Solution estimates for linear differential equations with delay

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Abstract

In this paper, we give explicit exponential estimates $|x(t)| \leq Me^{-\gamma(t-t_0)}$, where $t \geq t_0$, M > 0, for solutions of a linear scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = f(t), \quad t \ge t_0, \quad x(t) = \phi(t), \quad t \le t_0.$$

We consider two different cases: when $\gamma > 0$ (corresponding to exponential stability) and the case of $\gamma < 0$ when the solution is, generally, growing.

In the first case, together with the exponential estimate, we also obtain an exponential stability test, in the second case we get estimation for solution growth. Here both the coefficients and the delays are measurable, not necessarily continuous.

Keywords: linear delay differential equations, explicit solution estimates, variable delays and coefficients, exponential stability

AMS subject classification: 34K20, 34K25, 34K06

1. Introduction

Exponential or asymptotic stability of solutions is one of the most important properties of a functional differential equation (FDE). There are many publications on stability of a FDE. We cite here the papers [6, 7, 8, 9, 16, 17, 18, 22, 26, 28] and the monographs [4, 13, 14, 15, 21] fully or partially devoted to asymptotic stability of this class of equations. Asymptotic stability describes long time behavior of solutions. But in applications of FDEs, usually it is necessary to know estimates of solutions on finite intervals. The following well-known topics are close to one considered in the paper: Lyapunov exponents, asymptotic integration, and growth rates [2, 3, 5, 10, 11, 12, 19, 20, 23, 24, 25, 27].

In this paper, we propose exponential estimates for solutions of a scalar delay differential equation (DDE) with variable coefficients and delays

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0, \quad t \ge t_0, \ h_k(t) \le t, \ k = 1, \dots, m.$$
(1.1)

Estimates for non-homogeneous equations are also considered. We investigate two different cases, one when the equation is exponentially stable and the second when it is not, or when asymptotic behaviour of solutions is unknown.

For the equation with one delay

$$\dot{x}(t) + a(t)x(h(t)) = 0,$$

for example, the first case occurs when $a(t) \ge a_0 > 0$ and the equation is exponentially stable, the second case is for $a(t) \ge 0$ but the equation is not exponentially stable, or for a(t) being non-positive or oscillatory.

All our conditions can be applied to a wide class of scalar linear differential equations with variable coefficients and delays, without the assumption that these parameters are continuous functions, leading to solution estimates. For equations with variable coefficients and delays, such solution estimates are obtained for the first time, and for ordinary differential equations are sharp.

The paper is organized as follows. After some definitions and auxiliary results in Section 2, Section 3 contains an estimate for the fundamental function of an equation with a non-delay term. Section 4 is the main part of the paper. Here we consider solution estimates in the two above mentioned cases. Section 5 presents illustrating examples. Section 6 contains a discussion and suggests some projects for future research.

2. Preliminaries

We consider equation (1.1) under the following conditions:

- (a1) $b_k:[0,\infty)\to\mathbb{R}$ are Lebesgue measurable essentially bounded functions;
- (a2) the functions h_k are Lebesgue measurable on $[0, \infty)$, and for some finite constants $\delta_k \leq \tau_k$, $0 \leq \delta_k \leq t h_k(t) \leq \tau_k$, $k = 1, \ldots, m, t \geq t_0 \geq 0$, $\tau = \max_k \tau_k$.

Together with equation (1.1), we consider for any $t_0 \ge 0$ an initial value problem for the non-homogeneous equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = f(t), \ t \ge t_0; \ x(t) = \phi(t), \ t \le t_0,$$
(2.1)

where

(a3) $f:[t_0,\infty)\to\mathbb{R}$ is a Lebesgue measurable locally essentially bounded function, the initial function $\phi:[t_0-\tau,t_0]\to\mathbb{R}$ is bounded and Borel measurable.

By the solution of problem (2.1) we mean a locally absolutely continuous function x: $[t_0, \infty) \to \mathbb{R}$ satisfying almost everywhere the equation, whenever $t \ge t_0$, and the initial conditions if $t \le t_0$.

For any fixed s, the solution X(t,s) of the problem with the zero initial function and the initial value being equal to one

$$\frac{\partial X(t,s)}{\partial t} + \sum_{k=1}^{m} b_k(t)X(h_k(t),s) = 0, \ t \ge s; \ X(t,s) = 0, \ t < s, \ X(s,s) = 1$$

is called the fundamental function.

According to [4, Theorem 4.3.1], the solution of (2.1) exists and is unique. Also, the solution has the representation

$$x(t) = X(t, t_0)x(t_0) - \sum_{k=1}^{m} \int_{t_0}^{t_0 + \tau_k} X(t, s)b_k(s)\phi(h_k(s))ds + \int_{t_0}^{t} X(t, s)f(s)ds.$$
 (2.2)

In (2.2), we assume $\phi(t) = 0, t > t_0$.

We call equation (1.1) **uniformly exponentially stable** if, for some positive numbers M and γ , not depending on t_0 and ϕ , a solution of homogeneous (2.1), where $f \equiv 0$, satisfies

$$|x(t)| \le Me^{-\gamma(t-t_0)} \sup_{t \in [t_0-\tau,t_0]} |\phi(t)|, \quad t \ge t_0.$$
 (2.3)

We will say that the fundamental function X(t,s) of (1.1) has an exponential estimate if there exist $M_0 > 0$ and $\gamma_0 > 0$ such that

$$|X(t,s)| \le M_0 e^{-\gamma_0(t-s)}, \text{ for any } t \ge s \ge 0.$$
 (2.4)

We will further apply the Bohl-Perron theorem stated below.

Lemma 1. [4, Theorem 4.7.1] Assume that (a1)-(a2) hold. If the solution of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = f(t), \ x(t) = 0, \ t \le t_0$$
(2.5)

is bounded on $[t_0, \infty)$ for any essentially bounded function $f: [t_0, \infty) \to \mathbb{R}$, equation (1.1) is uniformly exponentially stable.

The main objective of the present paper is to obtain explicit estimates (2.3)-(2.4) and their generalizations for homogeneous equation (1.1). We consider the case when the equation is exponentially stable and when it is unstable, or asymptotic behaviour is unknown. In the former case $\gamma > 0$, while in the latter case $\gamma < 0$. We also get estimates for non-homogeneous equations (2.1).

3. Estimates for an Auxiliary Equation

In this section, for the fundamental function Y(t,s) of the equation involving both delay terms and a non-delay term

$$\dot{y}(t) = c(t)y(t) - \sum_{k=1}^{m} d_k(t)y(h_k(t)), \ t \ge t_0$$
(3.1)

we deduce a uniform estimate

$$|Y(t,s)| \le K \quad \text{for} \quad t \ge s \ge t_0. \tag{3.2}$$

We assume that for c, d_k , assumption (a1) holds, while the delay functions h_k satisfy (a2).

Denote
$$||f||_J = \operatorname{ess\,sup}_{t \in J} |f(t)|$$
, where $J = [t_0, t_1]$ or $J = [t_0, \infty)$, $d(t) := \sum_{k=1}^m d_k(t)$.

Lemma 2. If there is an $\alpha_0 > 0$ such that

$$d(t) - c(t) \ge \alpha_0, \quad K_0 := \left(\|c\|_{[t_0, \infty)} + \sum_{k=1}^m \|d_k\|_{[t_0, \infty)} \right) \sum_{k=1}^m \tau_k \left\| \frac{d_k}{d - c} \right\|_{[t_0, \infty)} < 1$$
 (3.3)

then (3.1) is uniformly exponentially stable. Moreover, the fundamental function Y(t,s) of (3.1) satisfies (3.2) with $K = (1 - K_0)^{-1}$.

Proof. For brevity of notations, we set $y(t) = Y(t, t_0)$. Then, y satisfies (3.1), where the initial value is $y(t_0) = 1$, with the zero initial function. Let $J = [t_0, t_1]$, where $t_1 > t_0$ is arbitrary. Equality (3.1) implies the estimate

$$\|\dot{y}\|_{J} \le \left(\|c\|_{[t_{0},\infty)} + \sum_{k=1}^{m} \|d_{k}\|_{[t_{0},\infty)} \right) \|y\|_{J}. \tag{3.4}$$

Equation (3.1) can be written as

$$\dot{y}(t) = -[d(t) - c(t)]y(t) + \sum_{k=1}^{m} d_k(t) \int_{h_k(t)}^{t} \dot{y}(\xi) d\xi.$$

Integrating, we get for y

$$y(t) = e^{-\int_{t_0}^t [d(\xi) - c(\xi)] d\xi} + \int_{t_0}^t e^{-\int_s^t [d(\xi) - c(\xi)] d\xi} [d(s) - c(s)] \left[\sum_{k=1}^m \frac{d_k(s)}{d(s) - c(s)} \int_{h_k(s)}^s \dot{y}(\xi) d\xi \right] ds.$$

Then, by the above equality, (3.4) and the definition of K_0 in (3.3),

$$||y||_J \le 1 + \sum_{k=1}^m \tau_k \left\| \frac{d_k}{d-c} \right\|_{[t_0,\infty)} ||\dot{y}||_J \le 1 + K_0 ||y||_J.$$

Then $||Y(t,t_0)||_J \leq (1-K_0)^{-1}$, and the expression in the right-hand side does not depend on t_1 . Hence $||Y(t,t_0)||_{[t_0,\infty)} \leq (1-K_0)^{-1}$. Again, the number in the right-hand side does not depend on t_0 . Thus estimate (3.2) holds.

To prove exponential stability, we apply the Bohl-Perron theorem. Let $f:[t_0,\infty)\to\mathbb{R}$ be globally essentially bounded, and y be a solution of

$$\dot{y}(t) = c(t)y(t) - \sum_{k=1}^{m} d_k(t)y(h_k(t)) + f(t), \ t \ge t_0, \ y(t) = 0, t \le t_0.$$
(3.5)

Equality (3.5) implies

$$\|\dot{y}\|_{J} \le \left(\|c\|_{[t_{0},\infty)} + \sum_{k=1}^{m} \|d_{k}\|_{[t_{0},\infty)} \right) \|y\|_{J} + \|f\|_{[t_{0},\infty)}. \tag{3.6}$$

Similarly to the above argument, equation (3.5) can be rewritten and integrated:

$$\dot{y}(t) = -\left[d(t) - c(t)\right]y(t) + \sum_{k=1}^{m} d_k(t) \int_{h_k(t)}^{t} \dot{y}(\xi)d\xi + f(t),$$

$$y(t) = \int_{t_0}^{t} e^{-\int_{s}^{t} [d(\xi) - c(\xi)]d\xi} [d(s) - c(s)] \left[\sum_{k=1}^{m} \frac{d_k(s)}{d(s) - c(s)} \int_{h_k(s)}^{s} \dot{y}(\xi)d\xi\right] ds$$

$$+ \int_{t_0}^{t} e^{-\int_{s}^{t} [d(\xi) - c(\xi)]d\xi} [d(s) - c(s)] \frac{f(s)}{d(s) - c(s)} ds.$$

Since $d(s) - c(s) \ge \alpha_0 > 0$ and $||f||_{[t_0,\infty)} < \infty$, we have $||\frac{f}{d-c}||_{[t_0,\infty)} < \infty$. Then, by (3.6),

$$||y||_J \le \sum_{k=1}^m \tau_k \left\| \frac{d_k}{d-c} \right\|_{[t_0,\infty)} ||\dot{y}||_J + \left\| \frac{f}{d-c} \right\|_{[t_0,\infty)} \le K_0 ||y||_J + M_0,$$

where

$$M_0 = \sum_{k=1}^m \tau_k \left\| \frac{d_k}{d-c} \right\|_{[t_0,\infty)} \|f\|_{[t_0,\infty)} + \left\| \frac{f}{d-c} \right\|_{[t_0,\infty)}.$$

Hence for the constant $M = (1 - K_0)^{-1} M_0 > 0$, we have $||y||_J \leq M$, where M does not depend on the interval J. Therefore $||x||_{[t_0,\infty)} < \infty$. Appliing Lemma 1, we conclude that (3.1) is uniformly exponentially stable.

Remark 1. In Lemma 2, instead of K_0 , we can take any number $K_1 \in (K_0, 1)$.

Remark 2. Equation (3.1) is uniformly exponentially stable under the assumptions of the lemma. In contrast to a known result on exponential stability of an equation $\dot{y} + c(t)y(t) + d(t)y(h(t)) = 0$ with a dominating non-delay term $c(t) > |d(t)| \ge 0$, we consider the case $c(t) \le 0$, d(t) > c(t). In this sense, the delay term is dominating over the non-delay term.

4. Main Results

4.1. Exponentially stable equations

We start with an exponential estimate for an exponentially stable equation.

Theorem 1. Assume that there are constants $\lambda > 0$ and $\alpha > 0$ such that the following inequalities hold:

$$a(t) := \sum_{k=1}^{m} e^{\lambda(t - h_k(t))} b_k(t) - \lambda \ge \alpha, \quad t \ge t_0,$$
(4.1)

$$M_1 := \left(\lambda + \sum_{k=1}^m e^{\lambda \tau_k} \|b_k\|_{[t_0,\infty)}\right) \sum_{k=1}^m \tau_k \left\|\frac{a_k}{a}\right\|_{[t_0,\infty)} < 1, \tag{4.2}$$

where $a_k(t) = e^{\lambda(t-h_k(t))}b_k(t)$.

Then for the solution of problem (2.1), the following estimate is valid

$$|x(t)| \le M_0 e^{-\lambda(t-t_0)} \left[|x(t_0)| + \sum_{k=1}^m \frac{e^{\lambda \tau_k} - 1}{\lambda} ||b_k||_{[t_0,\infty)} ||\phi||_{[t_0-\tau,t_0]} \right] + \frac{M_0}{\lambda} ||f||_{[t_0,t]}, \tag{4.3}$$

where $M_0 := (1 - M_1)^{-1}$.

Proof. Consider first the case $f \equiv 0$. After the substitution $x(t) = e^{-\lambda(t-t_0)}z(t)$ into (2.1), we get

$$\dot{z}(t) = \lambda z(t) - \sum_{k=1}^{m} e^{\lambda(t - h_k(t))} b_k(t) z(h_k(t)). \tag{4.4}$$

Equation (4.4) has the form of (3.1) with

$$c(t) = \lambda, \quad d_k(t) = e^{\lambda(t - h_k(t))} b_k(t), \quad d(t) = \sum_{k=1}^m e^{\lambda(t - h_k(t))} b_k(t).$$

Then in (4.1), a(t) corresponds to d(t) - c(t). Let Z(t,s) be the fundamental function of (4.4). Inequalities (4.1) and (4.2) imply (3.3). By Lemma 2, $|Z(t,s)| \leq M_0$. Let X(t,s) be a fundamental function of (2.1). Then, for X(t,s) we have the exponential equality $X(t,s) = e^{-\lambda(t-s)}Z(t,s)$. Hence $|X(t,s)| \leq M_0 e^{-\lambda(t-s)}$. By (2.2), for the solution x of problem (2.1) we have

$$|x(t)| \leq |X(t,t_0)||x(t_0)| + \sum_{k=1}^m \int_{t_0}^{t_0+\tau_k} |X(t,s)||b_k(s)||\phi(h_k(s))|ds$$

$$\leq M_0 e^{-\lambda(t-t_0)}|x(t_0)| + \frac{M_0}{\lambda} \sum_{k=1}^m ||b_k||_{[t_0,\infty)} \left(e^{-\lambda(t-t_0-\tau_k)} - e^{-\lambda(t-t_0)}\right) ||\phi||_{[t_0-\tau,t_0]},$$

which implies (4.3) with $f \equiv 0$.

For the general case we apply (2.2), the estimate for X(t,s) and the inequalities

$$\left| \int_{t_0}^t X(t,s)f(s) \, ds \right| \le M_0 \|f\|_{[t_0,t]} \int_{t_0}^t e^{-\lambda(t-t_0)} ds \le \frac{M_0}{\lambda} \|f\|_{[t_0,t]}.$$

Remark 3. Note that if the conditions of Theorem 1 hold, (1.1) is uniformly exponentially stable.

Remark 4. Let us apply Theorem 1 to a non-delay equation

$$\dot{x}(t) + b(t)x(t) = f(t), \quad t \ge t_0, \quad x(t_0) = x_0,$$
 (4.5)

where $b(t) \ge b_0 > 0$. If we take $\lambda = b_0$, conditions (4.1) and (4.2) hold, $M_0 = 1$. Estimate (4.3) becomes

$$|x(t)| \le e^{-b_0(t-t_0)}|x_0| + \frac{1}{b_0}||f||_{[t_0,t]},$$

where for constant positive b and f the equality holds. Hence estimate (4.3) in some sense is sharp.

The continuity of the functions $u(\lambda) = e^{\lambda \tau_k}$ which tend to one as $\lambda \to 0^+$ implies the following result.

Corollary 1. Assume that for some $\lambda_0 > 0$, $\alpha > 0$,

$$\sum_{k=1}^{m} e^{\lambda_0(t-h_k(t))} b_k(t) - \lambda_0 \ge \alpha, \quad \sum_{k=1}^{m} \|b_k\|_{[t_0,\infty)} \sum_{k=1}^{m} \tau_k \|b_k\|_{[t_0,\infty)} < \alpha.$$

Then equation (1.1) is uniformly exponentially stable, and for some $\lambda \in (0, \lambda_0]$, the solution of problem (2.1) satisfies estimate (4.3).

Theorem 1 and its corollary give only implicit conditions for the estimation of solutions of equation (1.1). The following theorem contains explicit conditions for the estimates.

Theorem 2. Assume that

$$b_k(t) \ge 0, \quad b_0 := \liminf_{t \to \infty} \sum_{k=1}^m b_k(t) > 0, \quad \sum_{k=1}^m \|b_k\|_{[t_0,\infty)} \sum_{k=1}^m \tau_k \left\| \frac{b_k}{\sum_{i=1}^m b_i} \right\|_{[t_0,\infty)} < 1.$$
 (4.6)

Then there exists a unique solution $\lambda_0 \in (0, b_0)$ of the equation

$$\left(\lambda + \sum_{k=1}^{m} e^{\lambda \tau_k} \|b_k\|_{[t_0,\infty)}\right) \sum_{k=1}^{m} \tau_k e^{\lambda \tau_k} \left\| \frac{b_k}{\sum_{k=1}^{m} b_k - \lambda} \right\|_{[t_0,\infty)} = 1.$$
 (4.7)

If $0 < \lambda < \lambda_0$, the solution of problem (2.1) satisfies (4.3).

Proof. For $0 < \lambda < b_0$, where b_0 is defined in (4.6), denote

$$g(\lambda) := \left(\lambda + \sum_{k=1}^{m} e^{\lambda \tau_k} \|b_k\|_{[t_0,\infty)}\right) \sum_{k=1}^{m} \tau_k e^{\lambda \tau_k} \left\| \frac{b_k}{\sum_{k=1}^{m} b_k - \lambda} \right\|_{[t_0,\infty)}.$$

This function is well defined as $\sum_{k=1}^{m} b_k - \lambda > 0$ for $\lambda \in (0, b_0)$, the terms $e^{\lambda \tau_k}$ increase in λ , while $\sum_{k=1}^{m} b_k - \lambda$ in the denominator is positive and decreasing, leading to the increase in the fraction, thus g is monotone increasing in λ . Since by (4.6), g(0) < 1, $g(b_0) = \infty$ and g is monotone increasing, the equation $g(\lambda) = 1$ has a unique solution $\lambda \in (0, b_0)$, which is denoted by λ_0 . By the

definition of g, $\left|\sum_{k=1}^{m} b_k(t) - \lambda_0\right| \ge \varepsilon b_k(t)$ for $k = 1, \dots, m$ and some $\varepsilon > 0$. Let $\lambda \in (0, \lambda_0)$. Then

$$\left| \frac{b_k(t)}{\sum_{k=1}^m e^{\lambda(t-h_k(t))} b_k(t) - \lambda} \right| = \frac{b_k(t)}{\sum_{k=1}^m e^{\lambda(t-h_k(t))} b_k(t) - \lambda} < \left\| \frac{b_k(t)}{\sum_{k=1}^m b_k(t) - \lambda_0} \right\|_{[t_0, \infty)}.$$

Hence

$$\left(\lambda + \sum_{k=1}^{m} e^{\lambda \tau_k} \|b_k\|_{[t_0,\infty)}\right) \sum_{k=1}^{m} \tau_k e^{\lambda \tau_k} \left\| \frac{b_k}{a} \right\|_{[t_0,\infty)} < g(\lambda_0) = 1.$$

By Theorem 1, the solution of problem (2.1) satisfies (4.3).

Corollary 2. Assume that $b_k(t) \equiv b_k > 0$, k = 1, ..., m, $b_0 := \sum_{k=1}^m b_k$ and $\sum_{k=1}^m \tau_k b_k < 1$. Then there is a unique solution $\lambda_0 \in (0, b_0)$ of

$$\left(\lambda + \sum_{k=1}^{m} e^{\lambda \tau_k} b_k\right) \sum_{k=1}^{m} \tau_k e^{\lambda \tau_k} \frac{b_k}{\sum_{k=1}^{m} b_k - \lambda} = 1.$$

If $0 < \lambda < \lambda_0$, the solution of problem (2.1) satisfies (4.3), where $||b_k||$ is replaced by b_k .

4.2. The equation is not exponentially stable

We proceed to the case when the equation in (2.1) is not exponentially stable, or its asymptotic behaviour is unknown.

Theorem 3. Let there exist $\lambda > 0$ and $\alpha > 0$ such that

$$a(t) := \sum_{k=1}^{m} e^{-\lambda(t - h_k(t))} b_k(t) + \lambda \ge \alpha,$$
 (4.8)

$$M_2 := \left(\lambda + \sum_{k=1}^m e^{-\lambda \delta_k} \|b_k\|_{[t_0,\infty)}\right) \sum_{k=1}^m \tau_k \left\| \frac{a_k}{a} \right\|_{[t_0,\infty)} < 1, \tag{4.9}$$

where $a_k(t) = e^{-\lambda(t-h_k(t))}b_k(t)$. Then the solution of problem (2.1) satisfies

$$|x(t)| \le M_0 e^{\lambda(t-t_0)} \left[|x(t_0)| + \sum_{k=1}^m \frac{1 - e^{-\lambda \tau_k}}{\lambda} ||b_k||_{[t_0,\infty)} ||\phi||_{[t_0-\tau,t_0]} + \frac{||f||_{[t_0,t]}}{\lambda} \right],\tag{4.10}$$

where $M_0 := (1 - M_2)^{-1}$.

Proof. First, let $f \equiv 0$. After substituting $x(t) = e^{\lambda(t-t_0)}z(t)$ into (2.1), we get

$$\dot{z}(t) = -\lambda z(t) - \sum_{k=1}^{m} e^{-\lambda(t - h_k(t))} b_k(t) z(h_k(t)). \tag{4.11}$$

Equation (4.11) has the form of (3.1), where

$$c(t) = -\lambda, \ d_k(t) = e^{-\lambda(t - h_k(t))}b_k(t), \ d(t) = \sum_{k=1}^m e^{-\lambda(t - h_k(t))}b_k(t), \ a(t) := d(t) - c(t).$$

Let Z(t,s) be the fundamental function of equation (4.11). By (4.8), $d(t)-c(t) \ge \alpha$. Inequality (4.9) implies

$$K_0 = \left(\|c\|_{[t_0,\infty)} + \sum_{k=1}^m \|d_k\|_{[t_0,\infty)} \right) \sum_{k=1}^m \tau_k \left\| \frac{d_k}{d-c} \right\|_{[t_0,\infty)} \leq M_2 < 1.$$

By Lemma 2 and Remark 1, $|Z(t,s)| \leq M_0$. For the fundamental function of (2.1) we have $X(t,s) = e^{\lambda(t-s)}Z(t,s)$. Hence $|X(t,s)| \leq M_0e^{\lambda(t-s)}$. By (2.2), the solution x of problem (2.1) satisfies

$$|x(t)| \leq |X(t,t_0)| |x(t_0)| + \sum_{k=1}^m \int_{t_0}^{t_0+\tau_k} |X(t,s)| |b_k(s)| |\phi(h_k(s))| ds$$

$$\leq M_0 e^{\lambda(t-t_0)} |x(t_0)| - \frac{M_0}{\lambda} \sum_{k=1}^m ||b_k||_{[t_0,\infty)} \left(e^{\lambda(t-t_0-\tau_k)} - e^{\lambda(t-t_0)} \right) ||\phi||_{[t_0-\tau,t_0]},$$

which implies (4.10) with $f \equiv 0$. For the general case, we apply

$$\left| \int_{t_0}^t X(t,s)f(s) \, ds \right| \le \|f\|_{[t_0,t]} \int_{t_0}^t M_0 e^{\lambda(t-s)} ds \le \frac{M_0}{\lambda} e^{\lambda(t-t_0)} \|f\|_{[t_0,t]}.$$

Remark 5. Note that for m = 1, $h_1(t) \equiv t$, $f \equiv 0$, $-b_0 \leq b_1(t) \leq 0$ almost everywhere, where $b_0 > 0$, for a solution x of (2.1) the inequality

$$x(t) = x(t_0) \exp\left\{-\int_{t_0}^t b_1(s) \, ds\right\} \le |x(t_0)| e^{b_0(t-t_0)}$$

holds. For $\lambda = -b_0 > 0$, Theorem 3 implies this estimate which is sharp for $b_1 \equiv -b_0$.

Remark 6. By [1, Corollary B.1], for X(t,s), which is a fundamental function of (1.1), the inequality

$$|X(t,s)| \le \exp\left\{ \int_s^t \sum_{k=1}^m |b_k(\xi)| d\xi \right\}$$

holds. Hence $\sum_{k=1}^{m} |b_k(t)| \leq \lambda_0$ implies inequality (4.10) for the solution of problem (2.1), where $M_0 = 1$, $\lambda = \lambda_0$. Theorem 3 allows to obtain estimates for solutions of problem (2.1) with smaller than λ_0 exponents, see Examples 2 and 3.

Corollary 3. Assume that there are constants $\lambda > 0$ and $\alpha > 0$ such that

$$a(t) := \sum_{k=1}^{m} e^{-\lambda(t - h_k(t))} b_k(t) + \lambda \ge \alpha$$
 (4.12)

and

$$M_3 := \left(\lambda + \sum_{k=1}^m e^{-\lambda \delta_k} \|b_k\|_{[t_0,\infty)}\right) \sum_{k=1}^m \tau_k e^{-\lambda \delta_k} \|b_k\|_{[t_0,\infty)} < \alpha. \tag{4.13}$$

Then for the solution of problem (2.1), estimate (4.10) holds, where $M_0 := \frac{\alpha}{\alpha - M_3}$.

5. Examples

In this section, we illustrate Theorems 1, 2 and 3 with examples. In Example 1, we construct an exponential estimate with a negative exponent using explicit conditions of Theorem 2. Examples 2 and 3 apply Theorem 3 to an unstable equation. In Example 4, we consider a delay equation with oscillating coefficients and two terms. We apply Theorem 1 to establish uniform exponential stability of this equation, and construct an exponential estimate for solutions of the equation.

Example 1. For the non-homogeneous equation with one delay

$$\dot{x}(t) + 0.2(2 - \sin t)x \left(t - \frac{2 - \sin t}{12}\right) = f(t), \ t \ge 0,$$
(5.1)

 $we\ calculate$

$$\tau = \frac{1}{12} \max_{t} (2 - \sin t) = 0.25, \quad ||b||_{[0,\infty)} = 0.2 \max_{t} (2 - \sin t) = 0.6.$$

We will apply here Theorem 2.

To find λ_0 we first consider the function $w(t) = \frac{b(t)}{b(t) - \lambda} = \frac{0.2(2 - \sin t)}{0.2(2 - \sin t) - \lambda}$, where $0 < \lambda < 0.2$. The critical values of this function are solutions of the equation $\cos t = 0$. Hence $||w||_{[t_0,\infty)} = w(\frac{\pi}{2}) = \frac{1}{1 - 5\lambda}$.

Equation (4.7) for (5.1) has the form

$$\left(\lambda + 0.6e^{\lambda/4}\right) \frac{1}{4}e^{\lambda/4} \frac{1}{1 - 5\lambda} = 1.$$

Numerically we compute the solution of this equation $\lambda_0 \approx 0.159229$, the assumptions of Theorem 2 hold for $\lambda < 0.159229$, say, for $\lambda = 0.15 < \lambda_0$. It is possible to check numerically that for $\lambda = 0.15 < \lambda_0$,

$$a(t) := 0.2(2 - \sin t)e^{(2 - \sin t)\lambda/12} - \lambda \ge \alpha > 0.05,$$

$$\left\| \frac{b}{a} \right\|_{[0,\infty)} = \left\| \frac{0.2(2 - \sin t)}{0.2(2 - \sin t)e^{(2 - \sin t)\lambda/12} - \lambda} \right\|_{[0,\infty)} < 3.85 < 4.$$

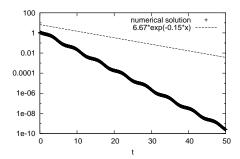
Since
$$M_1 = \left(\lambda + e^{\lambda \tau} \|b\|_{[0,\infty)}\right) \tau e^{\lambda \tau} \left\|\frac{b}{a}\right\|_{[0,\infty)} \le \left(0.15 + 0.6e^{0.15 \cdot 0.25}\right) 0.25e^{0.15 \cdot 0.25} \cdot 4 < 0.803 < 1,$$

the assumptions of Theorem 1 are also satisfied, $M_0 < 1/(1-0.803) < 5.1$, and exponential estimates imply. We have $||b||_{[0,\infty)}(e^{\lambda\tau}-1)/\lambda < 0.153$, thus the solution of (5.1) with x(0)=1, $\phi(t)=\sin(20t)-1$, $||\phi||_{[-0.25,0]}=2$, $f\equiv 0$ satisfies

$$|x(t)| \le 5.1e^{-0.15t} [|x(0)| + 0.153||\phi||_{[-0.25,0]}] \le 5.1 \cdot 1.306e^{-0.15t} \le 6.67e^{-0.15t}.$$

For $f(t) \equiv 0$, the comparison with the numerical solution is illustrated in Fig. 1, left.

Next, apply Theorem 1, where the right-hand side is $f(t) = 0.05(1 + \sin(2t))$, $||f||_{[0,\infty)} = 0.1$, $M_0/\lambda ||f||_{[0,\infty)} \le 0.1 \ 5.1/0.15 = 3.4$, and we get $|x(t)| \le 6.67e^{-0.15t} + 3.4$. See Fig. 1, right, for the comparison with numerical results.



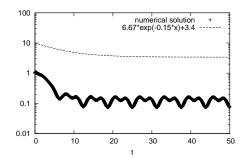


Figure 1: The absolute value of the numerical solution of (5.1) with x(0) = 1, $\phi(t) = \sin(20t) - 1$ compared to the theoretical estimate from Theorem 1 for (left) $f \equiv 0$; (right) $f(t) = 0.05(1 + \sin(2t))$. The scale in x is logarithmic.

Example 2. As it is well known, the autonomous equation

$$\dot{x}(t) + 2x(t-1) = 0 (5.2)$$

is unstable. Let $a(t) = \lambda + 2e^{-\lambda} = \alpha$. The assumptions of Corollary 3 are satisfied whenever

$$\frac{M_2}{\alpha} = \left(\lambda + 2e^{-\lambda}\right)e^{-\lambda}\frac{2}{\lambda + 2e^{-\lambda}} = 2e^{-\lambda} < 1,$$

or $\lambda > \ln 2 \approx 0.693147$. Take $\lambda = 0.8 > \ln 2$, then $\alpha = 2e^{-\lambda} + \lambda > 1.6986$, $M_2 = 2(\lambda + 2e^{-\lambda})e^{-\lambda} < 1.527 < \alpha$, $M_0 = \alpha/(\alpha - M_2) < 10$. In particular, $|x(t)| \leq 10e^{0.8t}|x(0)|$ for the zero initial function. By Remark 6, we can get $|x(t)| \leq e^{2t}$, which is a less sharp estimate for any $t \geq 1.92$.

Example 3. Further, consider the problem

$$\dot{x}(t) + 0.2(2 - \sin t)x(t - 15 + \sin t) = 0, \ t \ge 0, \ x(t) = 0, \ t < 0, x(0) = 1.$$
 (5.3)

Here $||b||_{[0,\infty)} = 0.6$, $\tau = 16$, $\delta = 14$, take $\lambda = 0.2$. Then $a(t) = e^{-0.2(15-\sin t)}0.2(2-\sin t) + 0.2 > 0.2 = \alpha$, we evaluate numerically $||b/a||_{[0,\infty)} < 2.6735$. Thus

$$M_2 = 16e^{-2.8} \left\| \frac{b}{a} \right\|_{[0,\infty)} (0.2 + 0.6e^{-0.2 \cdot 14}) \approx 0.61515 < 1.$$

Then $M_0 < 2.6$, and we get the estimate $|x(t)| < 2.6e^{0.2t}$ for the solution of (5.3), see Fig. 2 for the comparison with the absolute value of the numerical solution.

By Remark 6, we get $|x(t)| < e^{0.6t}$, which is less sharp for $t \ge 2.389$.

For the chosen initial conditions, the exponential estimate computed numerically has $\lambda \approx 0.09$. However, higher growth rates are possible for the same or even smaller delays and the same coefficient. Let |t| be the maximal integer not exceeding t. In the problem

$$\dot{x}(t) + 0.2(2 - \sin t)x(h(t)) = 0, \ t \ge 0, \ x(t) = 0, \ t < 0, x(0) = 1, \ h(t) = 4\pi \left\lfloor \frac{t}{4\pi} \right\rfloor, \tag{5.4}$$

the maximal delay of 4π is less than 14. Direct computation implies

$$x(t) = 1 - \int_0^t 0.2(2 - \sin s) \ ds, \ s \in [0, 4\pi], \ x(4\pi) = 1 - 0.2 \int_0^{4\pi} (2 - \sin s) \ ds = 1 - 1.6\pi < 0.$$

Next, $x(8\pi) = (1 - 1.6\pi) - (1 - 1.6\pi)1.6\pi = (1 - 1.6\pi)^2$, $x(4\pi k) = (1 - 1.6\pi)^k$. Here the growth coefficient $\lambda = \ln(1.6\pi - 1)/(4\pi) \approx 0.11$.

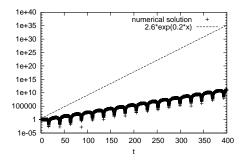


Figure 2: The absolute value of the solution of (5.3) with x(0) = 1, $\phi(t) = 0$ compared to the theoretical estimate from Theorem 3. The scale in x is logarithmic.

For both (5.2) and (5.3), Theorem 3 gives an estimate with a lower exponent than Remark 6. Finally, consider an example with oscillating coefficients.

Example 4. For the equation

$$\dot{x}(t) + 0.2(0.5 - \sin(t))x\left(t - \frac{2 - \sin t}{6}\right) + 0.2(0.5 + \sin(t))x\left(t - \frac{2 - \sin t}{12}\right) = 0, \ t \ge 0, \quad (5.5)$$

we have $\tau_1 = 0.5$, $\tau_2 = 0.25$, $||b_1||_{[0,\infty)} = ||b_2||_{[0,\infty)} = 0.3$. Take $\lambda = 0.02$, then we estimate numerically that

$$a(t) \ge \alpha \approx 0.18, \ \|b_1/a\|_{[0,\infty)} < 1.691, \ \|b_2/a\|_{[0,\infty)} < 1.669,$$

in (4.2) we have

$$M_1 < (0.02 + 0.3e^{0.01} + 0.3e^{0.005}) (0.5e^{0.01} \cdot 1.691 + 0.25e^{0.005} \cdot 1.669) < 0.7953,$$

and $M_0 < 4.89$, thus for x(0) = 1, $\phi(t) \equiv 0$, the solution of (5.5) satisfies

$$|x(t)| \le 4.89e^{-0.02t}. (5.6)$$

Remark 7. Inequality (5.6) implies that equation (5.5) with two oscillating coefficients is uniformly exponentially stable. We do not know other stability tests for this class of equations.

6. Conclusions and Discussion

As we mentioned in Introduction, there are several related topics, in particular, asymptotic integration or obtaining asymptotic formulas for solutions of a given equation. Such asymptotic expressions of the form $x(t) = (C + o(1))e^{\alpha t}$ have been constructed for various classes of differential equations: scalar and vector, linear and nonlinear, equations with concentrated and distributed delays, including integro-differential equations. The review paper [3] includes a detailed outline of these results with historical notes. Asymptotic formulas give a good approximation for long time behavior of solutions, but are useless for finite intervals. Estimates obtained in the present paper apply to finite intervals, and constitute a natural addition to asymptotic formulas. Most papers on asymptotic integration consider specific classes of equations, such as equations with a dominating non-delay term, autonomous equations, equations with a single delay. We have considered scalar equations with several variable delays and measurable coefficients. Another advantage of our results is an explicit form for these estimates, which we illustrated by examples of different types.

When investigating asymptotic stability, most authors considered equations with positive coefficients. If both positive and negative coefficients were involved, it was usually assumed that a positive part dominates over the negative one, see, for example, [4, 6, 7, 9, 16, 18, 22]. In Theorem 1, we got uniform exponential stability conditions, when the equation contains several delays, and all the coefficients may be oscillatory, see Example 5.5. We suggest that in this case, not only the estimate, but also the stability test is new.

Assume that b_k in Theorem 2 are proportional. Then (4.6) leads to the following uniform exponential stability condition

$$0 \le b_k(t) \le \beta_k, \quad \liminf_{t \to \infty} \sum_{k=1}^m b_k(t) > 0, \quad t - \tau_k \le h_k(t) \le t, \quad k = 1, \dots, m, \quad \sum_{k=1}^m b_k \tau_k < 1$$
 (6.1)

for (2.1), and to a relevant estimate of solutions. Note that for (2.1) with several variable delays and coefficients, one in the right-hand side of (6.1) is the best possible constant [22].

Let us state possible extensions of the results of the present paper.

- 1. Obtain explicit estimates of solutions for nonlinear equations in general, and for specific cases, for example, models of population dynamics. These include, but not limited to, the Hutchinson, the Mackey-Glass and the Nicholson blowflies equations.
- 2. Extend the estimates to a vector DDE. Consider other types of delay, such as distributed, as well as equations of the second and higher order, and stochastic differential equations.
- 3. In this paper, we presented pointwise estimates. It would be interesting to obtain estimates in an integral form.
- 4. For the case of a single delay

$$\dot{x}(t) + b(t)x(h(t)) = f(t), \ b(t) \ge b_0 > 0, t - h(t) \le \tau,$$

we obtain estimates for $||b||\tau < 1$. Relax this condition, replacing 1 with either $\frac{3}{2}$ or a sharp constant in (1, 1.5).

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