

Relations among spheroidal and spherical harmonics

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Abstract

A *contragenic* function in a domain $\Omega \subseteq \mathbb{R}^3$ is a reduced-quaternion-valued (i.e the last coordinate function is zero) harmonic function, which is orthogonal in $L_2(\Omega)$ to all monogenic functions and their conjugates. The notion of contragenicity depends on the domain and thus is not a local property, in contrast to harmonicity and monogenicity. For spheroidal domains of arbitrary eccentricity, we relate standard orthogonal bases of harmonic and contragenic functions for one domain to another via computational formulas. This permits us to show that there exist nontrivial contragenic functions common to the spheroids of all eccentricities.

Keywords: spherical harmonics, spheroidal harmonics, quaternionic analysis, monogenic function, contragenic function.

Classification: 30G35; 33D50

1 Introduction

In certain physical problems in nonspherical domains, it has been found convenient to replace the classical solid spherical harmonics with harmonic functions better adapted to the domain in question. For example, spheroidal harmonics are used in [7] for modeling potential fields around the surface of the earth.

A systematic analysis of harmonic functions on spheroidal domains was initiated by Szegő [19], followed by Garabedian [5] who produced orthogonal bases with respect to certain natural inner products associated to prolate and oblate spheroids, among them the L^2 -Hilbert space structures on the interior and on the boundary of the spheroid. Some aspects of the generation of harmonic functions which are orthogonal in the region exterior to a prolate spheroid were considered in [14] and generalized recently in [15].

The main question which interests us is to relate systems of harmonic functions associated with the spheroid Ω_μ (defined in (1) below) to those associated with the unit ball Ω_0 . Our starting point is a fundamental formula for spheroidal harmonics which was worked out in the short but beautiful paper [2] and is discussed thoroughly in Chapter 22 of the monumental text [7]. In classical books such as [6, 16, 18], these expansions in terms of these bases are used separately without specifying relations between them.

We complete the above formulas by relating different systems of harmonic functions associated with spheroids of different eccentricity. While the manipulation of the coefficients is essentially algebraic, it must be borne in mind that we are dealing with continuously varying families of function spaces which are determined by integration over varying domains.

This study is then extended to include the contragenic functions, which are those harmonic functions orthogonal to both the monogenic functions and the antimonogenic functions in the domain under consideration. In [4] a short table of contragenic polynomials was provided, which included some which did not depend on the parameter describing the eccentricity of the spheroid. Such polynomials are thus contragenic for all spheroids. Our main result, Theorem 4.6, describes the intersection of the spaces of contragenic functions.

2 Background on spheroidal harmonics

As a preliminary to the discussion of monogenic and contragenic functions on spheroids, we establish the basic facts for harmonics in this and the next section. Consider the family of coaxial spheroidal domains Ω_μ , scaled so that the major axis is of length 2:

$$\Omega_\mu = \{x \in \mathbb{R}^3 \mid x_0^2 + \frac{x_1^2 + x_2^2}{e^{2\nu}} < 1\}, \quad (1)$$

where $\nu \in \mathbb{R}$ and where following the notation in [4] the parameter $\mu = (1 - e^{2\nu})^{1/2}$ will be useful in later formulas. The equations relating the Cartesian coordinates of a point $x = (x_0, x_1, x_2)$ in Ω_μ to spheroidal coordinates (u, v, ϕ) are

$$x_0 = \mu \cos u \cosh v, \quad x_1 = \mu \sin u \sinh v \cos \phi, \quad x_2 = \mu \sin u \sinh v \sin \phi, \quad (2)$$

where in the case of the prolate spheroid ($\nu < 0$) the coordinates range over $u \in [0, \pi]$, $v \in [0, \operatorname{arctanh} e^\nu]$, $\phi \in [0, 2\pi)$ and the eccentricity is $0 <$

$\mu < 1$, while for the oblate spheroid ($\nu > 0$) we have $u \in [0, \pi]$ and $v \in [0, \operatorname{arctanh} e^\nu]$, $\phi \in [0, 2\pi)$ and μ is imaginary, $\mu/i > 0$. The spheroids reduce to the unit ball Ω_0 for $\nu = 0$. In many other treatments of spheroidal functions, which discuss the two (confocal) families separately, the ball is not represented. See [4] for a discussion of this question.

In terms of the coordinates (2), the *solid spheroidal harmonics* are

$$U_{n,m}^\pm[\mu](x) = \widehat{U}_{n,m}[\mu](u, v) \Phi_m^\pm(\phi), \quad (3)$$

where

$$\Phi_m^+(\phi) = \cos(m\phi), \quad \Phi_m^-(\phi) = \sin(m\phi) \quad (4)$$

and for $\mu \neq 0$,

$$\widehat{U}_{n,m}[\mu](u, v) = \frac{(n-m)!}{2^n (1/2)_n} \mu^n P_n^m(\cos u) P_n^m(\cosh v). \quad (5)$$

Here P_n^m are the associated Legendre functions of the first kind [6, Ch. III] of degree n and order m , and the (rising) Pochhammer symbol is $(a)_n = a(a+1)\cdots(a+n-1)$ with $(a)_0 = 1$ by convention. To avoid repetition, we state once and for all that $U_{n,m}^-[\mu]$ is only defined for $m \geq 1$, i.e. $U_{n,0}^-[\mu]$ is expressly excluded from all statements of theorems.

It was shown in [4] that with the scale factor which has been included in (5), the $U_{n,m}^\pm[\mu]$ are polynomials in (x_0, x_1, x_2) which are normalized so that the limiting case $\mu \rightarrow 0$ gives the classical *solid spherical harmonics*,

$$U_{n,m}^\pm[0](x) = |x|^n P_n^m(x_0/|x|) \Phi_m^\pm(\phi).$$

It is known from [5] that while the $U_{n,m}^\pm[\mu]$ are mutually orthogonal with respect to the Dirichlet norm on Ω_μ , the closely related functions, which we will call the *Garabedian spheroidal harmonics*,

$$V_{n,m}^\pm[\mu](x) = \frac{\partial}{\partial x_0} U_{n+1,m}^\pm[\mu](x) \quad (6)$$

form an orthogonal basis for $\mathcal{H}_2(\Omega_\mu)$, the linear subspace of real-valued harmonic functions in $L_2(\Omega_\mu)$. This property makes the $V_{n,m}^\pm[\mu]$ of greater interest for many considerations. The corresponding boundary Garabedian harmonics $\widehat{V}_{n,m}[\mu]$ in Ω_μ are characterized by the relation

$$V_{n,m}^\pm[\mu](x) = \widehat{V}_{n,m}[\mu](u, v) \Phi_m^\pm(\phi). \quad (7)$$

We recall [9] that for spherical harmonics, there is a formula analogous to Appell differentiation of monomials,

$$\frac{\partial}{\partial x_0} U_{n+1,m}^\pm[0](x) = (n+m+1)U_{n,m}^\pm[0](x). \quad (8)$$

However, $V_{n,m}^\pm[\mu]$ is not so simply related to $U_{n,m}^\pm[\mu]$ for $\mu \neq 0$, as was explained in [10]. We examine such relations in the next section.

3 Conversions among orthogonal spheroidal harmonics and spherical harmonics

3.1 Garabedian harmonics expressed by classical harmonics

As mentioned in the Introduction, it is of interest to express the orthogonal basis of harmonic functions for one spheroid Ω_μ in terms of those for another spheroid. It is natural to use the unit ball Ω_0 as a point of reference, which will be the case in the first results. We begin the calculation of the coefficients for the relationships among the various classes of harmonic functions by presenting various known formulas in a uniform manner. For $n \geq 0$, consider the rational constants

$$u_{n,m,k} = \frac{(1/2)_{n-k} (n+m-2k+1)_{2k}}{(-4)^k (1/2)_n k!} \quad (9)$$

for $0 \leq m \leq n$, $0 \leq 2k \leq n$, and let $u_{n,m,k} = 0$ otherwise. In the present notation, the main result of [2] may be expressed as follows (i.e. the factor $\alpha_{m,n} = (n-m)!/(2n-1)!!$ has been incorporated into (9)).

Proposition 3.1 ([2]). *Let $n \geq 0$ and $0 \leq m \leq n$. Then*

$$\widehat{U}_{n,m}[\mu] = \sum_{0 \leq 2k \leq n-m} u_{n,m,k} \mu^{2k} \widehat{U}_{n-2k,m}[0].$$

An important characteristic of this relation is that the same coefficients $u_{n,m,k}$ work for the “+” and “−” cases (cosines and sines) and, strikingly, for all values of μ . By (3), an equivalent form of expressing Proposition 3.1 is

$$U_{n,m}^\pm[\mu] = \sum_{0 \leq 2k \leq n-m} u_{n,m,k} \mu^{2k} U_{n-2k,m}^\pm[0]. \quad (10)$$

Since $\partial/\partial x_0$ in (6) is a linear operator, (10) gives automatically the corresponding result for the Garabedian harmonics,

$$V_{n,m}^\pm[\mu] = \sum_{0 \leq 2k \leq n-m+1} v_{n,m,k} \mu^{2k} V_{n-2k,m}^\pm[0], \quad (11)$$

where $v_{n,m,k} = u_{n+1,m,k}$. This in turn gives via (8) the following expression in terms of the spherical harmonics:

Corollary 3.2. *Let $n \geq 0$ and $0 \leq m \leq n$. Then*

$$\widehat{V}_{n,m}[\mu] = \sum_{0 \leq 2k \leq n-m+1} w_{n,m,k} \mu^{2k} \widehat{U}_{n-2k,m}[0],$$

where

$$w_{n,m,k} = (n + m - 2k + 1)v_{n,m,k}.$$

The coefficients

$$t_{n,m,k} = \frac{(n + m + 1)! (1/2)_{n-2k+1}}{4^k (n + m - 2k)! (1/2)_{n+1}}$$

give a similar expression for the Garabedian basic harmonics $V_{n,m}^\pm[\mu]$ in terms of the standard harmonics $U_{n,m}^\pm[\mu]$ for the same spheroid, rather than in terms of $\widehat{U}_{n,m}[0]$:

Theorem 3.3 ([10]). *Let $n \geq 0$ and $0 \leq m \leq n$. Then*

$$\widehat{V}_{n,m}[\mu] = \sum_{0 \leq 2k \leq n-m} t_{n,m,k} \mu^{2k} \widehat{U}_{n-2k,m}[\mu].$$

In [2] the inverse relation of (10) was also derived, expressing $U_{n,m}^\pm[0]$ in terms of $U_{n,m}^\pm[\mu]$, via

$$\widehat{U}_{n,m}[0] = \sum_{0 \leq k \leq n-m} u_{n,m,k}^0 \mu^{2k} \widehat{U}_{n-2k,m}[\mu], \quad (12)$$

where the coefficients can be written as

$$u_{n,m,k}^0 = \frac{4^{n-2k} (2n - 4k + 1) (n - k)! (m + n)! (1/2)_{n-2k}}{k! (2n - 2k + 1)! (n + m - 2k)!}, \quad (13)$$

again independent of μ . In consequence, applying the operator $\partial/\partial x_0$ and using (8), we have the following result.

Proposition 3.4. *Let $n \geq 0$ and $0 \leq m \leq n$. Then*

$$\widehat{U}_{n,m}[0] = \sum_{0 \leq 2k \leq n-m} w_{n,m,k}^0 \mu^{2k} \widehat{V}_{n-2k,m}[\mu],$$

where

$$w_{n,m,k}^0 = \frac{u_{n+1,m,k}^0}{n+m+1}.$$

The inverse relation for Theorem 3.3 is a much simpler formula, given as follows:

Corollary 3.5 ([10]). *For $n \geq 0$ and $0 \leq m \leq n$,*

$$\widehat{U}_{n,m}[\mu] = \frac{1}{n+m+1} \widehat{V}_{n,m}[\mu] + \frac{n+m}{4n^2-1} \mu^2 \widehat{V}_{n-2,m}[\mu].$$

This uses the convention $\widehat{V}_{n-2,m}[\mu] = 0$ when $m > n$; i.e.

$$\begin{aligned} \widehat{U}_{n,n-1}[\mu] &= \frac{1}{2n} \widehat{V}_{n,n-1}[\mu], \\ \widehat{U}_{n,n}[\mu] &= \frac{1}{2n+1} \widehat{V}_{n,n}[\mu]. \end{aligned}$$

3.2 Conversion among Garabedian harmonics

The preceding subsection does not include the inverse relation of (11) of the form

$$\widehat{V}_{n,m}[0] = \sum_{0 \leq 2k \leq n-m} v_{n,m,k}^0 \mu^{2k} \widehat{V}_{n-2k,m}[\mu]. \quad (14)$$

Instead of deriving it directly, we verify first the following remarkable conversion formula, which relates the spheroidal harmonics associated with Ω_μ to those associated with any other $\Omega_{\tilde{\mu}}$. Write

$$b_{n,m,k} = \frac{(n+m+1)!(1/2)_{n-2k+2}}{4^k k! (n+m-2k+1)!(1/2)_{n-k+2}}$$

when $0 \leq 2k \leq n-m+2$, otherwise $b_{n,m,k} = 0$.

Theorem 3.6. Let $n \geq 0$, $0 \leq m \leq n$, and let $\mu, \tilde{\mu} \in [0, 1) \cup i\mathbb{R}^+$ such that $\mu \neq 0$. The coefficients $v_{n,m,k}[\tilde{\mu}, \mu]$ in the relation

$$\widehat{V}_{n,m}[\tilde{\mu}] = \sum_{0 \leq 2k \leq n-m} v_{n,m,k}[\tilde{\mu}, \mu] \widehat{V}_{n-2k,m}[\mu]$$

are given by

$$v_{n,m,k}[\tilde{\mu}, \mu] = {}_2F_1(-k, -n+k-3/2; -n-1/2; (\tilde{\mu}/\mu)^2) b_{n,m,k} \mu^{2k},$$

with ${}_2F_1$ denoting the classical Gaussian hypergeometric function.

Proof. We begin by replacing μ with $\tilde{\mu}$ in Corollary 3.2 and substituting the terms on the right-hand side according to Proposition 3.4. By linear independence of the harmonic basis elements, it follows that

$$v_{n,m,k}[\tilde{\mu}, \mu] = \mu^{2k} \sum_{l=0}^k w_{n,m,l} w_{n-2l,m,k-l}^0 \left(\frac{\tilde{\mu}}{\mu} \right)^{2l} \quad (15)$$

in which we note that all terms are real valued. Using reductions such as $(2n-4k+3)(1/2)_{n-2k+1} = 2(1/2)_{n-2k+2}$ and recalling $0 \leq l \leq k$, one easily sees that

$$\begin{aligned} w_{n,m,l} &= \frac{(1/2)_{n-l+1} (n+m-2l+1)_{2l+1}}{(-4^l) l! (1/2)_{n+1}}, \\ w_{n-2l,m,k-l}^0 &= \frac{2 \cdot 4^{n-2k+1} (n+m-2l)! (n-k-l+1)! (1/2)_{n-2k+2}}{(k-l)! (2n-2k-2l+3)! (n+m-2k+1)!}. \end{aligned}$$

Therefore the product can be expressed as

$$w_{n,m,l} w_{n-2l,m,k-l}^0 = b_{n,m,k} c_{n,k,l}$$

where

$$\begin{aligned} c_{n,k,l} &= \frac{2 \cdot 4^{n-2k+1} (n+m+1)! (n-k-l+1)! (1/2)_{n-2k+2}}{(-4^l) l! (k-l)! (2n-2k-2l+3)! (n+m-2k+1)!} \\ &= \frac{(-k)_l (-n+k-3/2)_l}{l! (-n-1/2)_l} \end{aligned}$$

is the coefficient in the polynomial ${}_2F_1(-k, -n+k-3/2; -n-1/2; (\tilde{\mu}/\mu)^2) = \sum_{l=0}^k c_{n,k,l} (\tilde{\mu}/\mu)^{2l}$. \square

Corollary 3.7. For each $n \geq 0$, $0 \leq m \leq n$, the limits

$$\lim_{\tilde{\mu} \rightarrow 0} v_{n,m,k}[\tilde{\mu}, \mu], \quad \lim_{\mu \rightarrow 0} v_{n,m,k}[\tilde{\mu}, \mu]$$

exist and are given, respectively, by

$$v_{n,m,k}[0, \mu] = (n + m + 1)w_{n,m,k}^0 \mu^{2k}, \quad v_{n,m,k}[\tilde{\mu}, 0] = \frac{w_{n,m,k}}{n + m - 2k + 1} \tilde{\mu}^{2k}.$$

Proof. We may write (15) as

$$\begin{aligned} v_{n,m,k}[\tilde{\mu}, \mu] &= \sum_{l=1}^{k-1} w_{n,m,l} w_{n-2l,m,k-l}^0 \mu^{2(k-l)} \tilde{\mu}^{2l} \\ &\quad + w_{n,m,k} w_{n-2k,m,0}^0 \tilde{\mu}^{2k} + w_{n,m,0} w_{n,m,k}^0 \mu^{2k} \end{aligned}$$

and then simply take $\mu = 0$ or $\tilde{\mu} = 0$ to obtain the desired limit. \square

Referring to (14), we have

$$v_{m,n,k}^0 = \frac{w_{n,m,k}}{(n + m - 2k + 1)}.$$

4 Application to orthogonal monogenic and contragenic functions

The standard bases for spheroidal harmonics have their counterparts for the spaces of orthogonal monogenic polynomials taking values in \mathbb{R}^3 . Monogenic functions are defined by considering \mathbb{R}^3 as the real linear subspace of the quaternions $\mathbb{H} = \{\sum_{i=0}^3 x_i e_i : x_i \in \mathbb{R}\}$ for which the last coordinate x_3 vanishes. (Quaternionic multiplication is defined, as usual, so that $e_1^2 = e_2^2 = e_3^2 = -1$ and $e_1 e_2 = e_3 = -e_2 e_1$, $e_2 e_3 = e_1 = -e_3 e_2$, $e_3 e_1 = e_2 = -e_1 e_3$.) For background on quaternionic analysis in \mathbb{R}^3 , see [3, 8, 9, 11, 13]. A function $f: \Omega_\mu \rightarrow \mathbb{R}^3$ is *monogenic* when it is annihilated by the quaternionic differential operator $\partial = \partial/\partial x_0 + e_1 \partial/\partial x_1 + e_2 \partial/\partial x_2$ acting from the left. The *basic spheroidal monogenic polynomials* are constructed [10, 12] as

$$X_{n,m}^\pm[\mu] = \bar{\partial}(U_{n+1,m}^\pm[\mu]), \quad (16)$$

where $\bar{\partial} = \partial/\partial x_0 - e_1 \partial/\partial x_1 - e_2 \partial/\partial x_2$. This is analogous to the definition (6) for harmonic polynomials. $X_{n,m}^\pm[\mu]$ is monogenic because $\partial \bar{\partial}$ is equal to the Laplacian operator. We continue with the convention that $m \geq 1$ when the “-” sign appears in a superscript.

Theorem 4.1 ([10, 12]). *For all $n \geq 0$, the basic spheroidal monogenic polynomial (16) is equal to*

$$X_{n,0}^+[\mu] = V_{n,0}^+[\mu] - \frac{1}{n+2} (V_{n,1}^+[\mu]e_1 + V_{n,1}^-[\mu]e_2)$$

for $m = 0$, and

$$\begin{aligned} X_{n,m}^\pm[\mu] &= V_{n,m}^\pm[\mu] + \left[(n+m+1)V_{n,m-1}^\pm[\mu] - \frac{1}{n+m+2}V_{n,m+1}^\pm[\mu] \right] \frac{e_1}{2} \\ &\mp \left[(n+m+1)V_{n,m-1}^\mp[\mu] + \frac{1}{n+m+2}V_{n,m+1}^\mp[\mu] \right] \frac{e_2}{2} \end{aligned}$$

for $1 \leq m \leq n+1$. The polynomials $X_{n,m}^\pm[\mu]$ are orthogonal in $L^2(\Omega_\mu)$, i.e. in the sense of the scalar product defined by

$$\langle f, g \rangle_{[\mu]} = \int_{\Omega_\mu} \text{Sc}(\bar{f}g) dV.$$

4.1 Bases for monogenics in distinct spheroids

Analogously to (11) and (14), we now express $X_{n,m}^\pm[\mu]$ in terms of the spherical monogenics $X_{n,m}^\pm[0]$.

Theorem 4.2. *For $n \geq 0$ and $0 \leq m \leq n+1$,*

$$\begin{aligned} X_{n,m}^\pm[\mu] &= \sum_{0 \leq 2k \leq n-m+1} v_{n,m,k} \mu^{2k} X_{n-2k,m}^\pm[0], \\ X_{n,m}^\pm[0] &= \sum_{0 \leq 2k \leq n-m+1} v_{n,m,k}^0 \mu^{2k} X_{n-2k,m}^\pm[\mu], \\ X_{n,m}^\pm[\tilde{\mu}] &= \sum_{0 \leq 2k \leq n-m+1} v_{n,m,k}[\tilde{\mu}, \mu] X_{n-2k,m}^\pm[\mu], \end{aligned}$$

where $v_{n,m,k}$, $v_{n,m,k}^0$, and $v_{n,m,k}[\mu, \tilde{\mu}]$ are as in the previous section.

Proof. Fix a value of μ . Note that for given n , the collections $\{X_{k,m}^\pm[0]: k \leq n, 0 \leq m \leq k\}$ and $\{X_{k,m}^\pm[\mu]: k \leq n, 0 \leq m \leq k\}$ are bases for the same linear space, namely the monogenic \mathbb{R}^3 -valued polynomials in the variables (x_0, x_1, x_2) of degree $\leq n$. Therefore there must exist real coefficients a_k^\pm such that $X_{n,m}^\pm[\mu] = \sum_k \sum_m a_k^+ X_{n,k}^+[0] + \sum_k \sum_m a_k^- X_{n,k}^-[0]$. By Theorem 4.1, the

scalar part of this equation expresses the spheroidal harmonics $V_{n,m}^\pm[\mu]$ as a linear combination of the spherical harmonics $V_{k,m}^\pm[0]$. By the uniqueness of the representation (11) we have that $a_k^\pm = v_{n,m,k}\mu^{2k}$. The second formula follows by the same reasoning, and then the relationship between $X_{n,m}^\pm[\mu]$ and $X_{n,m}^\pm[\tilde{\mu}]$ is a consequence of the fact that by Theorem 3.6 the matrix $(v_{n,m,k}[\tilde{\mu}, \mu])_{n,k}$ is essentially the product of $(v_{n,m,k}\tilde{\mu}^{2k})_{n,k}$ and the inverse of $(v_{n,m,k}^0\mu^{2k})_{n,k}$. \square

4.1.1 Spheroidal ambigenic polynomials

Antimonogenic functions (quaternionic conjugates of monogenics, i.e. annihilated by $\bar{\partial}$) are generally not studied independently, since their properties may be obtained by taking the conjugate of facts about monogenic functions. For example, the basic antimonogenic polynomials satisfy essentially the same relation as given in Theorem 4.2,

$$\bar{X}_{n,m}^\pm[\mu] = \sum_{0 \leq 2k \leq n-m} v_{n,m,k}[\mu, \tilde{\mu}] \bar{X}_{n-2k,m}^\pm[\tilde{\mu}].$$

However, the subspace of the \mathbb{R}^3 -valued harmonic functions generated by the monogenic and antimonogenic functions together, that is, the *ambigenic* functions [1], is of interest.

An ambigenic function is not represented uniquely as a sum of a monogenic and an antimonogenic function because one may add and subtract a *monogenic constant*, that is, a function which is simultaneously monogenic and antimonogenic. A collection of ambigenic polynomials denoted $\{Y_{n,m}^{\pm,\pm}[\mu]\}$ was constructed in [4] and shown to be a basis of $2n(n+3)+3$ elements for the ambigenic polynomials of degree no greater than n , mutually orthogonal in $L^2(\Omega_\mu)$. For our purposes we will only need the particular ambigenic functions

$$A_{n,m}^\pm[\mu] = 2 \operatorname{Vec} X_{n,m}^\pm[\mu] = X_{n,m}^\pm[\mu] - \bar{X}_{n,m}^\pm[\mu], \quad (17)$$

where $q = \operatorname{Sc} q + \operatorname{Vec} q$ denotes the decomposition of a quaternionic quantity into its scalar and vector parts. It is simple to verify that for fixed μ , the $A_{n,m}^\pm[\mu]$ are linearly independent.

4.2 Relations among contragenic functions for distinct spheroids

The notion of contragenic harmonic functions was introduced in [1], arising from the previously unobserved fact that in contrast to \mathbb{C} -valued or \mathbb{H} -valued functions, there exist \mathbb{R}^3 -valued harmonic functions which are not ambigenic. Thus a function is called *contragenic* for a given domain Ω when it is orthogonal in $L^2(\Omega)$ to all monogenic and antimonogenic functions in Ω . In contrast to monogenicity and antimonogenicity, this is not a local property and therefore cannot be characterized in general by direct application of any differential operator. It is of interest to have a basis for the contragenic functions, in order to express an arbitrary harmonic function in a calculable way as a sum of an ambigenic function and a contragenic function. In the following, we will write

$$\mathcal{N}_*^{(n)}[\mu] = \{\text{polynomials of degree } \leq n \text{ in } x_0, x_1, x_2 \text{ which are orthogonal in } L_2(\Omega_\mu) \text{ to all ambigenic functions in } \Omega_\mu\},$$

for $n \geq 1$ (nonzero constant harmonic functions are never contragenic, so we will have no use for $\mathcal{N}_*^{(0)}[\mu] = \{0\}$), and we have the successive orthogonal complements

$$\mathcal{N}^{(n)}[\mu] = \mathcal{N}_*^{(n)}[\mu] \ominus \mathcal{N}_*^{(n-1)}[\mu],$$

which are composed of polynomials of degree precisely n . Thus $\mathcal{N}_*^{(n)}[\mu] = \bigoplus_{k=1}^n \mathcal{N}^{(k)}[\mu]$ and there is a Hilbert space orthogonal decomposition $\mathcal{N}_*[\mu] = \bigoplus_{k=1}^\infty \mathcal{N}^{(k)}[\mu]$ of the full collection of contragenic functions in $L^2(\Omega_\mu)$. The following explicit construction of a basis of the $\mathcal{N}^{(n)}[\mu]$, using as building blocks the scalar components of the monogenic functions, can be found in [4]. Write

$$\begin{aligned} a_{n,0}[\mu] &= 1, \\ a_{n,m}[\mu] &= \left(\frac{1}{(n+m+1)_2} \frac{\|V_{n,m+1}^+[\mu]\|_{[\mu]}}{\|V_{n,m-1}^+[\mu]\|_{[\mu]}} \right)^2 \end{aligned} \quad (18)$$

for $1 \leq m \leq n-1$, and $a_{n,m}[\mu] = 0$ for $m \geq n$ since then $V_{n,m}^\pm[\mu] = 0$ (this definition involves a slight modification of the notation in [4]), where integration over the ellipsoid gives explicitly

$$\|V_{n,m}^\pm[\mu]\|_{[\mu]}^2 = (1 + \delta_{0,m}) \beta_{n,m} \pi \mu^{2n+3} \int_1^{\frac{1}{\mu}} P_n^m(t) P_{n+2}^m(t) dt.$$

Here $\delta_{m,m'}$ is the Kronecker symbol and

$$\beta_{n,m} = \frac{(n+m+1)(n+m+1)!(n-m+2)!}{2^{2n+1}(1/2)_{n+1}(1/2)_{n+2}}.$$

Definition 4.3. For all $n \geq 1$, the *basic contragenic polynomials* $Z_{n,m}^\pm[\mu]$ associated to Ω_μ are

$$Z_{n,0}^+[\mu] = -A_{n,0}^+[\mu]e_3$$

for $m = 0$, and

$$Z_{n,m}^\pm[\mu] = \frac{1}{2}(\mp(a_{n,m}[\mu] + 1)A_{n,m}^\pm[\mu] + (a_{n,m}[\mu] - 1)A_{n,m}^\mp[\mu]e_3)$$

for $1 \leq m \leq n-1$, where $A_{n,m}^\pm[\mu]$ are defined by (17).

In [4] it was shown that $\{Z_{n,m}^\pm[\mu]: 0 \leq m < n-1\}$ is an orthonormal basis for $\mathcal{N}^{(n)}[\mu]$, and that the harmonic polynomials of degree $\leq n$ in Ω_μ decompose as orthogonal direct sums of the ambigenic and contragenic polynomials of degree $\leq n$. With the further notation

$$\begin{aligned}\Psi_{+,m}^\pm &= \Phi_m^\pm(\phi)e_1 \pm \Phi_m^\mp(\phi)e_2, \\ \Psi_{-,m}^\pm &= \Phi_m^\pm(\phi)e_1 \mp \Phi_m^\mp(\phi)e_2,\end{aligned}$$

which satisfy the obvious relations $\Psi_{+,m}^\pm e_3 = \pm \Psi_{+,m}^\mp$, $\Psi_{-,m}^\pm e_3 = \mp \Psi_{-,m}^\mp$, $e_1 V_{n,m}^\pm[\mu] + e_2 V_{n,m}^\mp[\mu] = \widehat{V}_{n,m} \Psi_{\pm,m}^\pm[\mu]$, $e_1 V_{n,m}^\pm[\mu] - e_2 V_{n,m}^\mp[\mu] = \widehat{V}_{n,m} \Psi_{\mp,m}^\pm[\mu]$ (where the $\widehat{V}_{n,m}[\mu]$ are given by (7)), the definitions give us almost immediately that

$$\begin{aligned}A_{n,0}^+[\mu] &= \frac{-2}{n+2} \widehat{V}_{n,1}[\mu] \Psi_{+,1}^+, \\ A_{n,m}^\pm[\mu] &= (n+m+1) \widehat{V}_{n,m-1}[\mu] \Psi_{-,m-1}^\pm \\ &\quad - \frac{1}{n+m+2} \widehat{V}_{n,m+1}[\mu] \Psi_{+,m+1}^\pm,\end{aligned}\tag{19}$$

$$\begin{aligned}Z_{n,0}^+[\mu] &= \frac{2}{n+2} \widehat{V}_{n,1}[\mu] \Psi_{+,1}^-, \\ Z_{n,m}^\pm[\mu] &= (n+m+1) a_{n,m}[\mu] \widehat{V}_{n,m-1}[\mu] \Psi_{-,m-1}^\mp \\ &\quad + \frac{1}{n+m+2} \widehat{V}_{n,m+1}[\mu] \Psi_{+,m+1}^\mp,\end{aligned}\tag{20}$$

where $1 \leq m \leq n - 1$.

Adding and subtracting instances of (19) and (20) gives by cancellation decompositions of the harmonic polynomials $\widehat{V}_{n,m}\Psi_{+,m}^\pm$ and $\widehat{V}_{n,m}\Psi_{-,m}^\pm$ as the sum of a contragenic and an ambigenic:

Lemma 4.4. *Let $n \geq 1$ and $1 \leq m \leq n + 1$. Then*

$$\widehat{V}_{n,m-1}[\mu]\Psi_{-,m-1}^\pm = \frac{1}{(n+m+1)(a_{n,m}[\mu]+1)}(Z_{n,m}^\mp[\mu] + A_{n,m}^\pm[\mu]),$$

and

$$\widehat{V}_{n,m+1}[\mu]\Psi_{+,m+1}^\pm = \frac{n+m+2}{a_{n,m}[\mu]+1}(Z_{n,m}^\mp[\mu] - a_{n,m}[\mu]A_{n,m}^\pm[\mu]).$$

The definition of contragenic function does not imply that an L^2 -function which belongs to the space $\mathcal{N}_*^{(n)}[\tilde{\mu}]$ should also be in $\mathcal{N}_*^{(n)}[\mu]$ when $\tilde{\mu} \neq \mu$, because the notion of orthogonality is different for different spheroids. In other words, we may not expect a formula like “ $Z_{n,m}^\pm[\tilde{\mu}] = \sum z_{n,m,k}[\tilde{\mu}, \mu]Z_{n-2k,m}^\pm[\mu]$.” The following result will enable us to give many examples for which $Z_{n,m}^\pm[\tilde{\mu}] \notin \mathcal{N}_*^{(n)}[\mu]$ for $m \geq 1$. However, it also shows that the intersection of all of the $\mathcal{N}_*^{(n)}[\mu]$ is nontrivial, giving what may be called *universal contragenic functions* in the context of spheroids.

We will use the coefficients

$$\begin{aligned} z_{n,0,k}^C[\tilde{\mu}, \mu] &= \frac{n-2k+2}{n+2}v_{n,1,k}[\tilde{\mu}, \mu], \\ z_{n,m,k}^C[\tilde{\mu}, \mu] &= \begin{cases} \frac{a_{n,m}[\tilde{\mu}]+1}{a_{n-2k,m}[\mu]+1}v_{n,m,k}[\tilde{\mu}, \mu], & 0 \leq 2k \leq n-m-1, \\ \frac{a_{n,m}[\tilde{\mu}]}{a_{n-2k,m}[\mu]+1}v_{n,m,k}[\tilde{\mu}, \mu], & n-m \leq 2k \leq n-m+1; \end{cases} \\ z_{n,m,k}^A[\tilde{\mu}, \mu] &= \begin{cases} \frac{a_{n,m}[\tilde{\mu}] - a_{n,m}[\mu]}{a_{n-2k,m}[\mu]+1}v_{n,m,k}[\tilde{\mu}, \mu], & 0 \leq 2k \leq n-m-1, \\ \frac{a_{n,m}[\tilde{\mu}]}{a_{n-2k,m}[\mu]+1}v_{n,m,k}[\tilde{\mu}, \mu], & n-m \leq 2k \leq n-m+1; \end{cases} \end{aligned} \tag{21}$$

($1 \leq m \leq n-1$) to express the decomposition of contragenics for one spheroid in terms of contragenics and ambigenics of any other.

Proposition 4.5. *Let $n \geq 1$. Then*

$$Z_{n,0}^+[\tilde{\mu}] = \sum_{0 \leq 2k \leq n-1} z_{n,k}^C[\tilde{\mu}, \mu] Z_{n-2k,0}[\mu];$$

and for $1 \leq m \leq n-1$,

$$Z_{n,m}^\pm[\tilde{\mu}] = \sum_{0 \leq 2k \leq n-m+1} (z_{n,m,k}^C[\tilde{\mu}, \mu] Z_{n-2k,m}^\pm[\mu] + z_{n,m,k}^A[\tilde{\mu}, \mu] A_{n-2k,m}^\pm[\mu]).$$

Proof. Apply Theorem 3.6 to the first formula of (20) with $\tilde{\mu}$ in place of μ to obtain that

$$Z_{n,0}^+[\tilde{\mu}] = \frac{2}{n+2} \sum_{0 \leq 2k \leq n-1} v_{n,1,k}[\tilde{\mu}, \mu] \widehat{V}_{n-2k,1}[\mu] \Psi_{+,1}^-,$$

which after another application of (20) reduces to the first statement. In the same way, for $m \geq 1$,

$$\begin{aligned} Z_{n,m}^\pm[\tilde{\mu}] &= (n+m+1)a_{n,m}[\tilde{\mu}] \sum_{0 \leq 2k \leq n-m+1} v_{n,m-1,k}[\tilde{\mu}, \mu] \widehat{V}_{n-2k,m-1}[\mu] \Psi_{-,m-1}^\pm \\ &+ \frac{1}{n+m+2} \sum_{0 \leq 2k \leq n-m-1} v_{n,m+1,k}[\tilde{\mu}, \mu] \widehat{V}_{n-2k,m+1}[\mu] \Psi_{+,m+1}^\pm. \end{aligned} \quad (22)$$

We observe from the definitions leading to Proposition 3.4 that

$$w_{n,m-1,l} w_{n-2l,m-1,k-l}^0 = \frac{n+m-2k+1}{n+m+1} w_{n,m,l} w_{n-2l,m,k-l}^0,$$

so (15) tells us that

$$\begin{aligned} \frac{n+m+1}{n+m-2k+1} v_{n,m-1,k}[\tilde{\mu}, \mu] &= v_{n,m,k}[\tilde{\mu}, \mu] \\ &= \frac{n+m-2k+2}{n+m+2} v_{n,m+1,k}[\tilde{\mu}, \mu]. \end{aligned}$$

From this and Lemma 4.4 we have that

$$\begin{aligned} &(n+m+1)v_{n,m-1,k}[\tilde{\mu}, \mu] \widehat{V}_{n-2k,m-1}[\mu] \Psi_{-,m-1}^\pm \\ &= \frac{1}{a_{n-2k,m}[\mu] + 1} v_{n,m,k}[\tilde{\mu}, \mu] (Z_{n-2k,m}^\mp[\mu] + A_{n-2k,m}^\pm[\mu]), \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{n+m+2} v_{n,m+1,k}[\tilde{\mu}, \mu] \widehat{V}_{n-2k,m+1}[\mu] \Psi_{+,m+1}^{\pm} \\ &= \frac{1}{a_{n-2k,m}[\mu] + 1} v_{n,m,k}[\tilde{\mu}, \mu] (Z_{n-2k,m}^{\mp}[\mu] - a_{n-2k,m}[\mu] A_{n-2k,m}^{\pm}[\mu]). \end{aligned}$$

Inserting these two relations into the respective sums of (22) gives the desired result. \square

Proposition 4.5 provides us with some information about the intersection of the spaces of contragenic functions up to degree n .

Theorem 4.6. *Let $n \geq 1$. The following statements hold:*

- (i) $Z_{n,0}^+[\mu] \in \mathcal{N}_*^{(n)}[0]$ for all μ ;
- (ii) $Z_{n,m}^{\pm}[\mu] \notin \mathcal{N}_*^{(n)}[0]$ when $\mu \neq 0$ and $1 \leq m \leq n-1$.

Proof. The first statement is an immediate consequence of the first formula of Proposition 4.5.

Now consider a basic element $Z_{n,m}^{\pm}[\mu]$ of $\mathcal{N}_*^{(n)}[\mu]$, with $\mu \neq 0$ and $1 \leq m \leq n-1$. A particular instance of the second formula of Proposition 4.5 is

$$Z_{n,m}^{\pm}[\mu] = \sum_{0 \leq 2k \leq n-m+1} (z_{n,m,k}^C[\mu, 0] Z_{n-2k,m}^{\pm}[0] + z_{n,m,k}^A[\mu, 0] A_{n-2k,m}^{\pm}[0]).$$

Suppose that $Z_{n,m}^{\pm}[\mu] \in \mathcal{N}_*^{(n)}[0]$. Then since the right hand side is orthogonal to all Ω_0 -ambigenics,

$$\sum_{0 \leq 2k \leq n-m+1} z_{n,m,k}^A[\mu, 0] A_{n-2k,m}^{\pm}[0] = 0,$$

and so by the linear independence, $z_{n,m,k}^A[\mu, 0] = 0$ for all k . The case in (21) where $2k$ is $n-m$ or $n-m+1$ tells us that $a_{n,m}[\mu] = 0$, which is manifestly false by (18). Consequently, $Z_{n,m}^{\pm}[\mu] \notin \mathcal{N}_*^{(n)}[0]$ as claimed. \square

Note that Theorem 4.6 does not assert that $Z_{n,0}^+[\mu]$ lies in the top-level slice $\mathcal{N}^{(n)}[0]$ of $\mathcal{N}_*^{(n)}[0]$.

Corollary 4.7. *Let $n \geq 1$. Then*

$$\dim \bigcap_{\mu \in [0,1) \cup \mathbb{R}^+} \mathcal{N}_*^{(n)}[\mu] \geq n.$$

Proof. The result is an immediate consequence of the fact that Theorem 4.6 is applicable to arbitrary μ , so the intersection contains a fixed n -dimensional subspace of $\mathcal{N}_*^{(n)}[0]$. \square

It also follows from Theorem 4.6 that the common intersection $\mathcal{N}_0 = \bigcap \mathcal{N}_*[\mu]$ of the full spaces of contragenic functions on spheroids is infinite dimensional, containing all of the contragenic polynomials $Z_{n,m}^+[\mu]$ for which $m = 0$. It seems likely that these contragenic polynomials have special characteristics because of their simpler structure, cf. (20). This phenomenon is not yet fully understood. Further questions relating to the exact relations among the spaces $\mathcal{N}_*^{(n)}[\mu]$ still remain open. If the method of the proof of Theorem 4.6 is applied to linear combinations of the $Z_{n,m}^\pm[\mu]$ instead of just to these generators individually, transcendental equations related to (18) appear. It is not yet known how the angles between the orthogonal complements of the mode-0 subspace $\mathcal{N}_0^n[0]$ in $\mathcal{N}_*^{(n)}[\mu]$, or of their union $\mathcal{N}_0[0]$ in $\mathcal{N}[\mu]$, vary with μ .

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