New multiplicative perturbation bounds for the generalized polar decomposition

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Abstract

Some new Frobenius norm bounds of the unique solution to certain structured Sylvester equation are derived. Based on the derived norm upper bounds, new multiplicative perturbation bounds are provided both for subunitary polar factors and positive semi-definite polar factors. Some previous results are then improved.

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1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ is the set of $m\times n$ complex matrices, $\mathbb{C}^{m\times n}_r$ is the subset of $\mathbb{C}^{m\times n}$ consisting of matrices with rank r, and I_n is the identity matrix in $\mathbb{C}^{n\times n}$. When A is a square matrix, $tr(A)$ stands for the trace of A. Let P_X denote the orthogonal projection from $\mathbb{C}^n$ onto its linear subspace X. For any $A\in\mathbb{C}^{m\times n}$, let A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\|A\|_2$, $\|A\|_F$ and $A^\dagger$ denote the conjugate transpose, the range, the null space, the spectral norm, the Frobenius norm and the Moore-Penrose inverse of A respectively, where $A^\dagger$ is the unique element of $\mathbb{C}^{n\times m}$ which satisfies

$$AA^\dagger A=A,\ A^\dagger AA^\dagger=A^\dagger,\ (AA^\dagger)^*=AA^\dagger \text{ and } (A^\dagger A)^*=A^\dagger A.$$

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It is known that $(A^*)^\dagger = (A^\dagger)^*$ and $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$. Furthermore, if A is Hermitian positive semi-definite, then $A^\dagger$ is also Hermitian positive semi-definite such that $(A^\dagger)^{\frac{1}{2}} = (A^{\frac{1}{2}})^\dagger$.

The polar decomposition is one of the most important factorizations, which occurs in various contexts. For any $T \in \mathbb{C}^{m\times n}$, the polar decomposition of T [1, 6, 11, 15, 18] is a factorization $T = UH$, where $H \in \mathbb{C}^{n\times n}$ is Hermitian positive semi-definite and $U \in \mathbb{C}^{m\times n}$ satisfies one of the following conditions:

$$\begin{aligned} U^*U &= I_n, \ \text{if } m \geq n, \\ UU^* &= I_m, \ \text{if } m < n. \end{aligned}$$

In particular, U is a unitary matrix when $m = n$. As a generalization of the trigonometric representation of a complex number, this decomposition for complex matrices is closely related to the Singular Value Decomposition (SVD). More precisely, let $T \in \mathbb{C}_r^{m\times n}$ with $m \geq n$, then $T^*T = HU^*UH = H^2$, hence H is unique such that $H = |T| = (T^*T)^{\frac{1}{2}}$. Furthermore, it follows from [7, Theorem 8.1] that all possible U are given by

$$U = P \begin{pmatrix} I_r & 0 \\ 0 & W \end{pmatrix} Q^*, \tag{1.1}$$

where

$$T = P \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0_{m-r,n-r} \end{pmatrix} Q^*$$

is the SVD of T and $W \in \mathbb{C}^{m-r,n-r}$ is arbitrary subject to having orthonormal columns.

The generalized polar decomposition of $T \in \mathbb{C}^{m\times n}$ [3, 9, 14, 17, 22] (also called the canonical polar decomposition in [8]) is the case where W in (1.1) is the zero matrix, which can be characterized as

$$T = U|T| \text{ and } \mathcal{N}(T) = \mathcal{N}(U), \tag{1.2}$$

or equivalently,

$$T = U|T| \text{ and } \mathcal{R}(T^*) = \mathcal{R}(U^*) \tag{1.3}$$

since $\mathcal{N}(T)^\perp = \mathcal{R}(T^*)$ and $\mathcal{N}(U)^\perp = \mathcal{R}(U^*)$ in the finite-dimensional case. Such a matrix U is unique [22], which is called a partial isometry (also called the subunitary polar factor of T in many literatures). The matrix $|T|$ is usually called the positive semi-definite polar factor of T.

There are other types of polar decompositions associated to the finite-dimensional spaces, such as the weighted generalized polar decomposition for matrices [10, 24, 25], the polar decomposition for Lie groups [12, 20, 26], and the polar decomposition for matrices acting on indefinite inner spaces [2, 8, 19]. The polar decomposition also works for bounded linear operators on Hilbert spaces. Let H, K be two Hilbert spaces and $\mathbb{B}(H, K)$ be the set of bounded linear operators from H to K. It is well-known that any $T \in \mathbb{B}(H, K)$ has the unique polar decomposition (1.2), where $U \in \mathbb{B}(H, K)$ is a partial isometry [5]. So the polar decomposition for elements of $\mathbb{B}(H, K)$ is exactly the direct generalization of the generalized polar decomposition for matrices. Note that if $H = K$, then $\mathbb{B}(H, H)$, abbreviated to $\mathbb{B}(H)$, is a von Neumann algebra. It follows from [21, Proposition 2.2.9] that the polar decomposition also works for a general von Neumann algebra. Nevertheless, it may fail to work for a general C^*-algebra; see [21, Remark 1.4.6]. For some applications of the polar decomposition, the reader is referred to [6, 13].

In this paper, we restrict our attention to the generalized polar decomposition for matrices. As mentioned above, the generalized polar decomposition of a matrix $A \in \mathbb{C}^{m\times n}$ is formulated by $A = U|A|$, where $|A| = (A^*A)^{\frac{1}{2}}$ and $U \in \mathbb{C}^{m\times n}$ is a partial isometry such that $U^*U = P_{\mathcal{R}(A^*)}$. Since U is a partial isometry, we have $UU^*U = U$. Furthermore, it can be deduced from (1.2) and (1.3) that

$$UU^* = P_{\mathcal{R}(A)} = AA^\dagger, U^*U = A^\dagger A, A = |A^*|U \tag{1.4}$$
$$A^* = U^*|A^*| = |A|U^*, |A^*| = U|A|U^*, |A| = U^*|A^*|U. \tag{1.5}$$

One research field of the generalized polar decomposition is its perturbation theory. Let $A \in \mathbb{C}^{m\times n}$ be given and $B \in \mathbb{C}^{m\times n}$ be a perturbation of A. Clearly, $\text{rank}(A) = \text{rank}(B)$ if and only if there exist $D_1 \in \mathbb{C}^{m\times m}$ and $D_2 \in \mathbb{C}^{n\times n}$ such that

$$B = D_1^* A D_2, \text{ where } D_1 \text{ and } D_2 \text{ are both nonsingular.} \tag{1.6}$$

The matrix B given by (1.6) is called a multiplicative perturbation of A. Multiplicative perturbation Frobenius norm upper bounds for subunitary polar factors and positive semi-definite polar factors are carried out in [4] and [10] respectively as follows:

Let $\Omega \in \mathbb{C}^{m\times m}, \Gamma \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{m\times n}$. It is well-known [11] that the Sylvester equation $\Omega X - X\Gamma = S$ has a unique solution $X \in \mathbb{C}^{m\times n}$ if and only if $\lambda(\Omega) \cap \lambda(\Gamma) = \emptyset$, where $\lambda(\Omega)$ and $\lambda(\Gamma)$ denote the spectrums of Ω and Γ, respectively. The Sylvester equation appears in many problems in science and technology, e.g., control theory, model reduction, the numerical solution of Riccati equations, image processing and so on. To deal with eigenspace and singular subspace variations, the structured Sylvester equation $\Omega X - X\Gamma = S$ with $S = \Omega C + D\Gamma$ is considered in [16] for any $C, D \in \mathbb{C}^{m\times n}$, and a Frobenius norm upper bound of the unique solution X is obtained as follows:

Lemma 1.1. [16, Lemma 2.2] *Let $\Omega \in \mathbb{C}^{m\times m}$ and $\Gamma \in \mathbb{C}^{n\times n}$ be two Hermitian matrices, and let $C, D \in \mathbb{C}^{m\times n}$ be arbitrary. If $\lambda(\Omega) \cap \lambda(\Gamma) = \emptyset$, then the Sylvester equation $\Omega X - X\Gamma = \Omega C + D\Gamma$ has a unique solution $X \in \mathbb{C}^{m\times n}$ such that*

$$\|X\|_F \leq \sqrt{\|C\|_F^2 + \|D\|_F^2}\Big/\eta, \tag{1.7}$$

where

$$\eta = \min_{\omega\in\lambda(\Omega),\gamma\in\lambda(\Gamma)} \frac{|\omega-\gamma|}{\sqrt{|\omega|^2+|\gamma|^2}}. \tag{1.8}$$

A direct application of the preceding lemma is as follows:

Corollary 1.2. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be two Hermitian positive definite matrices. Then for any $C, D \in \mathbb{C}^{m\times n}$, the Sylvester equation*

$$AX + XB = AC + DB \tag{1.9}$$

has a unique solution $X \in \mathbb{C}^{m\times n}$ such that

$$\|X\|_F \leq \sqrt{\|C\|_F^2 + \|D\|_F^2}. \tag{1.10}$$

The proof of Corollary 1.2 is easy. Indeed, if we put $\Omega = A$ and $\Gamma = -B$ in Lemma 1.2, then $\lambda(\Omega) \subseteq (0, +\infty)$ and $\lambda(\Gamma) \subseteq (-\infty, 0)$, therefore $\lambda(\Omega) \cap \lambda(\Gamma) = \emptyset$ and the number η defined by (1.8) is greater than one. Norm upper bound (1.10) then follows immediately from (1.7).

Although norm upper (1.10) has the advantage of the simpleness in its form, it usually turns out to be much coarse. For instance, even in the special case that both A and B are identity matrices, this norm upper bound fails to be accurate, since in this case $X = \frac{C+D}{2}$ and

$$\|X\|_F^2 \leq \frac{\|C+D\|_F^2 + \|C-D\|_F^2}{4} = \frac{\|C\|_F^2 + \|D\|_F^2}{2} \leq \|C\|_F^2 + \|D\|_F^2.$$

This norm upper bound may also fail to be accurate in another special case that $C = D$, for in this case $X = C$ is the unique solution to (1.9), whereas this norm upper bound gives the number $\sqrt{2}\,\|C\|_F$ rather than the exact value $\|C\|_F$.

The main tools employed in [4, 10] are the SVD of the associated matrices, together with the Frobenius norm upper bound (1.10) of the unique solution to the structured Sylvester equation (1.9).

The purpose of this paper is to improve norm upper bound (1.10), and then with no use of the SVD to derive new multiplicative perturbation bounds both for subunitary polar factors and positive semi-definite polar factors. Recently, a new kind of multiplicative perturbation called the weak perturbation is studied in [23]. It is notable that a weak perturbation may fail to be rank-preserving, so it is somehow complicated to use the SVD to handle weak perturbations. Nevertheless, the method employed in Section 3 of this paper can still be used to deal with the weak perturbation bounds for the generalized polar decomposition.

The rest of this paper is organized as follows. In Section 2, we study Frobenius norm bounds of the solution X to the structured Sylvester equation (1.9), and obtain new upper bounds (2.7), (2.8) and (2.20) of X. As an application, in Section 3 we study multiplicative perturbation bounds both for subunitary polar factors and positive semi-definite polar factors. A systematic improvement is made by using the improved upper bound instead of upper bound (1.10); see Theorems 3.1 and 3.3 for the details.

2. Frobenius norm bounds of the solution to the structured Sylvester equation

In this section, we study Frobenius norm bounds of the solution X to the structured Sylvester equation (1.9). To begin with, we recall some well-known results on the Frobenius norm of matrices. For any $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{n\times m}$, it holds that

$$|tr(XY)| \le \|X\|_F\,\|Y\|_F \text{ and } \|XY\|_F \le \min\{\|X\|_2\,\|Y\|_F, \|X\|_F\,\|Y\|_2\}.$$

If $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ are two orthogonal projections, then for any $M, N \in \mathbb{C}^{m\times n}$, the following equations hold:

$$\|PM + (I_m - P)N\|_F^2 = \|PM\|_F^2 + \|(I_m - P)N\|_F^2, \tag{2.1}$$

$$\|MQ + N(I_n - Q)\|_F^2 = \|MQ\|_F^2 + \|N(I_n - Q)\|_F^2. \tag{2.2}$$

Lemma 2.1. *Suppose that $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ are two Hermitian positive semi-definite matrices. Let $C, D \in \mathbb{C}^{m\times n}$ be such that $A^\dagger AC = C$ and $DBB^\dagger = D$. If $X \in \mathbb{C}^{m\times n}$ is a solution to (1.9) such that $A^\dagger AX = X = XBB^\dagger$. Then*

$$\begin{aligned}\|D - C\|_F^2 = \|D - X\|_F^2 + \|X - C\|_F^2 + \left\| A^{\frac{1}{2}}(X - C)(B^\dagger)^{\frac{1}{2}}\right\|_F^2 \\ + \left\|(A^\dagger)^{\frac{1}{2}}(D - X)B^{\frac{1}{2}}\right\|_F^2.\end{aligned} \tag{2.3}$$

Proof. By assumption, we have

$$A^\dagger A(X-C) = X-C \text{ and } (D-X)BB^\dagger = D-X, \tag{2.4}$$

which is combined with (1.9) to get

$$\begin{aligned}
(X-C)^* \cdot A(X-C) \cdot B^\dagger &= (X-C)^* \cdot (D-X)B \cdot B^\dagger \\
&= (X-C)^* \cdot (D-X) \\
&= (X-C)^*(D-C) - (X-C)^*(X-C).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|A^{\frac{1}{2}}(X-C)\big(B^\dagger\big)^{\frac{1}{2}}\|_F^2 &= tr\left[\big[A^{\frac{1}{2}}(X-C)\big(B^\dagger\big)^{\frac{1}{2}}\big]^*\big[A^{\frac{1}{2}}(X-C)\big(B^\dagger\big)^{\frac{1}{2}}\big]\right] \\
&= tr\left[\big(B^\dagger\big)^{\frac{1}{2}} \cdot (X-C)^* A(X-C)\big(B^\dagger\big)^{\frac{1}{2}}\right] \\
&= tr\left[(X-C)^* A(X-C)\big(B^\dagger\big)^{\frac{1}{2}} \cdot \big(B^\dagger\big)^{\frac{1}{2}}\right] \\
&= tr\left[(X-C)^* \cdot A(X-C) \cdot B^\dagger\right] \\
&= tr\left[(X-C)^*(D-C)\right] - \|X-C\|_F^2. \qquad (2.5)
\end{aligned}$$

Similarly, from (1.9) and (2.4) we can get

$$A^\dagger (D-X)B(D-X)^* = (X-C)(D-X)^*,$$

and thus

$$\begin{aligned}
\|\big(A^\dagger\big)^{\frac{1}{2}}(D-X)B^{\frac{1}{2}}\|_F^2 &= tr\left[A^\dagger(D-X)B(D-X)^*\right] \\
&= tr\left[(X-C)(D-X)^*\right] \\
&= tr\left[(D-X)^*(X-C)\right] \\
&= tr\left[(D-X)^*\big(D-C-(D-X)\big)\right] \\
&= tr\left[(D-X)^*(D-C)\right] - \|D-X\|_F^2. \qquad (2.6)
\end{aligned}$$

Since

$$\begin{aligned}
\|D-C\|_F^2 &= tr\left[(D-C)^*(D-C)\right] \\
&= tr\left[(D-X)^*(D-C)\right] + tr\left[(X-C)^*(D-C)\right],
\end{aligned}$$

the desired equation follows immediately from (2.5) and (2.6). $\square$

Now we provide a technique result of this section as follows:

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be two non-zero Hermitian positive semi-definite matrices, and let $C, D \in \mathbb{C}^{m\times n}$ be such that $A^\dagger AC = C$ and $DBB^\dagger = D$. If $X \in \mathbb{C}^{m\times n}$ is a solution to (1.9) such that $A^\dagger AX = X = XBB^\dagger$. Then*

$$\frac{\|C+D\|_F - \|C-D\|_F}{2} \le \|X\|_F \le \frac{\|C+D\|_F + \|C-D\|_F}{2}. \tag{2.7}$$

If furthermore $CBB^\dagger = C$ and $A^\dagger AD = D$, then

$$\frac{\|aC+bD\|_F - c\|C-D\|_F}{a+b} \le \|X\|_F \le \frac{\|aC+bD\|_F + c\|C-D\|_F}{a+b}, \tag{2.8}$$

where

$$\lambda_1 = \|A^\dagger\|_2 \cdot \|B\|_2, \quad \lambda_2 = \|A\|_2 \cdot \|B^\dagger\|_2, \tag{2.9}$$

$$a = 1 + \frac{1}{\lambda_1}, \quad b = 1 + \frac{1}{\lambda_2} \ \textit{and} \ c = \sqrt{1 - \frac{1}{\lambda_1\lambda_2}}. \tag{2.10}$$

Proof. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{C}^{m\times n}$ defined by

$$\langle U, V\rangle = tr(UV^*) \text{ for any } U, V \in \mathbb{C}^{m\times n}.$$

Then

$$\|U\|_F = \sqrt{\langle U, U\rangle} \text{ for any } U \in \mathbb{C}^{m\times n},$$

hence

$$\|C+D\|_F^2 + \|C-D\|_F^2 = 2\|C\|_F^2 + 2\|D\|_F^2, \tag{2.11}$$

$$\frac{\|C+D\|_F^2 - \|C-D\|_F^2}{4} = \mathrm{Re}\left(\langle C, D\rangle\right), \tag{2.12}$$

where $\mathrm{Re}\left(\langle C, D\rangle\right)$ denotes the real part of $\langle C, D\rangle$.

Firstly, we prove inequalities in (2.7). By Lemma 2.1 we know that (2.3) is satisfied, which leads obviously to

$$\|X-C\|_F^2 + \|D-X\|_F^2 \le \|D-C\|_F^2;$$

or equivalently,

$$\langle X-C, X-C\rangle + \langle D-X, D-X\rangle \le \langle D-C, D-C\rangle,$$

which can be simplified to

$$\|X\|_F^2 - \mathrm{Re}\left(\langle C+D, X\rangle\right) + \mathrm{Re}\left(\langle C, D\rangle\right) \le 0,$$

and thus

$$\|X\|_F^2 - \|C+D\|_F \cdot \|X\|_F + \operatorname{Re}\left(\langle C, D\rangle\right) \le 0, \tag{2.13}$$

since $\operatorname{Re}\left(\langle C+D, X\rangle\right) \le \|C+D\|_F \cdot \|X\|_F$. Then by (2.12) and (2.13), we obtain

$$\|X\|_F^2 - \|C+D\|_F \cdot \|X\|_F + \frac{\|C+D\|_F^2 - \|C-D\|_F^2}{4} \le 0,$$

which clearly gives (2.7).

Secondly, we prove inequalities in (2.8). Since A and B are Hermitian positive semi-definite, we have

$$\begin{aligned} (A^\dagger)^{\frac{1}{2}} &= (A^{\frac{1}{2}})^\dagger \text{ and } A^{\frac{1}{2}}(A^{\frac{1}{2}})^\dagger = (A^{\frac{1}{2}})^\dagger A^{\frac{1}{2}} = A^\dagger A = AA^\dagger, \\ (B^\dagger)^{\frac{1}{2}} &= (B^{\frac{1}{2}})^\dagger \text{ and } B^{\frac{1}{2}}(B^{\frac{1}{2}})^\dagger = (B^{\frac{1}{2}})^\dagger B^{\frac{1}{2}} = B^\dagger B = BB^\dagger. \end{aligned}$$

If in addition $CBB^\dagger = C$ and $A^\dagger AD = D$, then

$$\begin{aligned} \|X-C\|_F &= \left\|(A^\dagger)^{\frac{1}{2}}\left[A^{\frac{1}{2}}(X-C)(B^\dagger)^{\frac{1}{2}}\right]B^{\frac{1}{2}}\right\|_F \\ &\le \left\|(A^\dagger)^{\frac{1}{2}}\right\|_2 \cdot \left\|B^{\frac{1}{2}}\right\|_2 \cdot \left\|A^{\frac{1}{2}}(X-C)(B^\dagger)^{\frac{1}{2}}\right\|_F, \end{aligned}$$

which leads to

$$\left\|A^{\frac{1}{2}}(X-C)(B^\dagger)^{\frac{1}{2}}\right\|_F^2 \ge \frac{\|X-C\|_F^2}{\left\|(A^\dagger)^{\frac{1}{2}}\right\|_2^2 \cdot \left\|B^{\frac{1}{2}}\right\|_2^2} = \frac{\|X-C\|_F^2}{\lambda_1}. \tag{2.14}$$

Similarly,

$$\left\|(A^\dagger)^{\frac{1}{2}}(D-X)B^{\frac{1}{2}}\right\|_F^2 \ge \frac{\|D-X\|_F^2}{\left\|A^{\frac{1}{2}}\right\|_2^2 \cdot \left\|(B^\dagger)^{\frac{1}{2}}\right\|_2^2} = \frac{\|D-X\|_F^2}{\lambda_2}. \tag{2.15}$$

Let a and b be defined by (2.10). Note that

$$\lambda_1\lambda_2 = \left(\left\|A^\dagger\right\|_2 \cdot \left\|A\right\|_2\right) \cdot \left(\left\|B\right\|_2 \cdot \left\|B^\dagger\right\|_2\right) \ge \|A^\dagger A\|_2 \cdot \|BB^\dagger\|_2 = 1,$$

so we have

$$a + b - ab = 1 - \frac{1}{\lambda_1\lambda_2} \ge 0,$$

which indicates that the number c is well-defined such that $c = \sqrt{a+b-ab}$. By (2.3), (2.14), (2.15) and (2.10), we obtain

$$a\|X-C\|_F^2 + b\|D-X\|_F^2 \le \|D-C\|_F^2. \tag{2.16}$$

Using the same technique as in the derivation of (2.13), from (2.16) we can get

$$(a+b)\cdot\|X\|_F^2 - 2\|aC+bD\|_F\cdot\|X\|_F + d \le 0, \tag{2.17}$$

where

$$d = (a-1)\cdot\|C\|_F^2 + (b-1)\cdot\|D\|_F^2 + 2\mathrm{Re}\left(\langle C, D\rangle\right).$$

It follows from (2.17) that

$$\frac{\|aC+bD\|_F - \sqrt{e}}{a+b} \le \|X\|_F \le \frac{\|aC+bD\|_F + \sqrt{e}}{a+b}, \tag{2.18}$$

where

$$\begin{aligned} e &= \|aC+bD\|_F^2 - (a+b)d = a^2\|C\|_F^2 + b^2\|D\|_F^2 + 2ab\,\mathrm{Re}\langle C, D\rangle - (a+b)d \\ &= (a+b-ab)\left[\|C\|_F^2 + \|D\|_F^2 - 2\,\mathrm{Re}\left(\langle C, D\rangle\right)\right] \\ &= (a+b-ab)\cdot\|C-D\|_F^2 \text{ by (2.11) and (2.12).} \end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18) yields (2.8). ◻

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be two non-zero Hermitian positive semi-definite matrices, and let $C, D \in \mathbb{C}^{m\times n}$ be such that*

$$A^\dagger AC = C = CBB^\dagger \text{ and } A^\dagger AD = D = DBB^\dagger.$$

If $X \in \mathbb{C}^{m\times n}$ is a solution to (1.9) such that $A^\dagger AX = X = XBB^\dagger$. Then

$$\frac{\|C+D\|_F - \mu\|C-D\|_F}{2} \le \|X\|_F \le \frac{\|C+D\|_F + \mu\|C-D\|_F}{2}, \tag{2.20}$$

where

$$\lambda = \max\left\{\left\|A^\dagger\right\|_2\cdot\left\|B\right\|_2, \left\|A\right\|_2\cdot\left\|B^\dagger\right\|_2\right\} \text{ and } \mu = \sqrt{\frac{\lambda-1}{\lambda+1}}. \tag{2.21}$$

Proof. Following the notations as in the proof of Theorem 2.2, we have $\lambda = \max\{\lambda_1, \lambda_2\}$, therefore from the proof of Theorem 2.2 we know that (2.8) is satisfied if a, b and c therein be replaced by

$$a = b = 1 + \frac{1}{\lambda} \text{ and } c = \sqrt{a+b-ab} = \sqrt{1 - \frac{1}{\lambda^2}}.$$

The conclusion then follows from (2.21). ◻

When applied to the Hermitian positive definite matrices, a corollary can be derived directly as follows:

Corollary 2.4. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be two Hermitian positive definite matrices. Then for any $C, D \in \mathbb{C}^{m\times n}$, there exists a unique solution $X \in \mathbb{C}^{m\times n}$ to* (1.9) *such that*

$$\begin{aligned} \frac{\|aC+bD\|_F - c\|C-D\|_F}{a+b} \leq \quad \|X\|_F \quad &\leq \frac{\|aC+bD\|_F + c\|C-D\|_F}{a+b}, \\ \frac{\|C+D\|_F - \mu\|C-D\|_F}{2} \leq \quad \|X\|_F \quad &\leq \frac{\|C+D\|_F + \mu\|C-D\|_F}{2}, \end{aligned}$$

where

$$\begin{aligned} &\lambda_1 = \|A^{-1}\|_2 \cdot \|B\|_2, \lambda_2 = \|A\|_2 \cdot \|B^{-1}\|_2, \\ &\lambda = \max\{\lambda_1, \lambda_2\}, \mu = \sqrt{\frac{\lambda-1}{\lambda+1}}, \\ &a = 1 + \frac{1}{\lambda_1}, \quad b = 1 + \frac{1}{\lambda_2}, c = \sqrt{1 - \frac{1}{\lambda_1\lambda_2}}. \end{aligned}$$

Proof. The existence and uniqueness of the solution X to (1.9) follow from Lemma 1.1. The rest part of the assertion follows immediately from Theorems 2.2 and 2.3. □

Remark 2.1. It is notable that upper bounds (2.8) and (2.20) are accurate in the case that $C = D$. Furthermore, upper bound (2.8) is also accurate if $A = k_1 I_m$ and $B = k_2 I_n$ for any $m, n \in \mathbb{N}$ and $k_1, k_2 \in (0, +\infty)$. Indeed, in this case we have

$$a = 1 + \frac{k_2}{k_1} \text{ and } b = 1 + \frac{k_1}{k_2},$$

and thus $c = \sqrt{a+b-ab} = 0$, which leads obviously to the accuracy of upper bound (2.8).

Proposition 2.5. *Upper bounds* (2.7)*,* (2.8) *and* (2.20) *are all sharper than upper bound* (1.10)*.*

Proof. Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be Hermitian positive definite and $C, D \in \mathbb{C}^{m\times n}$ be arbitrary. Let a, b and c be defined by (2.10). Then $c = \sqrt{a+b-ab}$ and it can be shown that

$$c < \min\{a, b\} = \frac{a+b-|a-b|}{2}. \tag{2.22}$$

In fact, since $a > 1$ and $b > 1$, we have $a+b < a(a+b)$, which gives $c^2 < a^2$ and thus $c < a$. Similarly, we have $c < b$. It follows from (2.22) that

$$\frac{|a-b|+2c}{a+b} < 1. \tag{2.23}$$

Clearly, upper bound (2.20) is sharper that upper bound (2.7), since the number μ defined by (2.21) is less that 1. We show that the same is true for upper bound (2.8). To this end, we put

$$Y = \frac{C+D}{2} \text{ and } Z = \frac{C-D}{2}.$$

Then $C = Y + Z$ and $D = Y - Z$, hence by (2.23) we have

$$\begin{aligned} \frac{\|(a+b)Y + (a-b)Z\|_F + 2c\|Z\|_F}{a+b} &\leq \frac{(a+b)\|Y\|_F + \big(|a-b|+2c\big)\|Z\|_F}{a+b} \\ &\leq \|Y\|_F + \|Z\|_F, \end{aligned}$$

which means that upper bound (2.8) is sharper than upper bound (2.7).

So it remains to prove that

$$\theta \overset{def}{=} \frac{\|C+D\|_F + \|C-D\|_F}{2} \leq \sqrt{\|C\|_F^2 + \|D\|_F^2},$$

which can be verified easily, since

$$\theta^2 \leq \frac{\|C+D\|_F^2 + \|C-D\|_F^2}{2} = \|C\|_F^2 + \|D\|_F^2 \text{ by (2.11).} \quad \square$$

Remark 2.2. Numerical tests below show that upper bound (2.8) is sharper than upper bound (2.20) only in statistical sense.

Remark 2.3. It is interesting to make comparisons between upper bounds (1.7), (2.8) and (2.20) from statistical point of view. This can be illustrated by numerical tests as follows:

Numerical tests

(i) Let $A_1, B_1, C, D \in \mathbb{C}^{3\times 3}$ be random matrices produced by using Matlab command rand(3). Put $A = A_1^* A_1$ and $B = B_1^* B_1$. Each time run Matlab 10^5 times. For each time, let
   - α be the number of tests in which upper bound (2.8)$\leq$ upper bound (1.7);

- β be the number of tests in which upper bound (2.8)$\leq$upper bound (2.20);

- γ be the number of tests in which upper bound (2.20)$\leq$ upper bound (1.7).

Then $\alpha \approx 99986, \beta \approx 100000$ and $\gamma \approx 99976$[2].

(ii) Let A, B, C be the same as in (i) and let $D \in \mathbb{C}^{3\times 3}$ be the zero matrix. Then $\alpha \approx 39760, \beta \approx 99891$ and $\gamma \approx 0$.

(iii) Let A, B, D be the same as in (i) and let $C \in \mathbb{C}^{3\times 3}$ be the zero matrix. Then $\alpha \approx 39876, \beta \approx 99886$ and $\gamma \approx 0$.

(iv) Let A, B, C be the same as in (i) and let $D = -C$. Then $\alpha \approx 100000, \beta \approx 99792$ and $\gamma \approx 100000$.

(v) Let A, B, C be the same as in (i) and let $D = C$. Then $\alpha \approx 100000, \beta \approx 74899$ and $\gamma \approx 100000$.

Roughly speaking, when both C and D are random or if one of $C - D$ and $C + D$ is small in Frobenius norm, upper bounds (2.8) and (2.20) are statistically better than upper bound (1.7), whereas (1.7) is statistically better if C or D is small in Frobenius norm. A concrete example is constructed as follows, where $C - D$ is small in Frobenius norm.

Example 2.1. Let $A = I_2$, $B = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 4 \end{pmatrix}$, $C = S\left(\frac{5\pi}{32}\right)$ and $D = S\left(\frac{\pi}{6}\right)$, where

$$S(t) = \begin{pmatrix} \cos t & \frac{\sin t}{4} \\ \frac{\sin t}{4} & \cos t \end{pmatrix} \text{ for any } t \in (-\infty, +\infty).$$

Then

$$X = (C + DB)(I_2 + B)^{-1} = \begin{pmatrix} 0.8791 & 0.1190 \\ 0.1160 & 0.8723 \end{pmatrix} \text{ and thus } \|X\|_F = 1.2496.$$

Moreover, we have $\lambda = 4.7913$, $\mu = 0.8091$ by (2.21) and $\eta = 1.1832$ by (1.8). The relative errors of various upper bounds are listed in Table 1, which shows for this example, upper bounds (2.7), (2.8) and (2.20) are much better than the other two.

Remark 2.4. Before ending this section, we make a few comments on the newly obtained upper bounds (2.7), (2.8) and (2.20). One advantage of these

[2] These three average numbers depend on computer model, Matlab version and running times.

Table 1: Comparison of various Frobenius norm upper bounds

	u.b. (1.7)	u.b. (1.10)	u.b. (2.7)	u.b. (2.8)	u.b. (2.20)
Numerical value	1.4915	1.7648	1.2602	1.2578	1.2578
Relative error	19.36%	41.23%	0.85%	0.66%	0.66%

upper bounds are their sharpness under some circumstances. As shown in Proposition 2.5, all of them are sharper than upper bound (1.10), which is derived directly from a widely used upper bound (1.7). Some comparisons between upper bounds (1.7), (2.8) and (2.20) are presented based on numerical tests.

Another advantage of upper bounds (2.8) and (2.20) is their easiness to be determined. To deal with the Frobenius norm rather than the spectral norm, a parameter η is associated to upper bound (1.7). If the Hermitian positive-definite matrices A and B are both large in size, then this parameter η seems to be somehow inconvenient to be determined, since many eigenvalues of A and B have to be considered before getting this minimal value formulated by (1.8). By comparison, all parameters associated to upper bounds (2.8) and (2.20) are convenient to be determined.

In addition, literatures are rarely found on norm lower bounds of the solution X to (1.9). In this section we have managed to provide norm lower bounds in Theorem 2.2, Theorem 2.3 and Corollary 2.4, respectively. With the lower bound given in (2.8), it can be deduced immediately that upper bound (2.8) will be accurate if $C = D$ or the number c defined by (2.10) is zero, as is the case where A and B are positive scalar matrices.

3. New perturbation bounds for the generalized polar decomposition

In this section, we study perturbation bounds for the generalized polar decomposition. First, we provide the perturbation estimation for subunitary polar factors as follows:

Theorem 3.1. *Let B be the multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.6), and let $A = U|A|$ and $B = V|B|$ be the generalized polar decompositions of A and B, respectively. Then*

$$\|V - U\|_F \leq \inf_{s,t\in\mathbb{C}} \sqrt{\varphi_1^2(s,t) + \varphi_2^2(s,t) - \varphi_3^2(s,t)}, \tag{3.1}$$

where $s, t \in \mathbb{C}$ *are arbitrary,* λ *is defined by* (2.21) *and*

$$\begin{aligned}\varphi_1(s,t) &= \|V(I_n - tD_2^{-1}) + VV^*(\bar{s}D_1^{-1} - I_m)U\|_F,\\ \varphi_2(s,t) &= \|U^*(\bar{t}D_1 - I_m) + U^*U(I_n - sD_2)V^*\|_F,\\ \varphi_3(s,t) &= \frac{\|V(\bar{s}D_2^* - tD_2^{-1})U^*U + VV^*(\bar{s}D_1^{-1} - tD_1^*)U\|_F}{\sqrt{\lambda+1}}.\end{aligned}$$

Proof. It is clear that

$$\begin{aligned}V - U &= VV^*(V-U)U^*U + VV^*(V-U)(I_n - U^*U)\\ &\quad +(I_m - VV^*)(V-U)U^*U + (I_m - VV^*)(V-U)(I_n - U^*U)\\ &= \Omega_1 + \Omega_2 + \Omega_3, \qquad (3.2)\end{aligned}$$

where

$$\Omega_1 = VU^*U - VV^*U = VV^* \cdot \Omega_1 \cdot U^*U, \qquad (3.3)$$

$$\Omega_2 = V(I_n - U^*U) = VV^* \cdot \Omega_2 \cdot (I_n - U^*U), \qquad (3.4)$$

$$\Omega_3 = -(I_m - VV^*)U = (I_m - VV^*) \cdot \Omega_3. \qquad (3.5)$$

By (2.1) and (2.2) we have

$$\|V-U\|_F^2 = \|\Omega_1 + \Omega_2\|_F^2 + \|\Omega_3\|_F^2 = \|\Omega_1\|_F^2 + \|\Omega_2\|_F^2 + \|\Omega_3\|_F^2. \qquad (3.6)$$

In what follows we first deal with $\|\Omega_1\|_F^2$. Since $B = D_1^*AD_2$, we have

$$BD_2^{-1} = D_1^*A \text{ and } (D_1^{-1})^*B = AD_2, \qquad (3.7)$$

hence for any $t, s \in \mathbb{C}$, it holds that

$$B - A = B(I_n - tD_2^{-1}) + (tD_1^* - I_m)A, \qquad (3.8)$$

$$A - B = A(I_n - sD_2) + \big(s(D_1^{-1})^* - I_m\big)B. \qquad (3.9)$$

Therefore, by (3.8) we have

$$\begin{aligned}BA^\dagger A - BB^\dagger A &= BB^\dagger(B-A)A^\dagger A\\ &= BB^\dagger\left[B(I_n - tD_2^{-1}) + (tD_1^* - I_m)A\right]A^\dagger A\\ &= B(I_n - tD_2^{-1})A^\dagger A + BB^\dagger(tD_1^* - I_m)A. \qquad (3.10)\end{aligned}$$

Also it follows from (3.8) that

$$\begin{aligned}B(I_n - A^\dagger A) &= BB^\dagger(B-A)(I_n - A^\dagger A)\\ &= BB^\dagger\left[B(I_n - tD_2^{-1}) + (tD_1^* - I_m)A\right](I_n - A^\dagger A)\\ &= B(I_n - tD_2^{-1})(I_n - A^\dagger A), \qquad (3.11)\\ -(I_m - BB^\dagger)A &= (I_m - BB^\dagger)(B-A)A^\dagger A\\ &= (I_m - BB^\dagger)\left[B(I_n - tD_2^{-1}) + (tD_1^* - I_m)A\right]A^\dagger A\\ &= (I_m - BB^\dagger)(tD_1^* - I_m)A. \qquad (3.12)\end{aligned}$$

Since $A = U|A|$ and $B = V|B|$ are the generalized polar decompositions of A and B respectively, by (1.4) and (1.5) we have

$$A^\dagger A = U^*U, BB^\dagger = VV^*, A = U|A| \text{ and } B = |B^*|V.$$

The equations above together with (3.10) yield

$$\begin{aligned} |B^*| \cdot VU^*U - VV^*U \cdot |A| &= |B^*| \cdot V(I_n - tD_2^{-1})U^*U \\ &\quad + VV^*(tD_1^* - I_m)U \cdot |A|. \end{aligned} \tag{3.13}$$

Similarly, from (3.9) we can obtain

$$\begin{aligned} |A^*| \cdot UV^*V - UU^*V \cdot |B| &= |A^*| \cdot U(I_n - sD_2)V^*V \\ &\quad + UU^*\big(s(D_1^{-1})^* - I_m\big)V \cdot |B|, \end{aligned}$$

which gives

$$|A|V^* - U^*|B^*| = |A|(I_n - sD_2)V^* + U^*\big(s(D_1^{-1})^* - I_m\big)|B^*|, \tag{3.14}$$

since $U^*|A^*|U = |A|$ and $V|B|V^* = |B^*|$. In view of $U^*U|A| = |A|$ and $|B^*|VV^* = |B^*|$, from (3.14) we first take $*$-operation and then get

$$\begin{aligned} VU^*U \cdot |A| - |B^*| \cdot VV^*U &= V(I_n - \overline{s}D_2^*)U^*U \cdot |A| \\ &\quad + |B^*| \cdot VV^*(\overline{s}D_1^{-1} - I_m)U. \end{aligned} \tag{3.15}$$

Then the summation of (3.13) and (3.15) gives

$$|B^*|\Omega_1 + \Omega_1|A| = |B^*|C + D|A|, \tag{3.16}$$

where Ω_1 is given by (3.3) and

$$C = V(I_n - tD_2^{-1})U^*U + VV^*(\overline{s}D_1^{-1} - I_m)U, \tag{3.17}$$

$$D = VV^*(tD_1^* - I_m)U + V(I_n - \overline{s}D_2^*)U^*U. \tag{3.18}$$

Note that both $|B^*|$ and $|A|$ are Hermitian positive semi-definite, so we have

$$|A| \cdot |A|^\dagger = |A|^\dagger \cdot |A| = P_{\mathcal{R}(|A|)} = P_{\mathcal{R}(A^*)} = A^\dagger A = U^*U, \tag{3.19}$$

$$|B^*| \cdot |B^*|^\dagger = |B^*|^\dagger \cdot |B^*| = P_{\mathcal{R}(|B^*|)} = P_{\mathcal{R}(B)} = BB^\dagger = VV^*. \tag{3.20}$$

It follows from (3.17), (3.19) and (3.20) that

$$|B^*|^\dagger \cdot |B^*| \cdot C = VV^*C = C = CU^*U = C \cdot |A| \cdot |A|^\dagger.$$

Similarly, it can be deduced from (3.3), (3.18), (3.19) and (3.20) that

$$|B^*|^\dagger \cdot |B^*| \cdot D = D = D \cdot |A| \cdot |A|^\dagger,\ |B^*|^\dagger \cdot |B^*| \cdot \Omega_1 = \Omega_1 = \Omega_1 \cdot |A| \cdot |A|^\dagger.$$

Therefore, by Theorem 2.3 we have

$$\|\Omega_1\|_F \le \frac{\|C+D\|_F + \mu\|C-D\|_F}{2}, \tag{3.21}$$

where λ and μ are given by (2.21), since $\big\| \, |B^*| \, \big\| = \|B\|$, $\big\| \, |A| \, \big\| = \|A\|$ and

$$|B^*|^\dagger = \Big[(BB^*)^{\frac{1}{2}}\Big]^\dagger = \Big[(BB^*)^\dagger\Big]^{\frac{1}{2}} = \Big[(B^\dagger)^* B^\dagger\Big]^{\frac{1}{2}} \Longrightarrow \Big\| |B^*|^\dagger \Big\|_2 = \|B^\dagger\|_2,$$

$$|A|^\dagger = \Big[(A^*A)^{\frac{1}{2}}\Big]^\dagger = \Big[(A^*A)^\dagger\Big]^{\frac{1}{2}} = \Big[A^\dagger (A^\dagger)^*\Big]^{\frac{1}{2}} \Longrightarrow \Big\| |A|^\dagger \Big\|_2 = \|A^\dagger\|_2.$$

It is evident that

$$\frac{\Big[\|C+D\|_F + \mu\|C-D\|_F\Big]^2}{4} \le \frac{\|C+D\|_F^2 + \mu^2\|C-D\|_F^2}{2},$$

so by (3.21) we can obtain

$$\begin{aligned} \|\Omega_1\|_F^2 &\le \frac{\|C+D\|_F^2 + \|C-D\|_F^2}{2} - \frac{(1-\mu^2)\|C-D\|_F^2}{2} \\ &= \frac{\|C+D\|_F^2 + \|C-D\|_F^2}{2} - \frac{\|C-D\|_F^2}{\lambda+1}. \end{aligned} \tag{3.22}$$

Next, we deal with $\|\Omega_1\|_F^2 + \|\Omega_2\|_F^2 + \|\Omega_3\|_F^2$ based on (3.22). It follows from (3.17), (3.18), (3.4), (3.5), (2.1) and (2.2) that

$$\begin{aligned} &\frac{\|C+D\|_F^2 + \|C-D\|_F^2}{2} + \|\Omega_2\|_F^2 + \|\Omega_3\|_F^2 \\ =\ &\frac{\|C+D+\Omega_2+\Omega_3\|_F^2 + \|C-D+\Omega_2-\Omega_3\|_F^2}{2} \\ =\ &\frac{\|W_1+W_2\|_F^2 + \|W_1-W_2\|_F^2}{2} = \|W_1\|_F^2 + \|W_2\|_F^2, \end{aligned} \tag{3.23}$$

where

$$W_1 = C + \Omega_2 \text{ and } W_2 = D + \Omega_3. \tag{3.24}$$

Note that by (3.4), (3.5) and (3.7), we have

$$\begin{aligned} \Omega_2 &= V \cdot V^*V(I_n - U^*U) = V \cdot B^\dagger B(I_n - A^\dagger A) \\ &= V \cdot B^\dagger B(I_n - tD_2^{-1})(I_n - A^\dagger A) = V(I_n - tD_2^{-1})(I_n - U^*U), \\ \Omega_3 &= -(I_m - BB^\dagger)AA^\dagger \cdot U = (I_m - BB^\dagger)(tD_1^* - I_m)AA^\dagger \cdot U \\ &= (I_m - VV^*)(tD_1^* - I_m)U. \end{aligned}$$

The modified expressions of Ω_2 and Ω_3 above, together with (3.17), (3.18) and (3.24), yield

$$\begin{aligned} W_1 &= V(I_n - tD_2^{-1}) + VV^*(\bar{s}D_1^{-1} - I_m)U, &(3.25)\\ W_2 &= (tD_1^* - I_m)U + V(I_n - \bar{s}D_2^*)U^*U. &(3.26)\end{aligned}$$

Therefore, by (3.6), (3.22)–(3.23), (3.25)–(3.26) and (3.17)–(3.18), we conclude that

$$\|V-U\|_F^2 \le \|W_1\|_F^2 + \|W_2\|_F^2 - \frac{\|C-D\|_F^2}{\lambda+1} = \sum_{i=1}^{3}\varphi_i^2(s,t).$$

This completes the proof of (3.1). □

One of the main results of [4] turns out to be a corollary as follows:

Corollary 3.2. [4, Theorem 2.2] *Let B be the multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.6), and let $A = U|A|$ and $B = V|B|$ be the generalized polar decompositions of A and B, respectively. Then*

$$\|V-U\|_F \le \sqrt{\left(\|I_m - D_1^{-1}\|_F + \|I_n - D_2^{-1}\|_F\right)^2 + (\|I_m - D_1\|_F + \|I_n - D_2\|_F)^2}.$$

Proof. With the notations as in the proof of Theorem 3.1, we have

$$\begin{aligned} \varphi_1(1,1) &\le \|V\|_2 \cdot \|I_n - D_2^{-1}\|_F + \|VV^*\|_2 \cdot \|D_1^{-1} - I_m\|_F \cdot \|U\|_2\\ &\le \|I_n - D_2^{-1}\|_F + \|I_m - D_1^{-1}\|_F,\\ \varphi_2(1,1) &\le \|U^*\|_2 \cdot \|D_1 - I_m\|_F + \|U^*U\|_2 \cdot \|I_n - D_2\|_F \cdot \|V^*\|_2\\ &\le \|I_m - D_1\|_F + \|I_n - D_2\|_F.\end{aligned}$$

By (3.1) we have

$$\|V-U\|_F \le \sqrt{\varphi_1^2(1,1) + \varphi_2^2(1,1) - \varphi_3^2(1,1)} \le \sqrt{\varphi_1^2(1,1) + \varphi_2^2(1,1)}.$$

The desired upper bound then follows. □

Next, we provide the perturbation estimation for positive semi-definite polar factors as follows:

Theorem 3.3. *Let B be the multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.6), and let $A = U|A|$ and $B = V|B|$ be the generalized polar decompositions of A and B, respectively. Then*

$$\big\||B| - |A|\big\|_F \le \inf_{s,t\in\mathbb{C}} \sqrt{\gamma_1^2(s,t) + \gamma_2^2(s,t) - \gamma_3^2(s,t)}, \tag{3.27}$$

where $s,t\in\mathbb{C}$ *are arbitrary,* λ *is defined by* (2.21) *and*

$$\gamma_1(s,t)=\big\|\,|B|(I_n-tD_2^{-1})+V^*(tD_1^*-\bar{s}D_1^{-1})A\,\big\|_F,$$
$$\gamma_2(s,t)=\big\|\,|A|(sD_2-I_n)\big\|_F,$$
$$\gamma_3(s,t)=\frac{\big\||B|(I_n-tD_2^{-1})U^*U+V^*(tD_1^*-\bar{s}D_1^{-1})A-V^*V(\bar{s}D_2^*-I_n)|A|\big\|_F}{\sqrt{\lambda+1}}.$$

Proof. Since $|B|$ is Hermitian positive semi-definite, we have

$$|B|\cdot|B|^\dagger=|B|^\dagger\cdot|B|=P_{\mathcal{R}(|B|)}=P_{\mathcal{R}(B^*)}=B^\dagger B=V^*V. \tag{3.28}$$

As shown in the derivation of (3.2), by (3.19) and (3.28) we can obtain

$$|B|-|A|=\Upsilon_1+\Upsilon_2+\Upsilon_3,$$

where

$$\Upsilon_1=|B|\cdot U^*U-V^*V\cdot|A|=V^*V\cdot\Upsilon_1\cdot U^*U, \tag{3.29}$$
$$\Upsilon_2=|B|(I_n-U^*U)=V^*V\cdot\Upsilon_2\cdot(I_n-U^*U), \tag{3.30}$$
$$\Upsilon_3=-(I_n-V^*V)\cdot|A|=(I_n-V^*V)\cdot\Upsilon_3. \tag{3.31}$$

By (2.1) and (2.2) we have

$$\big\|\,|B|-|A|\,\big\|_F^2=\|\Upsilon_1+\Upsilon_2\|_F^2+\|\Upsilon_3\|_F^2=\|\Upsilon_1\|_F^2+\|\Upsilon_2\|_F^2+\|\Upsilon_3\|_F^2. \tag{3.32}$$

In what follows we first deal with $\|\Upsilon_1\|_F^2$. Note that $|B^*|V=B=V|B|$, so if Pre-multiply U and post-multiply V, then from (3.14) we can obtain

$$U|A|V^*V-UU^*V|B|=U|A|(I_n-sD_2)V^*V+UU^*\big(s(D_1^{-1})^*-I_m\big)V|B|,$$

which gives by taking $*$-operation that

$$V^*V|A|U^*-|B|V^*UU^*=V^*V(I_n-\bar{s}D_2^*)|A|U^*+|B|V^*(\bar{s}D_1^{-1}-I_m)UU^*.$$

As $U^*U|A|=|A|=|A|U^*U$, post-multiplying the equation above by $U|A|$ yields

$$\begin{aligned}V^*V|A|\cdot|A|-|B|V^*U|A| &= V^*V(I_n-\bar{s}D_2^*)|A|\cdot|A|\\ &\quad+|B|V^*(\bar{s}D_1^{-1}-I_m)\,U|A|.\end{aligned} \tag{3.33}$$

It is notable that we can get $A=(D_1^{-1})^*BD_2^{-1}$ from $B=D_1^*AD_2$, so it can be deduced from (3.33) that for any $t\in\mathbb{C}$,

$$\begin{aligned}U^*U|B|\cdot|B|-|A|U^*V|B| &= U^*U\big(I_n-\bar{t}(D_2^*)^{-1}\big)|B|\cdot|B|\\ &\quad+|A|U^*(\bar{t}D_1-I_m)\,V|B|,\end{aligned}$$

which gives by taking $*$-operation that

$$\begin{aligned}|B|\cdot|B|U^*U-|B|V^*U|A| &= |B|\cdot|B|(I_n-tD_2^{-1})U^*U \\ &\quad +|B|V^*(tD_1^*-I_m)U|A|. \qquad (3.34)\end{aligned}$$

Subtracting (3.33) from (3.34) leads to

$$|B|\cdot|B|U^*U-V^*V|A|\cdot|A|=|B|\cdot C+D\cdot|A|, \qquad (3.35)$$

where

$$C=|B|(I_n-tD_2^{-1})U^*U+V^*(tD_1^*-\bar{s}D_1^{-1})U|A|, \qquad (3.36)$$
$$D=V^*V(\bar{s}D_2^*-I_n)|A|. \qquad (3.37)$$

Moreover, we have

$$\begin{aligned}|B|\,|B|U^*U-V^*V|A|\,|A| &= |B|\,|B|U^*U-|B|\,|A|+|B|\,|A|-V^*V|A|\,|A| \\ &= |B|\cdot\Upsilon_1+\Upsilon_1\cdot|A|,\end{aligned}$$

which is combined with (3.35) to get

$$|B|\cdot\Upsilon_1+\Upsilon_1\cdot|A|=|B|\cdot C+D\cdot|A|$$

with the property that

$$|B|^\dagger\cdot|B|\cdot X=X=X\cdot|A|\cdot|A|^\dagger \text{ for any } X\in\{\Upsilon_1,C,D\}.$$

Then as in the proof of Theorem 3.1, we can obtain

$$\|\Upsilon_1\|_F^2\le\frac{\|C+D\|_F^2+\|C-D\|_F^2}{2}-\frac{\|C-D\|_F^2}{\lambda+1}, \qquad (3.38)$$

where λ is given by (2.21).

Next, we modify the expressions of Υ_2 and Υ_3. From (3.30), (3.31) and (3.7), we have

$$\begin{aligned}\Upsilon_2 &= V^*V|B|(I_n-U^*U)=V^*B(I_n-A^\dagger A)=V^*B(I_n-tD_2^{-1})(I_n-A^\dagger A) \\ &= V^*V|B|(I_n-tD_2^{-1})(I_n-U^*U)=|B|(I_n-tD_2^{-1})(I_n-U^*U), \\ \Upsilon_3 &= -(I_n-B^\dagger B)A^*U=-(I_n-B^\dagger B)(I_n-\bar{s}D_2^*)A^*U \\ &= -(I_n-B^\dagger B)(I_n-\bar{s}D_2^*)|A|U^*U=(I_n-V^*V)(\bar{s}D_2^*-I_n)|A|.\end{aligned}$$

The new expressions of Υ_2 and Υ_3 above, together with (3.36) and (3.37), yield

$$W_1 \stackrel{def}{=} C + \Upsilon_2 = |B| \cdot (I_n - tD_2^{-1}) + V^*(tD_1^* - \bar{s}D_1^{-1})A, \quad (3.39)$$

$$W_2 \stackrel{def}{=} D + \Upsilon_3 = (\bar{s}D_2^* - I_n) \cdot |A|. \quad (3.40)$$

As in the proof of Theorem 3.1, based on (3.38)–(3.40), we can obtain

$$\|V - U\|_F^2 \leq \|W_1\|_F^2 + \|W_2\|_F^2 - \frac{\|C - D\|_F^2}{\lambda + 1} = \sum_{i=1}^{3} \gamma_i^2(s,t).$$

This completes the proof of (3.27). ☐

A result of [10] can be derived from the preceding theorem as follows:

Corollary 3.4. [10, Corollary 3.3 (2)] *Let B be the multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.6), and let $A = U|A|$ and $B = V|B|$ be the generalized polar decompositions of A and B, respectively. Then*

$$\| \, |B| - |A| \, \|_F \leq \sqrt{\rho^2 + \|A\|_2^2 \cdot \|I_n - D_2\|_F^2},$$

where

$$\rho = \|B\|_2 \cdot \|I_n - D_2^{-1}\|_F + \|D_1^* - D_1^{-1}\|_F \cdot \|A\|_2.$$

Proof. With the notations as in the proof of Theorem 3.3, we have

$$\gamma_1(1,1) \leq \rho \text{ and } \gamma_2(1,1) \leq \|A\|_2 \cdot \|I_n - D_2\|_F.$$

From (3.27) we have

$$\| \, |B| - |A| \, \|_F \leq \sqrt{\gamma_1^2(1,1) + \gamma_2^2(1,1) - \gamma_3^2(1,1)} \leq \sqrt{\gamma_1^2(1,1) + \gamma_2^2(1,1)}.$$

The desired upper bound then follows. ☐

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