m}(u)$ but not the joint moment $\varphi_{\delta_{23},n,m}(u)$. It is further assumed that $w(x,y) = w_1(y)$. As δ_2 and δ_3 are no longer present in $\phi_{\delta_{123},n,m}(u)$, for convenience we write $\delta_1 = \delta$ and denote $\phi_{\delta_{123},n,m}(u)$ by $\phi_{\delta,n,m}(u)$ so that

$$\phi_{\delta,n,m}(u) = E[e^{-\delta\tau} N_-^n(\tau) N_+^m(\tau) w_1(|U(\tau)|) 1\{\tau < \infty\} | U(0) = u]. \tag{4.1}$$

Similarly, the moment-based discounted densities $h^*_{\delta_1,\delta_2,\delta_3,n,m}(x,y|u)$ and $h_{\delta_1,\delta_2,\delta_3,n,m}(y|u)$ are written as $h^*_{\delta,n,m}(x,y|u)$ and $h_{\delta,n,m}(y|u)$ respectively. Furthermore, $\phi_{\delta,n,m}(u)$ is abbreviated as $\phi_{\delta,n}(u)$ when $m = 0$.

4.1 Note on the discounted density $h_{\delta,n,m}(y|u)$

Assume that the density of the downward jump size admits the same factorization as (3.1). Analogous to the derivations of (3.2) and (3.3) (with some integrals changed to summations as $N_+(\tau)$ and $N_-(\tau)$ are discrete random variables), it is found that

$$h_{\delta,n,m}(y|u) = \sum_{k=1}^{r} D_{\delta,n,m,k}(u)\gamma_k(y) \tag{4.2}$$

and

$$\int_u^\infty \int_0^\infty h^*_{\delta,n,m}(x,y|0) w_1(y-u)\, dx\, dy = \sum_{k=1}^{r} E_{\delta,n,m,k}\gamma_k(u) \tag{4.3}$$

for some constants $D_{\delta,n,m,k}(u)$'s and $E_{\delta,n,m,k}$'s.

4.2 Detailed analysis when claims follow a combination of exponentials

In the remaining part of this section, we assume that each downward jump is distributed as a combination of exponentials with density

$$p_-(y) = \sum_{k=1}^{r} \chi_k \beta_k e^{-\beta_k y}, \tag{4.4}$$

where $\beta_k > 0$ for $k = 1, 2, \ldots, r$ and $\sum_{k=1}^r \chi_k = 1$. Without loss of generality, it is assumed that all β_k's are distinct and all χ_k's are non-zero. The class of combinations of exponentials is 'dense' such that it can approximate arbitrarily accurately any continuous distribution on $(0,\infty)$ (see e.g. Dufresne (2007) for its fitting). The density (4.4) satisfies (3.1), and one has that $\eta_k(x) = \chi_k\beta_k e^{-\beta_k x}$ and $\gamma_k(x) = e^{-\beta_k x}$. Note that the intermediate result (3.6) is still applicable with $\widetilde{h}_{\delta_1+i\delta_2+j\delta_3,\delta_2,\delta_3,n-i,m-j}(s|0)$ replaced by $\widetilde{h}_{\delta,n-i,m-j}(s|0)$ (for $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,m$), $h^*_{\delta_1,\delta_2,\delta_3,n,m}(x,y|0)$ by $h^*_{\delta,n,m}(x,y|0)$, and $\widetilde{\phi}_{\delta_{123},n,m}(s)$ by $\widetilde{\phi}_{\delta,n,m}(s)$. Rearrangements along with the use of (4.2) and (4.3) lead to

$$\widetilde{\phi}_{\delta,n,m}(s) = \frac{1}{1-\widetilde{h}_{\delta,0,0}(s|0)} \left(\sum_{(i,j)\in\Gamma(n,m)} \binom{n}{i}\binom{m}{j} \sum_{k=1}^{r} \frac{D_{\delta,n-i,m-j,k}(0)}{\beta_k + s} \widetilde{\phi}_{\delta,i,j}(s) + \sum_{k=1}^{r} \frac{E_{\delta,n,m,k}}{\beta_k + s} \right). \tag{4.5}$$

Using (4.2), one can write

$$\frac{1}{1-\widetilde{h}_{\delta,0,0}(s|0)} = \frac{1}{1-\sum_{k=1}^{r}\frac{D_{\delta,0,0,k}(0)}{\beta_k+s}} = \prod_{k=1}^{r}\frac{\beta_k+s}{R_k+s}, \tag{4.6}$$

where $\{-R_k\}_{k=1}^{r}$ are the r roots of

$$\prod_{k=1}^{r}(\beta_k+\xi) - \sum_{k=1}^{r} D_{\delta,0,0,k}(0)\prod_{i\neq k}^{r}(\beta_i+\xi) = 0$$

which is a polynomial equation in ξ. The roots of the above equation is equivalent to those of $\widetilde{h}_{\delta,0,0}(\xi|0) = 1$. By writing

$$\left|\widetilde{h}_{\delta,0,0}(\xi|0)\right| \leq \int_0^\infty |e^{-\xi y}| h_{\delta,0,0}(y|0)\,dy = \int_0^\infty e^{-\Re(\xi)y} h_{\delta,0,0}(y|0)\,dy = E[e^{-\delta\tau-\Re(\xi)|U(\tau)|}1\{\tau<\infty\}|U(0)=0]$$

and noting that the last expectation is strictly less than one for $\Re(\xi)\geq 0$ when $\delta>0$ or the positive loading condition holds, it is observed that $|\widetilde{h}_{\delta,0,0}(\xi|0)| < 1$ for $\Re(\xi)\geq 0$. Consequently, the roots of $\widetilde{h}_{\delta,0,0}(\xi|0) = 1$ must have negative real parts, i.e. $\Re(R_k) > 0$ for $k = 1, 2, \ldots, r$. Now, through substitution of (4.6) into (4.5), for $n, m \in \mathbb{N}$ it can be deduced inductively that

$$\widetilde{\phi}_{\delta,n,m}(s) = \sum_{k=1}^{r}\sum_{i=0}^{n+m}\frac{A_{n,m,i,k}}{(R_k+s)^{i+1}}$$

and therefore the solution form of $\phi_{\delta,n,m}(u)$ is given by

$$\phi_{\delta,n,m}(u) = \sum_{k=1}^{r}\sum_{i=0}^{n+m} A_{n,m,i,k}\frac{u^i e^{-R_k u}}{i!}, \qquad u \geq 0, \tag{4.7}$$

where $A_{n,m,i,k}$'s are constants to be determined.

To find the constants in (4.7), similar to the proof of Theorem 1 we proceed by putting the solution form (4.7) into (2.1) and (2.2) when $m = 0$ and $m \in \mathbb{N}^+$ respectively. This leads to Theorems 4 and 5 below. We only provide the proof of the slightly more complicated Theorem 5. It will be seen that $\{R_k\}_{k=1}^{r}$ are roots of the Lundberg's equation (in ξ)

$$\left(q_-\sum_{k=1}^{r}\frac{\chi_k\beta_k}{\beta_k-\xi} + q_+\widetilde{p}_+(\xi)\right)\widetilde{k}(\delta+c\xi) = 1, \tag{4.8}$$

which is known to have exactly r roots with positive real parts (e.g. Cheung (2011, p.16)).

Theorem 4 *Suppose that claim amounts follow a combination of exponentials with density* (4.4). *For* $n \in \mathbb{N}$, *the Gerber-Shiu function* $\phi_{\delta,n,0}(u) = \phi_{\delta,n}(u)$ *defined via* (4.1) *is given by* (4.7) *(with* $m = 0$*), where* $\{R_k\}_{k=1}^{r}$ *are the* r *roots of the Lundberg's equation* (4.8) *with positive real parts. The* $(n+1)r$ *constants* $\{A_{n,0,i,k} : i = 0, 1, \ldots, n; k = 1, 2, \ldots, r\}$ *can be solved from the system of* $(n+1)r$ *linear equations comprising*

$$A_{n,0,i,a} = q_-\sum_{k=1}^{r}\chi_k\beta_k\sum_{j=i}^{n}\sum_{b=i}^{j}\sum_{l=i}^{b}\binom{n}{j}\frac{(-1)^{b+l}A_{j,0,b,a}c^{l-i}}{(\beta_k-R_a)^{b+1-l}}\mathcal{T}_{\delta+cR_a}^{l-i+1}k(0)$$

$$+ q_+ \sum_{b=i}^{n} \sum_{l=i}^{b} A_{n,0,b,a} \mathcal{T}_{R_a}^{b-l+1} p_+(0) c^{l-i} \mathcal{T}_{\delta+cR_a}^{l-i+1} k(0),$$

for $i = 0, 1, \ldots, n-1$ and $a = 1, 2, \ldots, r$, together with

$$\sum_{k=1}^{r} \sum_{i=0}^{n} A_{n,0,i,k} \frac{(-1)^{i+1}}{(\beta_a - R_k)^{i+1}} + \widetilde{w}_1(\beta_a) 1\{n = 0\} = 0, \tag{4.9}$$

for $a = 1, 2, \ldots, r$. The procedure is recursive in terms of the constants pertaining to the lower-order Gerber-Shiu functions. The recursion starts itself when $n = 0$ with the constant $A_{0,0,0,0}$ computed by (4.9).

Theorem 5 *Suppose that claim amounts follow a combination of exponentials with density (4.4). For $n \in \mathbb{N}$ and $m \in \mathbb{N}^+$, the Gerber-Shiu function $\phi_{\delta,n,m}(u)$ defined in (4.1) is given by (4.7), where $\{R_k\}_{k=1}^r$ are the r roots of the Lundberg's equation (4.8) with positive real parts. The $(n+m+1)r$ constants $\{A_{n,m,i,k} : i = 0, 1, \ldots, n+m; k = 1, 2, \ldots, r\}$ can be solved from the system of $(n+m+1)r$ linear equations comprising*

$$\begin{aligned} A_{n,m,i,a} = q_- \sum_{k=1}^{r} \chi_k \beta_k & \sum_{j=(i-m)\vee 0}^{n} \sum_{b=i}^{j+m} \sum_{l=i}^{b} \binom{n}{j} \frac{(-1)^{b+l} A_{j,m,b,a} c^{l-i}}{(\beta_k - R_a)^{b+1-l}} \mathcal{T}_{\delta+cR_a}^{l-i+1} k(0) \\ & + q_+ \sum_{j=(i-n)\vee 0}^{m} \binom{m}{j} \sum_{b=i}^{n+j} \sum_{l=i}^{b} A_{n,j,b,a} \mathcal{T}_{R_a}^{b-l+1} p_+(0) c^{l-i} \mathcal{T}_{\delta+cR_a}^{l-i+1} k(0), \end{aligned} \tag{4.10}$$

for $i = 0, 1, \ldots, n+m-1$ and $a = 1, 2, \ldots, r$, together with

$$\sum_{k=1}^{r} \sum_{i=0}^{n+m} A_{n,m,i,k} \frac{(-1)^{i+1}}{(\beta_a - R_k)^{i+1}} = 0, \tag{4.11}$$

for $a = 1, 2, \ldots, r$. The procedure is recursive in terms of the constants pertaining to the lower-order Gerber-Shiu functions (and this requires the application of Theorem 4 as a starting point).

Proof. Substituting (4.4) and (4.7), the first integral on the right-hand side of (2.2) is almost identical to Equation (3.21) in Cheung (2013). Quoting the result therein with slight modifications, we arrive at

$$\begin{aligned} & \int_0^\infty e^{-\delta t} \left(q_- \sum_{j=0}^{n} \binom{n}{j} \int_0^{u+ct} \phi_{\delta,j,m}(u+ct-y) p_-(y)\, dy \right) k(t)\, dt \\ &= q_- \sum_{a=1}^{r} \chi_a \beta_a \sum_{k=1}^{r} \sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{j+m} A_{j,m,i,k} \frac{(-1)^{i+1}}{(\beta_a - R_k)^{i+1}} \widetilde{k}(\delta + \beta_a c) e^{-\beta_a u} \\ &\quad + q_- \sum_{a=1}^{r} \sum_{i=0}^{n+m} \left(\sum_{k=1}^{r} \chi_k \beta_k \sum_{j=(i-m)\vee 0}^{n} \sum_{b=i}^{j+m} \sum_{l=i}^{b} \binom{n}{j} \frac{(-1)^{b+l} A_{j,m,b,a} c^{l-i}}{(\beta_k - R_a)^{b+1-l}} \mathcal{T}_{\delta+cR_a}^{l-i+1} k(0) \right) \frac{u^i e^{-R_a u}}{i!}. \end{aligned} \tag{4.12}$$

For the second term in (2.2), we first examine

$$\int_0^\infty \phi_{\delta,n,j}(u+ct+y) p_+(y)\, dy = \int_0^\infty \left(\sum_{k=1}^{r} \sum_{i=0}^{n+j} A_{n,j,i,k} \frac{(u+ct+y)^i e^{-R_k(u+ct+y)}}{i!} \right) p_+(y)\, dy$$

$$
\begin{aligned}
&= \sum_{k=1}^{r}\sum_{i=0}^{n+j} A_{n,j,i,k} \sum_{l=0}^{i} \left(\int_0^\infty \frac{y^{i-l} e^{-R_k y}}{(i-l)!} p_+(y)\, dy \right) \frac{(u+ct)^l e^{-R_k(u+ct)}}{l!} \\
&= \sum_{k=1}^{r}\sum_{i=0}^{n+j} A_{n,j,i,k} \sum_{l=0}^{i} \mathcal{T}_{R_k}^{i-l+1} p_+(0) \sum_{b=0}^{l} c^{l-b} \frac{t^{l-b} e^{-R_k ct}}{(l-b)!} \frac{u^b e^{-R_k u}}{b!} \\
&= \sum_{k=1}^{r}\sum_{i=0}^{n+j}\sum_{l=0}^{i}\sum_{b=0}^{l} A_{n,j,i,k} \mathcal{T}_{R_k}^{i-l+1} p_+(0) c^{l-b} \frac{t^{l-b} e^{-R_k ct}}{(l-b)!} \frac{u^b e^{-R_k u}}{b!}.
\end{aligned}
$$

Thus, one has that

$$
\begin{aligned}
&\int_0^\infty e^{-\delta t} \left(\int_0^\infty \phi_{\delta,n,j}(u+ct+y) p_+(y)\, dy \right) k(t)\, dt = \sum_{k=1}^{r}\sum_{i=0}^{n+j}\sum_{l=0}^{i}\sum_{b=0}^{l} A_{n,j,i,k} \mathcal{T}_{R_k}^{i-l+1} p_+(0) c^{l-b} \mathcal{T}_{\delta+cR_k}^{l-b+1} k(0) \frac{u^b e^{-R_k u}}{b!} \\
&= \sum_{k=1}^{r}\sum_{b=0}^{n+j}\sum_{i=b}^{n+j}\sum_{l=b}^{i} A_{n,j,i,k} \mathcal{T}_{R_k}^{i-l+1} p_+(0) c^{l-b} \mathcal{T}_{\delta+cR_k}^{l-b+1} k(0) \frac{u^b e^{-R_k u}}{b!} \\
&= \sum_{a=1}^{r}\sum_{i=0}^{n+j}\sum_{b=i}^{n+j}\sum_{l=i}^{b} A_{n,j,b,a} \mathcal{T}_{R_a}^{b-l+1} p_+(0) c^{l-i} \mathcal{T}_{\delta+cR_a}^{l-i+1} k(0) \frac{u^i e^{-R_a u}}{i!},
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_0^\infty e^{-\delta t} \left(q_+ \sum_{j=0}^{m} \binom{m}{j} \int_0^\infty \phi_{\delta,n,j}(u+ct+y) p_+(y)\, dy \right) k(t)\, dt \\
&= q_+ \sum_{a=1}^{r}\sum_{i=0}^{n+m} \left(\sum_{j=(i-n)\vee 0}^{m} \binom{m}{j} \sum_{b=i}^{n+j}\sum_{l=i}^{b} A_{n,j,b,a} \mathcal{T}_{R_a}^{b-l+1} p_+(0) c^{l-i} \mathcal{T}_{\delta+cR_a}^{l-i+1} k(0) \right) \frac{u^i e^{-R_a u}}{i!} \qquad (4.13)
\end{aligned}
$$

upon changing the order of summations $\sum_{j=0}^{m}$ and $\sum_{i=0}^{n+j}$. As (4.7) equals the sum of (4.12) and (4.13) because of (2.2), comparing the coefficients of $u^i e^{-R_a u}/i!$ reveals that (4.10) holds true for $i = 0, 1, \ldots, n+m$ and $a = 1, 2, \ldots, r$. In particular, when $i = n+m$, this implies that each R_a (for $a = 1, 2, \ldots, r$) satisfies the Lundberg's equation (4.8).

Next, equating the coefficients of $e^{-\beta_a u}$ leads to

$$
\sum_{k=1}^{r}\sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{j+m} A_{j,m,i,k} \frac{(-1)^{i+1}}{(\beta_a - R_k)^{i+1}} = 0. \qquad (4.14)
$$

for $a = 1, 2, \ldots, r$. The above equation holds true for $n \in \mathbb{N}$, and in particular putting $n = 0$ proves that (4.11) holds true when $n = 0$. And then substitution of $n = 1$ into (4.14) together with the validity of (4.11) at $n = 0$ proves (4.11) at $n = 1$. Recursively, it is clear that (4.11) is true for $n \in \mathbb{N}$. ■

5 Numerical illustrations

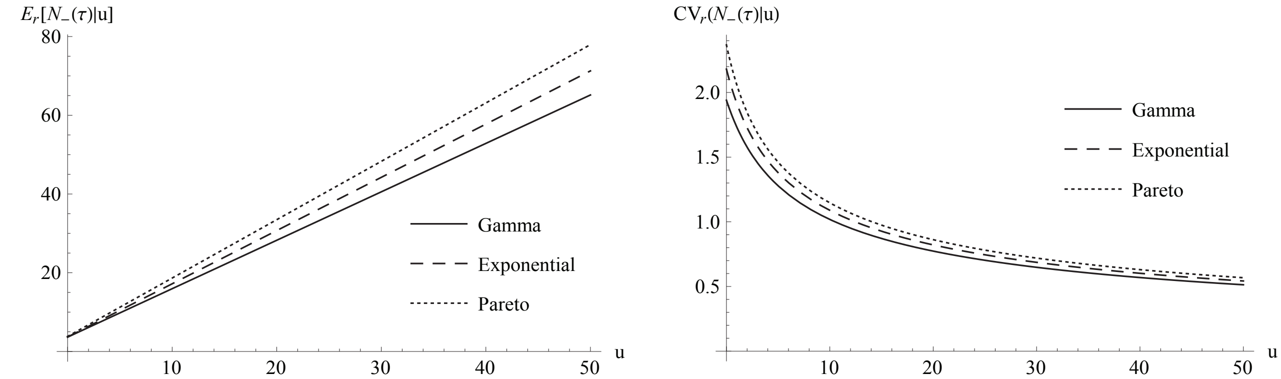
In this section, we shall compute some moment-based quantities involving (1) the number of upward jumps $N_+(\tau)$ and number of downward jumps $N_-(\tau)$ until ruin; and (2) the total discounted gains

$\sum_{i=1}^{N_+(\tau)} e^{-\delta_3 T_{+,i}} Y_{+,i}$ and total discounted losses $\sum_{i=1}^{N_-(\tau)} e^{-\delta_2 T_{-,i}} Y_{-,i}$ until ruin. For notational convenience, we shall use $E[X|u]$ to denote the expected value of a random variable X under an initial surplus $U(0) = u$. Similarly, the conditional expectation of a random variable X given the event of ruin $\{\tau < \infty\}$ (resp. the event of survival $\{\tau = \infty\}$) will be denoted by $E_r[X|u]$ (resp. $E_s[X|u]$), which can be calculated as $E_r[X|u] = E[X1\{\tau < \infty\}|u]/\psi(u)$ (resp. $E_s[X|u] = E[X1\{\tau = \infty\}|u]/(1 - \psi(u))$. The variance of X is $\mathrm{Var}_\bullet(X|u) = E_\bullet[X^2|u] - (E_\bullet[X|u])^2$, where $E_\bullet$ may represent the unconditional expectation E (see (1.6)) or the conditional expectation E_r or E_s. Then the coefficient of variation is $\mathrm{CV}_\bullet(X|u) = \sqrt{\mathrm{Var}_\bullet(X|u)}/E_\bullet[X|u]$. Further denoting the covariance of two random variables X_1 and X_2 by $\mathrm{Cov}_\bullet(X_1, X_2|u) = E_\bullet[X_1 X_2|u] - E_\bullet[X_1|u]E_\bullet[X_2|u]$, the corresponding correlation is given by $\mathrm{Corr}_\bullet(X_1, X_2|u) = \mathrm{Cov}_\bullet(X_1, X_2|u)/\sqrt{\mathrm{Var}_\bullet(X_1|u)\mathrm{Var}_\bullet(X_2|u)}$.

In all numerical illustrations, it is always assumed that the (net) premium rate is $c = 2$, and each jump Y_i follows a two-sided exponential distribution with density $p(y) = 0.2e^{-y}1\{y > 0\} + 0.8(0.5e^{0.5y})1\{y < 0\}$, i.e. with probability $q_+ = 0.2$ a jump represents an exponential gain with mean $E[Y_{+,1}] = 1$ and with probability $q_- = 0.8$ it is an exponential loss with mean $E[Y_{-,1}] = 2$. Three different inter-arrival time assumptions will be made on $\{V_i\}_{i=1}^{\infty}$: (1) a Gamma distribution with density $k(t) = 2.5^{2.5}t^{1.5}e^{-2.5t}/\Gamma(2.5)$; (2) an exponential distribution with $k(t) = e^{-t}$; and (3) a Pareto distribution with $k(t) = (4.5)3.5^{4.5}/(t+3.5)^{5.5}$. These three distributions have common mean of 1, but they have different variances of 0.4, 1 and 1.8 respectively. Note that the loading condition holds true as $cE[V_1] + q_+E[Y_{+,1}] - q_-E[Y_{-,1}] = 0.6 > 0$.

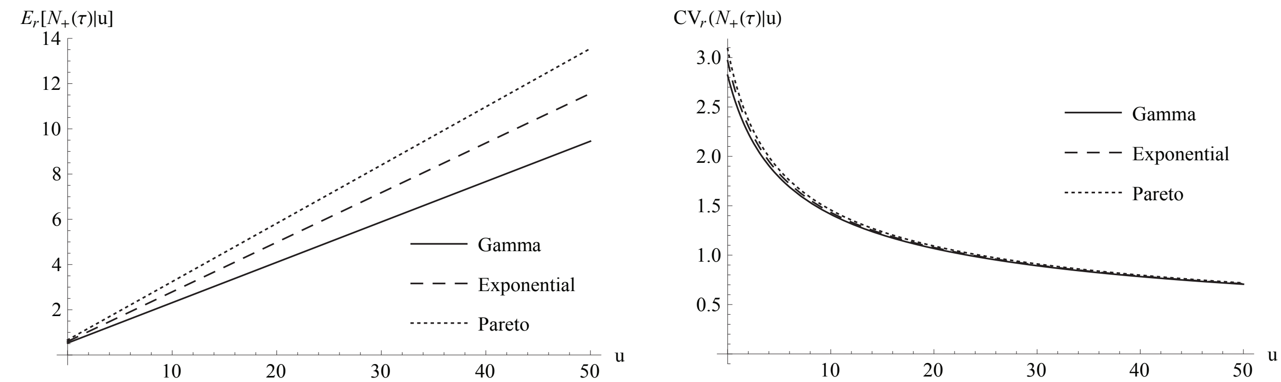
5.1 Numbers of upward and downward jumps

We first consider the number of downward jumps $N_-(\tau)$ and the number of upward jumps $N_+(\tau)$ conditional on ruin, where Theorems 4 and 5 can be applied (with $\delta_1 = 0$ and $w_1(\cdot) \equiv 1$). Figure 1 plots the conditional mean and coefficient of variation of $N_-(\tau)$ against the initial surplus u under different inter-arrival times. Figure 1(a) shows that (for each inter-arrival time distribution) $E_r[N_-(\tau)|u]$ increases linearly in u. This is because if the process starts with a larger amount of capital, then the surplus process will survive longer before ruin and therefore more claims are likely to occur. Indeed, it is easily checked that $E_r[N_-(\tau)|u] = (A_{1,0,0,1}/A_{0,0,0,1}) + (A_{1,0,1,1}/A_{0,0,0,1})u$ is linear in u by utilizing Theorem 4 twice ($n = 0$ and $n = 1$) under the current assumptions. For each fixed u, the expectation $E_r[N_-(\tau)|u]$ increases with the variance $\mathrm{Var}(V_1)$ of the inter-arrival time distribution. Concerning the variability of $N_-(\tau)$, a separate plot (which is not reproduced here for brevity) reveals that $\mathrm{Var}_r(N_-(\tau)|u)$ is also increasing linearly in u, and it increases with $\mathrm{Var}(V_1)$ as well. (The linearity of $\mathrm{Var}_r(N_-(\tau)|u)$ in u is a direct consequence of the identity $A_{2,0,2,1}/(2A_{0,0,0,1}) = (A_{1,0,1,1}/A_{0,0,0,1})^2$, which can be proved in a straightforward manner using the statement of Theorem 4 but is omitted for simplicity.) It is more interesting to look at the normalized measure of variability $\mathrm{CV}_r(N_-(\tau)|u)$ in Figure 1(b), which is decreasing in u. Since the inter-arrival times of the renewal process dictate the number of jumps, it is natural that $\mathrm{CV}_r(N_-(\tau)|u)$ increases when $\mathrm{Var}(V_1)$ increases (or when the coefficient of variation of V_1 increases since $E[V_1]$ is kept fixed across all three inter-arrival time distributions). The conditional mean and coefficient of variation of $N_+(\tau)$ in Figure 2 show the same patterns as those in Figure 1, and the same comments above are valid. Note that $E_r[N_-(\tau)|u]$ in Figure 1(a) is larger than $E_r[N_+(\tau)|u]$ in Figure 2(a) because when a jump arrives it is downward with probability $q_- = 0.8$ but upward only with probability $q_+ = 0.2$. Lastly, the conditional covariance and correlation between $N_-(\tau)$ and $N_+(\tau)$ are given in Figure 3. As with the individual variances, $\mathrm{Cov}_r(N_-(\tau), N_+(\tau)|u)$ increases linearly in u as well and is ordered according to the inter-arrival times' variance. The correlation $\mathrm{Corr}_r(N_-(\tau), N_+(\tau)|u)$ is close to 0.9 for all values of u under each inter-arrival time distribution, but it increases slightly with $\mathrm{Var}(V_1)$ (notice that the y-axis is magnified). Intuitively, there are two factors that can affect the sign of the covariance or



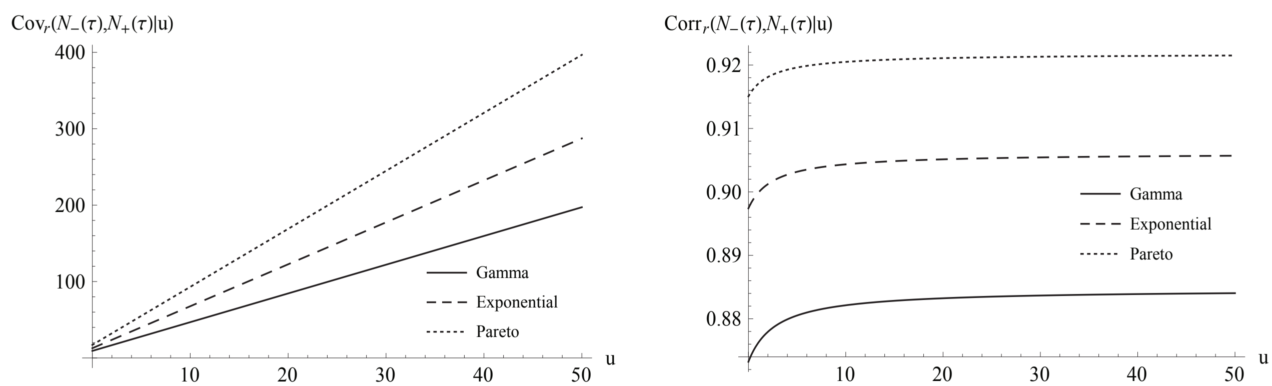
(a) (b)

Figure 1: (a) Expectation and (b) coefficient of variation of number of downward jumps conditional on ruin



(a) (b)

Figure 2: (a) Expectation and (b) coefficient of variation of number of upward jumps conditional on ruin



(a) (b)

Figure 3: (a) Covariance and (b) correlation of numbers of downward and upward jumps conditional on ruin

correlation. Recall that $N_-(\tau) + N_+(\tau) = N(\tau)$ is the total number of jumps until ruin. This means that both $N_-(\tau)$ and $N_+(\tau)$ come from same source, which may have the tendency to drive $N_-(\tau)$ and

$N_+(\tau)$ in opposite directions. In contrast, it also implies that both $N_-(\tau)$ and $N_+(\tau)$ can be large (resp. small) if $N(\tau)$ is large (resp. small) or the process survives longer (resp. shorter). The highly positive correlation in Figure 3(b) suggests that the latter effect dominates. Interestingly, although $\{N_-(t)\}_{t\geq 0}$ and $\{N_+(t)\}_{t\geq 0}$ are independent when the inter-arrival times $\{V_i\}_{i=1}^{\infty}$ are exponential, $N_-(\tau)$ and $N_+(\tau)$ are highly correlated via the ruin time τ.

5.2 Total discounted gains and losses

Now, we turn our attention to the total discounted claims until ruin $\sum_{i=1}^{N_-(\tau)} e^{-\delta_2 T_{-,i}} Y_{-,i}$ and the total discounted gains until ruin $\sum_{i=1}^{N_+(\tau)} e^{-\delta_3 T_{+,i}} Y_{+,i}$, which can be retrieved from $Z_{-,\delta_2}(\tau)$ and $Z_{+,\delta_3}(\tau)$ using the cost functions $f_-(x) = f_+(x) = x$. Theorems 1 and 2 are applicable to compute their joint moments on the ruin set (by letting $\delta_1 = 0$ and $w_1(\cdot) \equiv 1$) while Theorem 3 is applied for the case conditional on survival. The forces of interest $\delta_2 = \delta_3 = 0.01$ are assumed. Figure 4 shows the mean and variance of $Z_{-,0.01}(\tau)$ conditional on ruin occurrence. From Figure 4(a), it is observed that $E_r[Z_{-,0.01}(\tau)|u]$ increases and then converges to a finite value as u increases. The monotonicity of $E_r[Z_{-,0.01}(\tau)|u]$ in u is due to the fact that when u increases, (1) a higher number of claims occur before ruin (see Figure 1(a)); and (2) a larger total claim amount is needed to bring the surplus level below zero. However, for fixed u the value of $E_r[Z_{-,0.01}(\tau)|u]$ appears not to be very sensitive to the distribution of the inter-arrival times. See Figures 5 and 7 of Cheung (2013) for similar observations in an insurance risk model without upward jumps. Turning to Figure 4(b), although the variance $\mathrm{Var}_r(Z_{-,0.01}(\tau)|u)$ is not monotone in u it converges as u increases. A similar pattern can also be found in Cheung (2013, Figures 6 and 8). It is also noted that $\mathrm{Var}_r(Z_{-,0.01}(\tau)|u)$ is ordered according to the variance $\mathrm{Var}(V_1)$ of the inter-arrival time distribution, which makes sense because the factor $e^{-\delta_2 T_{-,i}}$ used to discount the ith claim has more variability if the inter-arrival times (and hence arrival times) are more likely to take on extreme values. Next, Figure 5 is concerned with the mean and variance of $Z_{+,0.01}(\tau)$ conditional on ruin. As u increases, there are two opposing effects on $Z_{+,0.01}(\tau)$. On one hand, a larger number of gains are expected to happen on the ruin set (see Figure 2(a)). On the other hand, sample paths for which the total gain amount is large are unlikely to lead to ruin and thus more likely to be excluded from the ruin set. The fact that $E_r[Z_{+,0.01}(\tau)|u]$ is increasing in u in Figure 5(a) means that the former effect dominates in our example. As opposed to $E_r[Z_{-,0.01}(\tau)|u]$, the value of $E_r[Z_{+,0.01}(\tau)|u]$ increases with $\mathrm{Var}(V_1)$. Looking at Figure 5(b), $\mathrm{Var}_r(Z_{+,0.01}(\tau)|u)$ first increases and then decreases in u and it converges as u increases further. Moreover, $\mathrm{Var}_r(Z_{+,0.01}(\tau)|u)$ increases with $\mathrm{Var}(V_1)$ for a similar reason used to explain Figure 4(b). Note that the magnitude in Figure 4 is considerably larger than that in Figure 5 because the probability $q_- = 0.8$ and the mean $E[Y_{-,1}] = 2$ of the downward jump are larger than the counterparts $q_+ = 0.2$ and $E[Y_{+,1}] = 1$ for the upward jump. Figure 6 depicts the behaviour of the covariance and correlation between $Z_{-,0.01}(\tau)$ and $Z_{+,0.01}(\tau)$ conditional on ruin. It can be seen that $\mathrm{Cov}_r(Z_{-,0.01}(\tau), Z_{+,0.01}(\tau)|u)$ and $\mathrm{Corr}_r(Z_{-,0.01}(\tau), Z_{+,0.01}(\tau)|u)$ are positive for small values of u. In this case, the dominant factor is that the numbers of jumps $N_-(\tau)$ and $N_+(\tau)$ are highly correlated on the ruin set (see Figure 3) and therefore the total jump amounts $Z_{-,0.01}(\tau)$ and $Z_{+,0.01}(\tau)$ tend to move in the same direction as well. However, the covariances and correlations under exponential and Gamma inter-arrival times turn negative when u is large. An explanation is that for fixed u if the process survives long this is possibly associated to big upward jumps and small downward jumps. (Even the downward jump causing ruin may be large, it occurs late at the ruin time and contributes little to $Z_{-,0.01}(\tau)$ due to discounting.) Similarly, if the process ruins early then the upward jump amounts are likely to be small and but big downward jumps happen early. Figure 6 suggests that such a negative relationship between $Z_{-,0.01}(\tau)$ and $Z_{+,0.01}(\tau)$ becomes more significant as u becomes large.

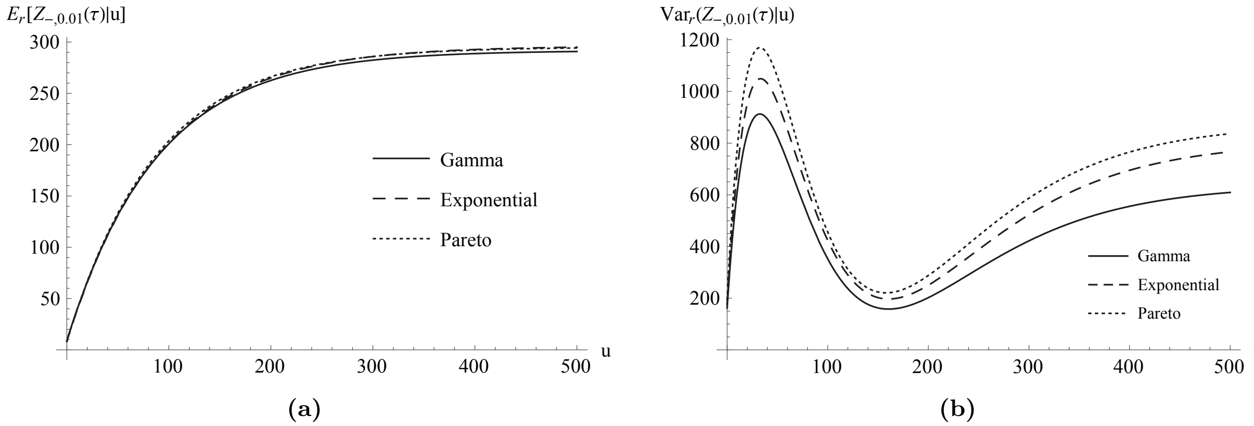


Figure 4: (a) Expectation and (b) variance of discounted losses conditional on ruin

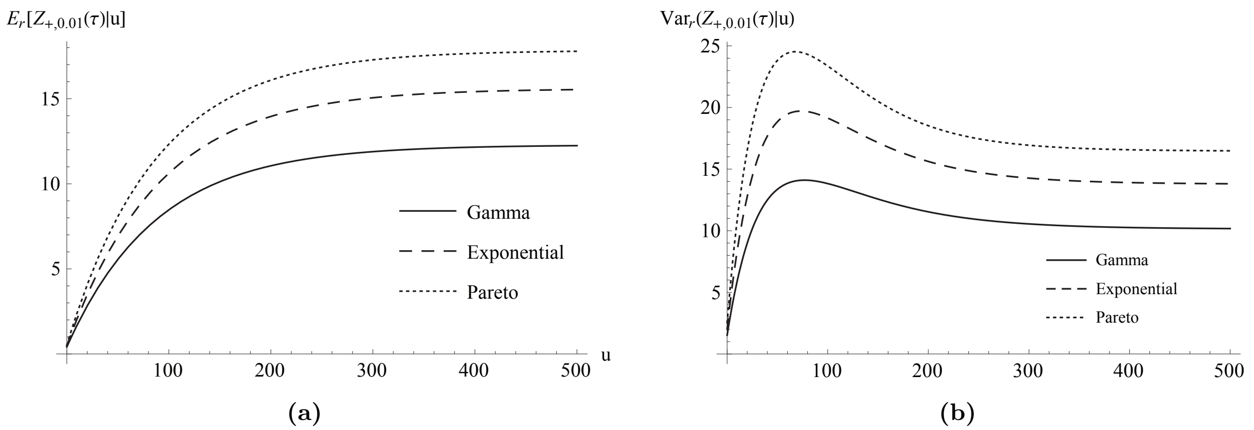


Figure 5: (a) Expectation and (b) variance of discounted gains conditional on ruin

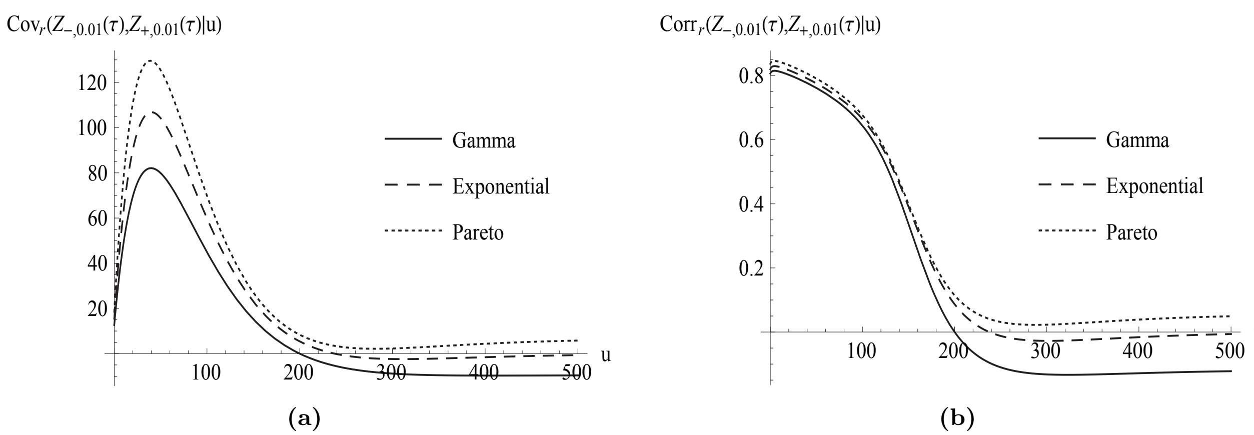
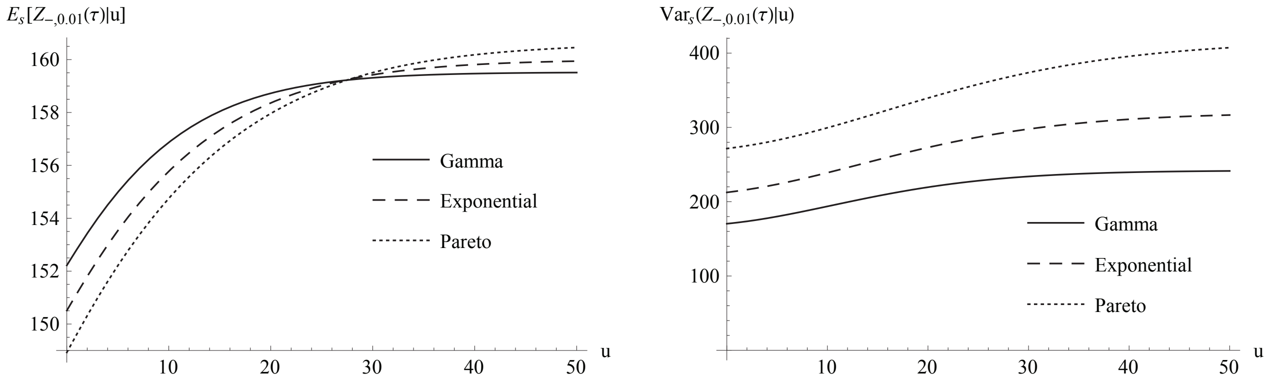


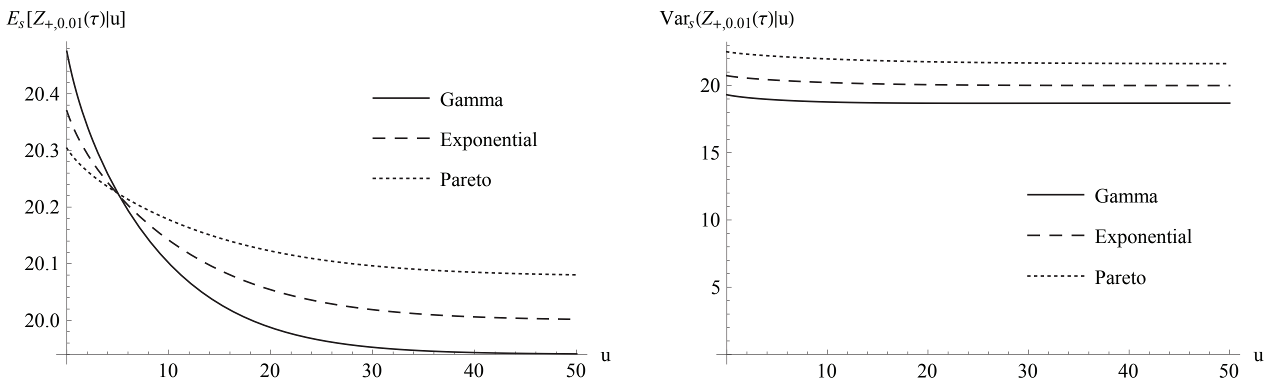
Figure 6: (a) Covariance and (b) correlation of discounted losses and discounted gains conditional on ruin

Next, the same measures as in Figures 4-6 concerning the total discounted claim and gain amounts $Z_{-,\delta_2}(\tau)$ and $Z_{+,\delta_3}(\tau)$ but conditional on survival of the process are provided in Figures 7-9. From Figures 7(a) and 8(a) (for which the y-axis has been magnified), the expectations $E_s[Z_{-,0.01}(\tau)|u]$ and



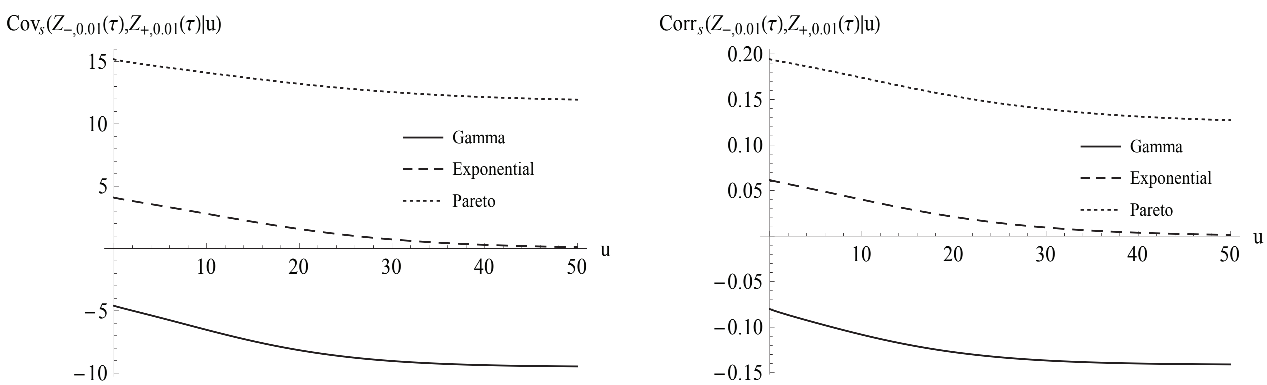
(a) **(b)**

Figure 7: (a) Expectation and (b) variance of discounted losses conditional on survival



(a) **(b)**

Figure 8: (a) Expectation and (b) variance of discounted gains conditional on survival



(a) **(b)**

Figure 9: (a) Covariance and (b) correlation of discounted losses and discounted gains conditional on survival

$E_s[Z_{+,0.01}(\tau)|u]$ are rather insensitive to the initial surplus u and the choice of inter-arrival time distribution. Similar to Figures 4(b) and 5(b), $\mathrm{Var}_s(Z_{-,0.01}(\tau)|u)$ and $\mathrm{Var}_s(Z_{+,0.01}(\tau)|u)$ in Figures 7(b) and 8(b) increase with the variance $\mathrm{Var}(V_1)$ of the inter-arrival times, and the interpretations on Figure 4(b) are applicable here. From Figure 9, both $\mathrm{Cov}_s(Z_{-,0.01}(\tau), Z_{+,0.01}(\tau)|u)$ and $\mathrm{Corr}_s(Z_{-,0.01}(\tau), Z_{+,0.01}(\tau)|u)$ decrease in u and then converges as u gets large. However, their sign varies with the inter-arrival time assumption. For Pareto and exponential (resp. Gamma) inter-arrival times, these quantities are always positive (resp. negative). Indeed, using Theorem 3 along with some straightforward algebra, the asymptotic covariance is found to be

$$\lim_{u\to\infty} \mathrm{Cov}_s(Z_{-,\delta_2}(\tau), Z_{+,\delta_3}(\tau)|u) = C^*_{1,1} - C^*_{1,0}C^*_{0,1}$$
$$= q_-q_+E[f_-(Y_{-,1})]E[f_+(Y_{+,1})]\left\{\frac{\widetilde{k}(\delta_2+\delta_3)}{1-\widetilde{k}(\delta_2+\delta_3)}\left(\frac{\widetilde{k}(\delta_2)}{1-\widetilde{k}(\delta_2)}+\frac{\widetilde{k}(\delta_3)}{1-\widetilde{k}(\delta_3)}\right)-\frac{\widetilde{k}(\delta_2)}{1-\widetilde{k}(\delta_2)}\frac{\widetilde{k}(\delta_3)}{1-\widetilde{k}(\delta_3)}\right\}, \tag{5.1}$$

for which the sign is determined by the term inside the curly brackets. In particular, since $\widetilde{k}(s) = \lambda/(\lambda+s)$ when $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate λ, it is easy to check that the above limit is identical to zero, and this agrees with the limiting behaviour of the exponential case in Figure 9. According to Remark 7, one also notes that the asymptotic (conditional) covariance (5.1) is identical to the (unconditional) covariance between the discounted sums $\sum_{i=1}^{\infty} e^{-\delta_2 T_{-,i}} f_-(Y_{-,i})$ and $\sum_{i=1}^{\infty} e^{-\delta_3 T_{+,i}} f_+(Y_{+,i})$, and in the Poisson case these random variables are well known to be independent.

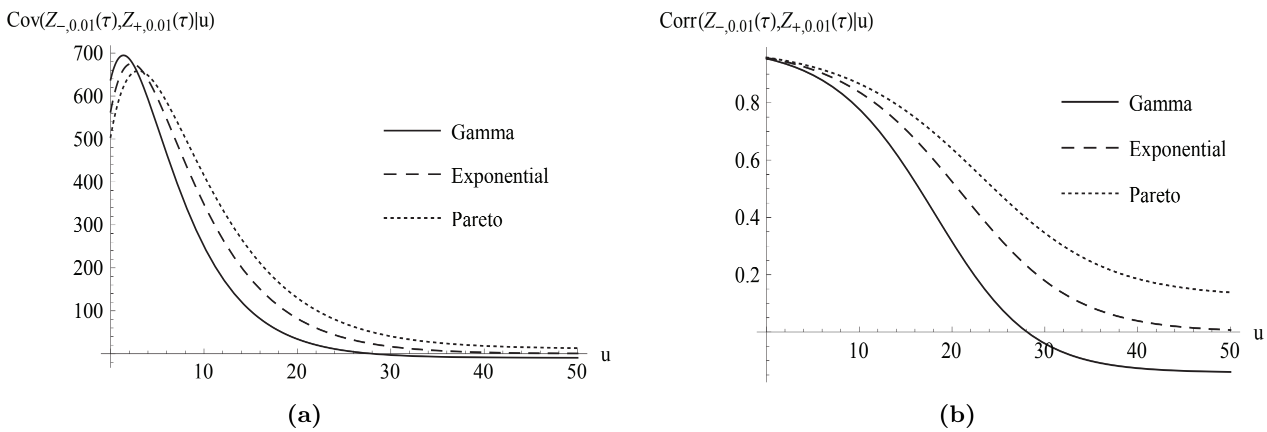


Figure 10: Unconditional (a) covariance and (b) correlation of discounted losses and discounted gains

Finally, Figure 10 shows the unconditional covariance $\mathrm{Cov}(Z_{-,0.01}(\tau), Z_{+,0.01}(\tau)|u)$ and correlation $\mathrm{Corr}(Z_{-,0.01}(\tau), Z_{+,0.01}(\tau)|u)$, and their shapes are essentially a combination of Figure 6 (conditional on ruin) and Figure 9 (conditional on survival). The contributions of the events $\{\tau < \infty\}$ and $\{\tau = \infty\}$ are in accordance with (although not linear in) the ruin probability and the survival probability respectively. As u increases, the ruin probability decreases and converges to zero while the survival probability increases and converges to one. Hence, the contribution of the former event dominates when u is small while the that of the latter dominates when u is large.

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