# On the Darboux transformations and sequences of p-orthogonal polynomials

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#### Abstract

For a fixed  $p \in \mathbb{N}$ , sequences of polynomials  $\{P_n\}, n \in \mathbb{N}$ , defined by a (p+2)term recurrence relation are related to several topics in Approximation Theory. A (p+2)-banded matrix J determines the coefficients of the recurrence relation of any of such sequences of polynomials. The connection between these polynomials and the concept of orthogonality has been already established through a p-dimension vector of functionals. This work goes further in this topic by analyzing the relation between such vectors for the set of sequences  $\{P_n^{(j)}\}, n \in N$ , associated with the Darboux transformations  $J^{(j)}, j = 1, ..., p$ , of a given (p+2)-banded matrix J.

#### **1** Introduction

For a fixed  $p \in \mathbb{N}$  we consider a sequence of polynomials  $\{P_n\}, n \in \mathbb{N}$ , defined by a (p+2)-term recurrence relation

$$P_{n+1}(z) + (a_{n,n} - z)P_n(z) + \sum_{j=1}^p a_{n,n-j}P_{n-j}(z) = 0, \quad n \in \mathbb{N},$$

$$P_{-p} = \dots = P_{-1} = 0, \quad P_0 \equiv 1.$$
(1)

These polynomials are related with several topics such as Hermite-Padé approximants and vector continued fractions ([5,9]). In particular, this kind of polynomials plays an essential role in the study of some integrable systems (see for instance [1-3]).

The relation between the concept of orthogonality and the sequences of polynomials verifying a (p + 2)-term recurrence relation was established in [13] in the following well-known result.

Lemma 1 With the above notation, the following statements are equivalent.

- (i)  $\{P_n\}, n \in \mathbb{N}$ , verify (1) with  $a_{n,n-p} \neq 0$  for all  $n \in \mathbb{N}$ .
- (ii) There exists a vector of functionals  $\nu = (\nu_1, \dots, \nu_p)$ , where each  $\nu_r \in \mathcal{P}'$ ,  $r = 1, \dots, p$ , is defined on the space of polynomials  $\mathcal{P}$  verifying

$$\begin{cases}
\nu_r \left[ z^k P_n(z) \right] = 0, & k = 0, 1, \dots, \quad kp + r \le n, \quad n \in \mathbb{N}, \\
\nu_r \left[ z^k P_{kp+r-1}(z) \right] \neq 0, & k = 0, 1, \dots
\end{cases}$$
(2)

In the sequel, we call vector of *p*-orthogonality to any vector of functionals  $\nu = (\nu_1, \ldots, \nu_p)$  verifying (2). In this case, we say that  $\{P_n\}$  is a sequence of *p*-orthogonal polynomials with respect to  $\nu$ .

For a sequence of polynomials  $\{P_n\}$  defined in (1), the sequence of linear functionals  $\{\mathcal{L}_n\}$ ,  $n = 0, 1, \ldots$ , given by

$$\mathcal{L}_j[P_i] = \delta_{i,j}, \quad i, j = 0, 1, \dots,$$
(3)

plays an relevant role in the study of the orthogonality.  $\{\mathcal{L}_n\}$  is called *dual sequence*, and it is the unique sequence of functionals verifying (3). It is easy to check that the *p* first terms  $\mathcal{L}_r$ , r = 0, 1..., p - 1, of the dual sequence verify the orthogonality conditions (2). This fact proves the existence of some vector of *p*-orthogonality associated with each arbitrary sequence  $\{P_n\}$  of polynomials. However, the uniqueness of a vector functional as in (2) is not guaranteed. In fact, P. Maroni characterized in [10] these vectors of *p*-orthogonality as  $(\nu_1, \ldots, \nu_p)$  such that

$$\nu_{1} = \lambda_{1,0}\mathcal{L}_{0}$$

$$\nu_{2} = \lambda_{2,0}\mathcal{L}_{0} + \lambda_{2,1}\mathcal{L}_{1}$$

$$\vdots$$

$$\nu_{p} = \lambda_{p,0}\mathcal{L}_{0} + \lambda_{p,1}\mathcal{L}_{1} + \dots + \lambda_{p,p-1}\mathcal{L}_{p-1}$$

$$(4)$$

being  $\lambda_{i,j} \in \mathbb{C}$  and  $\lambda_{i,i-1} \neq 0$  for  $i, j+1 \in \{1, \ldots, p\}$ .

Associated with (1), it is possible to define the (p+2)-banded matrix J whose entries are the coefficients of the recurrence relation,

$$J = \begin{pmatrix} a_{0,0} & 1 & & & \\ a_{1,0} & a_{1,1} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ a_{p,0} & a_{p,1} & \cdots & a_{p,p} & 1 & \\ 0 & a_{p+1,1} & & \ddots & \ddots & \ddots \\ & 0 & \ddots & & & \\ & & & \ddots & & & \end{pmatrix},$$
(5)

where we assume  $a_{j+p,j} \neq 0$ ,  $j = 0, 1, \ldots$ 

The use of discrete Darboux transformations was proposed in [11, 12] with the focus in the application to the Toda lattices. In [1-6] this study was extended to (p + 2)banded matrices as (5). In the present work we are concerned about finding relations between the vectors of *p*-orthogonality associated with the Darboux transformations of such matrices (5). Thereby this paper complements the analysis that has been done in [4] for the Geronimus transformations. We include here the following summary, with the more relevant concepts, for an independent reading.

Let  $C \in \mathbb{C}$  be such that the main determinants of the infinite matrix J - CI verify

$$\det\left(CI_n - J_n\right) \neq 0 \text{ for each } n \in \mathbb{N}$$
(6)

Due to the well-known fact that  $P_n(z) = \det(zI_n - J_n)$ , (6) is equivalent to  $C \in \mathbb{C}$  is not a zero of the sequence  $\{P_n\}$ . (Here and in the sequel, given a semi-infinite matrix A, we denote by  $A_n$ ,  $n \in \mathbb{N}$ , the finite matrix of order n formed with the first n rows and columns of A.) In these conditions, there exist two lower and upper triangular matrices, L and U respectively,

$$L = \begin{pmatrix} 1 & & & \\ l_{1,1} & 1 & & \\ \vdots & \ddots & \ddots & \\ l_{p,1} & l_{p,2} & \dots & 1 & \\ 0 & l_{p+1,2} & \ddots & \ddots & \\ & 0 & \ddots & \ddots & \ddots & \end{pmatrix},$$

and

$$U = \begin{pmatrix} \gamma_1 & 1 & & \\ & \gamma_{p+2} & 1 & \\ & & \gamma_{2p+3} & \ddots \\ & & & \ddots \end{pmatrix},$$

such that

$$J - CI = LU \tag{7}$$

is the unique factorization of J - CI in these conditions (see [8] for details). In the sequel we assume  $C \in \mathbb{C}$  fixed.

In [6] the following factorization of L was given.

**Lemma 2** [6, Theorem 1, pp. 118] In the above conditions, if  $l_{p+i,i+1} \neq 0$  for each i = 0, 1, ..., then there exist p bi-diagonal matrices

$$L^{(j)} = \begin{pmatrix} 1 & & & \\ \gamma_{j+1} & 1 & & \\ & \gamma_{p+j+2} & 1 & \\ & & \gamma_{2p+j+3} & \ddots \\ & & & \ddots \end{pmatrix}, \ j = 1, 2, \dots, p,$$
(8)

with  $\gamma_{(k-1)p+j+k} \neq 0$  for all  $k \in \mathbb{N}$ , verifying

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$$L = L^{(1)}L^{(2)}\cdots L^{(p)}.$$
(9)

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Moreover, for each  $j \in \{1, 2, ..., p-1\}$  it is possible to choose certain set of p-j elements  $\gamma_{j+1}, \gamma_{p+j+2}, ..., \gamma_{(p-j)p}$  of  $L^{(j)}$  such that the factorization (9) is unique for the fixed set of p(p-1)/2 points

(7) and (9) provide the so called *Darboux factorization* of J - CI, defined as

$$J - CI = L^{(1)}L^{(2)} \cdots L^{(p)}U,$$
(11)

where  $L^{(1)}, L^{(2)}, \ldots L^{(p)}$  are bidiagonal matrices as in (8) given in Lemma 2 for certain set (10) of p(p-1)/2 fixed entries. Each circular permutation of a Darboux factorization gives a new (p+2)-banded matrix,

$$J^{(j)} = CI + L^{(j+1)}L^{(j+2)} \cdots L^{(p)}UL^{(1)}L^{(2)} \cdots L^{(j)}, \ j = 1, 2, \dots, p.$$
(12)

**Definition 1** The permutations  $J^{(j)}$  given in (12) are called Darboux transformation of J - CI.

As a consequence of the Darboux transformations, for each  $j \in \{1, ..., p\}$  it is possible to define a new sequence of polynomials  $\{P_n^{(j)}\}$  verifying a (p+2) recurrence relation,

$$P_{n+1}^{(j)}(z) + (a_{n,n}^{(j)} - z)P_n^{(j)}(z) + \sum_{s=1}^p a_{n,n-s}^{(j)}P_{n-s}^{(j)}(z) = 0, \quad n \in \mathbb{N},$$

$$P_{-p}^{(j)} = \dots = P_{-1}^{(j)} = 0, \quad P_0^{(j)} \equiv 1.$$

$$(13)$$

Another way to write (13) is

$$(J^{(j)} - zI)v^{(j)} = 0,$$
 (14)

where

$$v^{(j)}(z) = \left(P_0^{(j)}(z), P_1^{(j)}(z), \ldots\right)^T, \qquad j = 0, 1, \dots, p.$$

Here and in the following, we extend the notation taking  $P_n^{(0)} = P_n$ ,  $n \in \mathbb{N}$  and  $J^{(0)} = J$ .

Since Lemma 1, in the above conditions there exists some vector of *p*-orthogonality

$$\nu^{(j)} = \left(\nu_1^{(j)}, \dots, \nu_p^{(j)}\right), \quad j = 1, \dots, p,$$
(15)

such that the corresponding conditions of orthogonality like (2) are verified.

In this work we analyze some relations between the vectors of *p*-orthogonality  $\nu = (\nu_1, \ldots, \nu_p)$  associated to the polynomials  $\{P_n\}$  and the vectors of *p*-orthogonality (15),  $j = 1, \ldots, p$ , for a Darboux factorization of J - CI under some conditions.

**Definition 2** Here and in what follows,  $(z - C)\nu_i$ , i = 1, ..., p, is a linear functional defined on the space  $\mathcal{P}[z]$  of polynomials as

$$(z - C)\nu_i[q] = \nu_i[(z - C)q]$$

for each  $q \in \mathcal{P}[z]$ .

We recall that it is possible to have several vectors of *p*-orthogonality associated with each sequence of polynomials verifying a (p+2)-term recurrence relation. If  $(\nu_1, \ldots, \nu_p)$ 

verifies (2) then for  $(c_1, \ldots, c_p) \in \mathbb{C}$  also  $(c_1\nu_1, \ldots, c_p\nu_p)$  verifies (2). In a more general procedure, the coefficients  $\lambda_{i,j}$  of (4) and the determinants

$$\Delta_{m}^{(j)} = \begin{vmatrix} \lambda_{j+2,0} & \lambda_{j+3,0} & \cdots & \ddots & \lambda_{m,0} & \cdots & \lambda_{j+m+1,0} \\ \vdots & \vdots & & \vdots & & \\ \lambda_{j+2,j+1} & \lambda_{j+3,j+1} & \cdots & \ddots & \lambda_{m,1} & \cdots & \lambda_{j+m+1,j+1} \\ 0 & \lambda_{j+3,j+2} & \vdots & & \vdots & \\ \vdots & 0 & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{m,m-1} & \cdots & \lambda_{j+m+1,m-1} \end{vmatrix}$$

j = 0, 1, ..., p-1, m = 1, 2, ..., p-j-1, play an important role in this paper. Our main contribution is the following.

**Theorem 1** Let  $\nu = (\nu_1, \ldots, \nu_p)$  be a vector of p-orthogonality for  $\{P_n\}$ ,  $n \in \mathbb{N}$ , as in (4) such that

$$\Delta_m^{(j)} \neq 0, \quad j = 0, \dots, p - 1, \quad m = 1, \dots, p - j - 1.$$
(16)

Then there exists a Darboux factorization (11) of J-CI such that, for each j = 1, 2, ..., p,

$$\nu^{(j)} = (\nu_{j+1}, \dots, \nu_p, (z - C)\nu_1, \dots, (z - C)\nu_j)$$
(17)

is a vector of p-orthogonality for the sequence of polynomials  $\{P_n^{(j)}\}, n \in \mathbb{N}$ , associated with the Darboux transformation  $J^{(j)}$  of J - CI given in (12) (where we understand  $\nu^{(p)} = ((z - C)\nu_1, \dots, (z - C)\nu_p)$  in (17)).

**Definition 3** In the conditions of Theorem 1 we say that the vectors of p-orthogonality (17) are the Darboux transformations of  $\nu$ .

In Section 2 some auxiliary results are established. In particular, some connections between the sequences  $\{P_n^{(j)}\}, j = 0, ..., p$ , are given for each fixed Darboux factorization. Finally, Theorem 1 is proved in Section 3.

#### 2 Orthogonality and Darboux transformations

In this section we assume that (11) is a fixed Darboux factorization of J-CI corresponding with a given set of entries (10) of the matrices  $L^{(1)}, \ldots, L^{(p)}$ .

The following result establishes some relationships between the various sequences of polynomials associated with the Darboux transformations of J - CI.

**Theorem 2** We have the following relations between the sequences of polynomials  $\{P_n^{(j)}\}, j = 0, 1, ..., p.$ 

$$P_n^{(j+1)}(z) = \frac{P_{n+1}^{(j)}(z) + \sum_{s=0}^{p-1} g_{n+1,n-s+1}^{(j)} P_{n-s}^{(j)}(z)}{z - C}, \quad j = 0, 1, \dots, p-1, \quad n \ge 0, (18)$$

$$P_n^{(p)}(z) = \frac{P_{n+1}(z) - \frac{P_{n+1}(C)}{P_n(C)} P_n(z)}{z - C}, \quad n \ge 0,$$
(19)

where  $g_{n+1,n-s+1}^{(j)} \in \mathbb{C}$  for  $j = 0, 1, \dots, p-1, n \ge 0$  and  $s = 0, \dots, p-1$ .

<u>**Proof.-**</u> In [6] was proved that

$$L^{(j+1)}L^{(j+2)}\cdots L^{(i)}v^{(i)}(z) = v^{(j)}(z), \quad 0 \le j < i \le p,$$
(20)

where the product of the triangular matrix  $L^{(j+1)}L^{(j+2)}\cdots L^{(i)}$  times the vector  $v^{(i)}(z)$  is understanding in a formal sense.

Moreover, from this, (12) and (14),

$$L^{(j+1)}L^{(j+2)}\cdots L^{(p)}UL^{(1)}\cdots L^{(j)}v^{(j)}(z) = (z-C)v^{(j)}(z), \quad j = 0, 1, \dots, p,$$
(21)

where we understand

$$ULv^{(p)}(z) = (z - C)v^{(p)}(z)$$
(22)

when j = p. Replacing j by j + 1 in (21),

$$L^{(j+2)}L^{(j+3)}\cdots L^{(p)}UL^{(1)}\cdots L^{(j)}L^{(j+1)}v^{(j+1)}(z) = (z-C)v^{(j+1)}(z), \quad j = 0, 1, \dots, p-1,$$

understanding  $L^{(j+2)}L^{(j+3)}\cdots L^{(p)} = I$  when j = p-1. From this and (20) (for i = j+1),

$$L^{(j+2)} \cdots L^{(p)} U L^{(1)} \cdots L^{(j)} v^{(j)}(z) = (z - C) v^{(j+1)}(z), \quad j = 0, 1, \dots, p - 1,$$
(23)

where  $L^{(2)} \cdots L^{(p)} U v^{(0)}(z) = (z - C) v^{(1)}(z)$ , this is,  $L^{(1)} \cdots L^{(j)} = I$  when j = 0.

On the other hand, it is easy to check that, for each  $k \in \mathbb{N}$ , the row k of the infinite matrix  $L^{(j+2)} \cdots L^{(p)} UL^{(1)} \cdots L^{(j)}$  is

$$\left(g_{k,1}^{(j)}, g_{k,2}^{(j)}, \ldots, g_{k,k+1}^{(j)}, 0, \ldots\right),$$

being  $g_{k,k+1}^{(j)} = 1$  and the entries  $g_{k,s}^{(j)}$ ,  $s = 1, \ldots, k$ , independent on z. Moreover,  $g_{k,s}^{(j)} = 0$ for  $s \leq k - p$  when k > p. In other words,  $L^{(j+2)} \cdots L^{(p)} U L^{(1)} \cdots L^{(j)}$  is a (p+1)-banded Hessenberg matrix, where it is easy to see that  $g_{k,k-p+1}^{(j)} \neq 0$  since  $\gamma_r \neq 0$   $(r \in \mathbb{N})$ . Hence, taking into account (23),

$$(z - C)P_{k-1}^{(j+1)}(z) = g_{k,k-p+1}^{(j)}P_{k-p}^{(j)}(z) + \dots + g_{k,k+1}^{(j)}P_k^{(j)}(z)$$

which drives to (18) when k = n + 1. Note that, in this case,

$$g_{n+1,n-p+2}^{(j)} \neq 0.$$
(24)

For j = 0 and i = p, (20) becomes

$$Lv^{(p)}(z) = L^{(1)} \cdots L^{(p)} v^{(p)}(z) = v^{(0)}(z).$$

Therefore

$$Uv^{(0)}(z) = (z - C)v^{(p)}(z)$$
(25)

(see (22)) and, comparing the (n + 1)-row in both sides of (25),

$$(z - C)P_n^{(p)}(z) = \gamma_{n(p+1)+1}P_n(z) + P_{n+1}(z)$$
(26)

The right hand side of (26) is a polynomial with a root in z = C. Thus

$$\gamma_{n(p+1)+1} = -\frac{P_{n+1}(C)}{P_n(C)}.$$
(27)

From (26) and (27) we arrive to (19).

**Remark 1** (18) and (19) coincide in the classic case p = 1. Both relations extend [7, (7.3), pp. 35], this is,

$$P_n^{(1)}(z) = \frac{P_{n+1}(z) - \frac{P_{n+1}(C)}{P_n(C)}P_n(z)}{z - C},$$

where the sequence of Kernel polynomials  $\{P_n^{(1)}\}\$  are defined in terms of  $\{P_n\}$ . In this sense  $\{P_n^{(j)}\}\$ ,  $j = 1, \ldots, p$ , are extensions of this classical sequence of Kernel polynomials.

The above remark justifies the following definition.

**Definition 4** For each j = 1, ..., p the polynomials  $\{P_n^{(j)}\}, n \in \mathbb{N}$ , are called *j*-Kernel polynomials.

For each sequence  $\{P_n^{(j)}\}, j = 0, 1, \dots, p$ , we denote by  $\{\mathcal{L}_n^{(j)}\}$  the corresponding dual sequence (taking  $\mathcal{L}_n^{(0)} = \mathcal{L}_n$ ). Equivalently to the behavior of the sequences of polynomials, the terms of the dual sequences are related to each other.

**Lemma 3** With the above notation, for each j = 0, 1, ..., p-1 and n = 0, 1, ... we have

$$\mathcal{L}_{n}^{(j+1)} = \mathcal{L}_{n}^{(j)} + \gamma_{n(p+1)+j+2} \mathcal{L}_{n+1}^{(j)}$$
(28)

$$(z-C)\mathcal{L}_{n}^{(j)} = \mathcal{L}_{n-1}^{(j+1)} + \sum_{s=0}^{p-1} g_{n+s+1,n+1}^{(j)} \mathcal{L}_{n+s}^{(j+1)}$$
(29)

$$(z - C)\mathcal{L}_n = \mathcal{L}_{n-1}^{(p)} - \frac{P_{n+1}(C)}{P_n(C)}\mathcal{L}_n^{(p)}$$
(30)

<u>Proof</u>.- In [6], the relation

$$P_{m+1}^{(j)} = P_{m+1}^{(j+1)} + \gamma_{m(p+1)+j+2} P_m^{(j+1)}, \quad m = -1, 0, 1, \dots,$$
(31)

was proved (here,  $\gamma_{-(p+1)+j+2} = 0$ ). Then

$$\mathcal{L}_{n}^{(j+1)}\left[P_{m+1}^{(j)}\right] = \mathcal{L}_{n}^{(j+1)}\left[P_{m+1}^{(j+1)}\right] + \gamma_{m(p+1)+j+2}\mathcal{L}_{n}^{(j+1)}\left[P_{m}^{(j+1)}\right] \\ = \begin{cases} 0 & , n \neq m, m+1 \\ 1 & , n = m+1 \\ \gamma_{m(p+1)+j+2} & , n = m. \end{cases}$$
(32)

Moreover,

$$\left(\mathcal{L}_{n}^{(j)} + \gamma_{n(p+1)+j+2}\mathcal{L}_{n+1}^{(j)}\right) \left[P_{m+1}^{(j)}\right] = \mathcal{L}_{n}^{(j)} \left[P_{m+1}^{(j)}\right] + \gamma_{n(p+1)+j+2}\mathcal{L}_{n+1}^{(j)} \left[P_{m+1}^{(j)}\right]$$

also drives to (32). That is, both sides of (28) coincide on the basis  $\{P_m^{(j)}\}$ ,  $m \in \mathbb{N}$ , of the space  $\mathcal{P}$  of polynomials. Therefore (28) is verified.

For  $m = 0, 1, \ldots$ , taking into account (18),

$$(z - C)\mathcal{L}_{n}^{(j)}\left[P_{m}^{(j+1)}\right] = \mathcal{L}_{n}^{(j)}\left[(z - C)P_{m}^{(j+1)}\right]$$
$$= \mathcal{L}_{n}^{(j)}\left[P_{m+1}^{(j)}\right] + \sum_{s=0}^{p-1} g_{m+1,m-s+1}^{(j)}\mathcal{L}_{n}^{(j)}\left[P_{m-s}^{(j)}\right]$$
$$= \begin{cases} 0 & , \quad n \neq m+1, m, m-1, \dots, m-p+1.\\ 1 & , \quad n = m+1\\ g_{n+s+1,n+1}^{(j)} & , \quad n = m-s , \quad s = 0, 1, \dots, p-1. \end{cases}$$
(33)

On the other hand,

$$\left(\mathcal{L}_{n-1}^{(j+1)} + \sum_{s=0}^{p-1} g_{n+s+1,n+1}^{(j)} \mathcal{L}_{n+s}^{(j+1)}\right) \left[P_m^{(j+1)}\right] = \mathcal{L}_{n-1}^{(j+1)} \left[P_m^{(j+1)}\right] + \sum_{s=0}^{p-1} g_{n+s+1,n+1}^{(j)} \mathcal{L}_{n+s}^{(j+1)} \left[P_m^{(j+1)}\right]$$

coincides with (33) for each  $m \in \mathbb{N}$ . Therefore, (29) holds. We underline that this is true even if n = 0 in (29), understanding  $\mathcal{L}_{-1}^{(j+1)} = 0$  in this case.

As in (28)-(29), we apply both sides of (30) to a basis of polynomials. Then using (19),

$$(z-C)\mathcal{L}_{n}\left[P_{m}^{(p)}\right] = \mathcal{L}_{n}\left[(z-C)P_{m}^{(p)}\right]$$

$$= \mathcal{L}_{n}\left[P_{m+1} - \frac{P_{m+1}(C)}{P_{m}(C)}P_{m}\right]$$

$$= \mathcal{L}_{n}\left[P_{m+1}\right] - \frac{P_{m+1}(C)}{P_{m}(C)}\mathcal{L}_{n}\left[P_{m}\right] = \begin{cases} 0 & , n \neq m, m+1 \\ 1 & , n = m+1 \\ -\frac{P_{m+1}(C)}{P_{m}(C)} & , n = m. \end{cases}$$
(34)

Further,

$$\left(\mathcal{L}_{n-1}^{(p)} - \frac{P_{n+1}(C)}{P_n(C)}\mathcal{L}_n^{(p)}\right) \left[P_m^{(p)}\right] = \mathcal{L}_{n-1}^{(p)} \left[P_m^{(p)}\right] - \frac{P_{n+1}(C)}{P_n(C)}\mathcal{L}_n^{(p)} \left[P_m^{(p)}\right],$$

which produces exactly the same result as in (34). Then (30) is proved.  $\Box$ 

In the classic case p = 1, the functionals of orthogonality  $\nu$  and  $\nu^{(1)}$ , associated respectively with  $\{P_n\}$  and the Kernel polynomials  $\{P_n^{(1)}\}$ , are related by

$$\nu^{(1)} = (z - C)\nu$$
.

The next result extends this fact to the general case  $p \in \mathbb{N}$ . In fact, this lemma is equivalently to Theorem 1 in the case j = p.

**Lemma 4** Let  $\nu = (\nu_1, \nu_2, \dots, \nu_p)$  be a vector of p-orthogonality for  $\{P_n\}$ . Then

$$\nu^{(p)} = ((z - C)\nu_1, (z - C)\nu_2, \dots, (z - C)\nu_p)$$

is a vector of p-orthogonality for the p-Kernel polynomials  $\{P_n^{(p)}\}$ .

<u>Proof</u>.- Due to (19), for each  $r = 1, 2, \ldots, p$  we have

$$((z-C)\nu_r) \left[ z^k P_{mp+i}^{(p)} \right] = \nu_r \left[ z^k (z-C) P_{mp+i}^{(p)} \right] = \nu_r \left[ z^k \left( P_{mp+i+1} - \frac{P_{mp+i+1}(C)}{P_{mp+i}(C)} P_{mp+i} \right) \right]$$
  
=  $\nu_r \left[ z^k P_{mp+i+1} \right] - \frac{P_{mp+i+1}(C)}{P_{mp+i}(C)} \nu_r \left[ z^k P_{mp+i} \right],$ 

where

$$\begin{cases}
\nu_r \left[ z^k P_{mp+i+1} \right] = 0 , \quad k \ge 0, \quad kp+r \le mp+i+1, \\
\nu_r \left[ z^k P_{mp+i} \right] = 0 , \quad k \ge 0, \quad kp+r \le mp+i.
\end{cases}$$
(35)

Therefore, if  $kp + r \leq mp + i$  then (35) holds and

$$((z-C)\nu_r)\left[z^k P_{mp+i}^{(p)}\right] = 0, \qquad k \ge 0, \qquad kp+r \le mp+i.$$

Moreover, using (18),

$$((z-C)\nu_r)\left[z^k P_{kp+r-1}^{(p)}\right] = \nu_r \left[z^k P_{kp+r}\right] - \frac{P_{kp+r}(C)}{P_{kp+r-1}(C)}\nu_r \left[z^k P_{kp+r-1}\right],$$

where  $\nu_r \left[ z^k P_{kp+r} \right] = 0$  (see (35)) and  $\nu_r \left[ z^k P_{kp+r-1} \right] \neq 0$  (see (2)). This is,

$$((z-C)\nu_r)\left[z^k P_{kp+r-1}^{(p)}\right] = -\frac{P_{kp+r}(C)}{P_{kp+r-1}(C)}\nu_r\left[z^k P_{kp+r-1}\right] \neq 0.$$

This proves that  $(z - C)\nu_r$  is the *r*-th entry of a vector of *p*-orthogonality associated with the sequence of polynomials  $\{P_n^{(p)}\}$  and, consequently,  $\nu^{(p)}$  is one of such vectors.  $\Box$ 

**Remark 2** We underline that, in the case j = p, we have proved that the statement of Theorem 1 is verified independently on the condition (16).

**Lemma 5** For each j = 0, 1, ..., p, let  $\nu^{(j)} = \left(\nu_1^{(j)}, \nu_2^{(j)}, \ldots, \nu_p^{(j)}\right)$  be a vector of p-orthogonality for  $\{P_n^{(j)}\}$ . Then, for j = 0, 1, ..., p - 1 we have:

(a) 
$$\tilde{\nu}^{(j+1)} = \left(\nu_1^{(j+1)}, \dots, \nu_{p-1}^{(j+1)}, (z-C)\nu_1^{(j)}\right)$$
 is a vector of p-orthogonality for  $\{P_n^{(j+1)}\}$ .  
(b)  $\tilde{\nu}^{(j)} = \left(\nu_1^{(j)}, \nu_1^{(j+1)}, \dots, \nu_{p-1}^{(j+1)}\right)$  is a vector of p-orthogonality for  $\{P_n^{(j)}\}$ .

<u>Proof</u>.- In the first place, because the first entries of the vector  $\tilde{\nu}^{(j+1)}$  coincide with the corresponding to  $\nu^{(j+1)}$ , to prove (a) it is enough to check

$$(z-C)\nu_1^{(j)} \left[ z^k P_n^{(j+1)} \right] = 0, \quad kp+p \le n,$$
(36)

$$(z-C)\nu_1^{(j)} \left[ z^k P_{(k+1)p-1}^{(j+1)} \right] \neq 0.$$
(37)

Indeed, using (18) of Theorem 2,

$$(z-C)\nu_1^{(j)}\left[z^k P_n^{(j+1)}\right] = \nu_1^{(j)}\left[z^k P_{n+1}^{(j)}\right] + \sum_{s=0}^{p-1} g_{n+1,n-s+1}^{(j)}\nu_1^{(j)}\left[z^k P_{n-s}^{(j)}\right],$$

where  $\nu_1^{(j)} \left[ z^k P_{n+1}^{(j)} \right] = 0$  for  $kp + 1 \le n+1$  and  $\nu_1^{(j)} \left[ z^k P_{n-s}^{(j)} \right] = 0$  for  $kp + 1 \le n-s$ ,  $s = 0, 1, \dots, p-1$ .

Then,

$$(z-C)\nu_1^{(j)}\left[x^k P_n^{(j+1)}\right] = 0$$

for  $kp + 1 \le n - p + 1$  or, what is the same, (36) holds.

For a similar reason,

$$(z - C)\nu_1^{(j)} \left[ z^k P_{kp+p-1}^{(j+1)} \right] = \nu_1^{(j)} \left[ z^k P_{kp+p}^{(j)} \right] + \sum_{s=0}^{p-1} g_{kp+p,kp-s+p}^{(j)} \nu_1^{(j)} \left[ z^k P_{kp-s+p-1}^{(j)} \right]$$
$$= g_{kp+p,kp+1}^{(j)} \nu_1^{(j)} \left[ z^k P_{kp}^{(j)} \right] \neq 0 ,$$

which, taking into account (24), gives (37). Thus (a) is verified.

In the second place, take  $r \in \{1, \ldots, p-1\}$ ,  $k \in \{0, 1, \ldots, \}$  and  $n \in \mathbb{N}$ . Using (31),

$$\nu_r^{(j+1)} \left[ z^k P_n^{(j)} \right] = \nu_r^{(j+1)} \left[ z^k P_n^{(j+1)} \right] + \gamma_{(n-1)(p+1)+j+2} \nu_r^{(j+1)} \left[ z^k P_{n-1}^{(j+1)} \right].$$
(38)

In (2) we see

$$\nu_r^{(j+1)} \left[ z^k P_n^{(j+1)} \right] = \nu_r^{(j+1)} \left[ z^k P_{n-1}^{(j+1)} \right] = 0, \quad kp+r \le n-1.$$
(39)

Hence (38) implies

$$\nu_r^{(j+1)} \left[ z^k P_n^{(j)} \right] = 0, \quad kp + r + 1 \le n.$$
(40)

From (38)-(39), taking n = kp + r,

$$\nu_r^{(j+1)} \left[ z^k P_{kp+r}^{(j)} \right] = \gamma_{(kp+r-1)(p+1)+j+2} \nu_r^{(j+1)} \left[ z^k P_{kp+r-1}^{(j+1)} \right] \neq 0.$$
(41)

Since (2), we have that (40) and (41) give  $\nu_r^{(j+1)} = \tilde{\nu}_{r+1}^{(j)}$ , which is the (r+1)-th entry of vector of *p*-orthogonality for  $\{P_n^{(j)}\}, n \in \mathbb{N}$  (we recall that  $r+1 \leq p$ ).

### 3 Proof of Theorem 1

Through Lemma 4 and Remark 2, the result is verified for j = p, independent on the factorization (11). Then we want to find a Darboux factorization (11) such that (17) is a vector of *p*-orthogonality for the corresponding sequence  $\{P_n^{(j)}\}$  of polynomials when  $j = 1, \ldots, p-1$ .

We proceed recursively on  $j = 1, 2, \ldots, p$ .

#### **3.1** First step: j = 1

In this case, in (17) we have the vector of functionals

$$\nu^{(1)} = (\nu_2, \dots, \nu_p, (z - C)\nu_1).$$
(42)

Due to Lemma 5, to prove Theorem 1 it is sufficient to show that  $\nu_2, \ldots, \nu_p$  are the first p-1 entries of a vector of p-orthogonality for  $\{P_n^{(j)}\}$ , where this sequence of polynomials is corresponding to some Darboux transformation  $J^{(1)}$  of J - CI. In this step, we choose  $L^{(1)}$  appropriately for our goal. This is, we will see how to fix the entries

$$\gamma_2, \gamma_{p+3}, \ldots, \gamma_{(p-1)p}$$

of  $L^{(1)}$  in (10) with the purpose to define  $\mathcal{L}_m^{(1)}$  as in (28) and to find  $\lambda_{m,k}^{(1)} \in \mathbb{C}$ ,  $k = 0, 1, \ldots, m-1$ , and  $\lambda_{m,m-1}^{(1)} \neq 0$  such that

$$\nu_{m+1} = \sum_{k=0}^{m-1} \lambda_{m,k}^{(1)} \mathcal{L}_k^{(1)}, \quad m = 1, 2, \dots, p-1$$
(43)

(see (4)).

Because  $\nu$  is a vector of *p*-orthogonality for  $\{P_n\}$ , we know

$$\nu_{m+1} = \sum_{k=0}^{m} \lambda_{m+1,k} \mathcal{L}_k, \quad m = 1, 2, \dots, p-1.$$
(44)

For any Darboux factorization, (28) holds taking j = 0. From this (43) is equivalent to

$$\nu_{m+1} = \lambda_{m,0}^{(1)} \mathcal{L}_0 + \sum_{k=1}^{m-1} \left( \gamma_{(k-1)(p+1)+2} \lambda_{m,k-1}^{(1)} + \lambda_{m,k}^{(1)} \right) \mathcal{L}_k + \gamma_{(m-1)(p+1)+2} \lambda_{m,m-1}^{(1)} \mathcal{L}_m.$$

Comparing the last expression with (44) for  $m = 1, \ldots, p - 1$ , we have

$$\lambda_{m,0}^{(1)} = \lambda_{m+1,0}, \tag{45}$$

$$\gamma_{(k-1)(p+1)+2}\lambda_{m,k-1}^{(1)} + \lambda_{m,k}^{(1)} = \lambda_{m+1,k}, \quad k = 1, \dots, m-1,$$
(46)

$$\gamma_{(m-1)(p+1)+2}\lambda_{m,m-1}^{(1)} = \lambda_{m+1,m}.$$
(47)

We rewrite (45)-(47) as

$$L_{m+1}^{(1)} \begin{pmatrix} \lambda_{m,0}^{(1)} \\ \lambda_{m,1}^{(1)} \\ \vdots \\ \lambda_{m,m-1}^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_{m+1,0} \\ \lambda_{m+1,1} \\ \vdots \\ \lambda_{m+1,m-1} \\ \lambda_{m+1,m} \end{pmatrix}, \ m = 1, \dots, p-1.$$
(48)

In other words, (45)-(46) is

$$\begin{pmatrix} \lambda_{m,0}^{(1)} \\ \lambda_{m,1}^{(1)} \\ \vdots \\ \lambda_{m,m-1}^{(1)} \end{pmatrix} = \begin{pmatrix} L_m^{(1)} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_{m+1,0} \\ \lambda_{m+1,1} \\ \vdots \\ \lambda_{m+1,m-1} \end{pmatrix}, m = 1, \dots, p-1,$$

or, what is the same,

$$\lambda_{m,k-1}^{(1)} = \lambda_{m+1,k-1} - \gamma_{(k-2)(p+1)+2}\lambda_{m+1,k-2} + \gamma_{(k-3)(p+1)+2}\gamma_{(k-2)(p+1)+2}\lambda_{m+1,k-3} - \dots + (-1)^{k-1}\gamma_2\gamma_{(p+1)+2}\dots\gamma_{(k-2)(p+1)+2}\lambda_{m+1,0}, \quad 1 \le k \le m, \quad 1 \le m \le p-1.$$
(49)

Furthermore, taking into account the expression of the last row of  $\left(L_{m+1}^{(1)}\right)^{-1}$ , the following relation joint with (49) are equivalent to (48),

$$\lambda_{m+1,m} - \gamma_{(m-1)(p+1)+2}\lambda_{m+1,m-1} + \dots + (-1)^m \gamma_2 \gamma_{(p+1)+2} \dots \gamma_{(m-1)(p+1)+2}\lambda_{m+1,0} = 0, m = 1, \dots, p-1.$$
(50)

Therefore, the proof of the case j = 1 is reduced to find entries  $\gamma_2, \ldots, \gamma_{(p-1)p}$  of  $L^{(1)}$  verifying the condition (50) such that the coefficients  $\lambda_{m,k-1}^{(1)}, k = 1, \ldots, m$ , provided in (49) define a vector of functionals

$$\nu^{(1)} = \left(\nu_1^{(1)}, \dots, \nu_{p-1}^{(1)}, (z-C)\nu_1\right)$$

as in (42). We proceed recursively for  $m = 1, 2, \ldots, p - 1$ .

For m = 1 in (50) we have  $\lambda_{2,1} - \gamma_2 \lambda_{2,0} = 0$ , where we know from (16) that  $\Delta_1 = \lambda_{2,0} \neq 0$ . Then, taking  $\Delta_0 := 1$ , we can define

$$\gamma_2 = \lambda_{2,1} \frac{\Delta_0}{\Delta_1}.$$

Moreover, since (49) we define  $\lambda_{1,0}^{(1)} = \lambda_{2,0}$  and  $\nu_1^{(1)} = \lambda_{2,0} \mathcal{L}_0$  has been constructed.

Now we will to prove that the first entries of  $L^{(1)}$  can be choosen as

$$\gamma_{(m-1)(p+1)+2} = \lambda_{m+1,m} \frac{\Delta_{m-1}}{\Delta_m}, \quad m = 1, 2, \dots, p-1.$$
(51)

Indeed, (51) is verified for m = 1. Assume that (51) holds for  $m \le s . Assume also that$ 

$$\gamma_2, \ \gamma_{(p+1)+2}, \ \dots, \ \gamma_{(s-1)(p+1)+2}$$

have been choosen verifying (50). Then  $\gamma_{s(p+1)+2}$  can be defined taking m = s + 1 in (50), this is,

$$0 = \lambda_{s+2,s+1} - \gamma_{s(p+1)+2} \left[ \lambda_{s+2,s} - \gamma_{(s-1)(p+1)+2} \lambda_{s+2,s-1} + \cdots + (-1)^{s+1} \gamma_2 \gamma_{(p+1)+2} \cdots \gamma_{(s-1)(p+1)+2} \lambda_{s+2,0} \right],$$

where, from (51), we see that

$$\Delta_s \left[ \lambda_{s+2,s} - \gamma_{(s-1)(p+1)+2} \lambda_{s+2,s-1} + \dots + (-1)^{s+1} \gamma_2 \gamma_{(p+1)+2} \dots \gamma_{(s-1)(p+1)+2} \lambda_{s+2,0} \right]$$

is the development of the determinant  $\Delta_{s+1}$  by its last column. Thus

$$\gamma_{s(p+1)+2} = \lambda_{s+2,s+1} \frac{\Delta_s}{\Delta_{s+1}}$$

and (51) is verified in m = s + 1 and, consequently, for all m = 1, 2, ..., p - 1.

In this way the entries

$$\gamma_2, \gamma_{(p+1)+2}, \ldots, \gamma_{(p-1)p}$$

of  $L^{(1)}$  are chosen verifying (51) and the coefficients  $\lambda_{m,k-1}^{(1)}, k = 1, \ldots, m$ , given in (49) define the vector of orthogonality

$$\nu^{(1)} = \left(\nu_1^{(1)}, \dots, \nu_{p-1}^{(1)}, (z-C)\nu_1\right)$$

for a new sequence of polynomials  $\{P_n^{(1)}\}$ .

#### **3.2** Steps $2, 3, \ldots, p-1$ .

In each one of the following steps, we want to find the first appropriate entries of the corresponding bidiagonal matrix. This is, in the step j + 1, for j = 1, 2, ..., p - 1, we assume that for each s = 1, 2, ..., j the entries

$$\gamma_{s+1}, \gamma_{(p+1)+s+1}, \dots, \gamma_{(p-s-1)(p+1)+s+1}$$

of  $L^{(s)}$  have been defined such that

$$\nu^{(s)} = (\nu_{s+1}, \dots, \nu_p, (z-C)\nu_1, \dots, (z-C)\nu_s)$$

is a vector of p-orthogonality for  $\{P_n^{(s)}\}$ . Then, we want to find the first p - j - 1 entries

$$\gamma_{j+2}, \gamma_{(p+1)+j+2}, \ldots, \gamma_{(p-j-1)p}$$

of  $L^{(j+1)}$  such that  $\nu^{(j+1)}$  in (17) is a vector of *p*-orthogonality for the corresponding sequence  $\{P_n^{(j+1)}\}$ . Due to the case j + 1 = p is solved in Lemma 4, we assume  $j \in \{1, 2, \ldots, p-2\}$  in the following. We differentiate two kind of entries in  $\nu^{(j+1)}$ . This is, we denote

$$\nu^{(j+1)} = \left(\nu_1^{(j+1)}, \dots, \nu_p^{(j+1)}\right)$$

where

$$\nu_k^{(j+1)} = \begin{cases} \nu_{j+k+1} & , \quad k = 1, \dots, p - j - 1, \\ (z - C)\nu_{j+k+1-p} & , \quad k = p - j, \dots, p. \end{cases}$$
(52)

In the first place, we analyze the entries  $\nu_k^{(j+1)}$ ,  $k = 1, \ldots, p-j-1$ , for which we want to define  $\mathcal{L}_s^{(j+1)}$ ,  $s = 0, \ldots, k-1$ , as in (28) and to find  $\lambda_{k,0}^{(j+1)}, \ldots, \lambda_{k,k-1}^{(j+1)}$  such that

$$\nu_k^{(j+1)} = \sum_{s=0}^{k-1} \lambda_{k,s}^{(j+1)} \mathcal{L}_s^{(j+1)}.$$
(53)

Since (17) and (52) we know that

$$\nu_k^{(j+1)} = \nu_{k+1}^{(j)}, \quad k = 1, \dots, p - j - 1.$$

Hence,

$$\nu_k^{(j+1)} = \sum_{r=0}^k \lambda_{k+1,r}^{(j)} \mathcal{L}_r^{(j)}, \quad k = 1, \dots, p - j - 1.$$
(54)

Using (28) in (53),

$$\begin{split} \nu_k^{(j+1)} &= \sum_{s=0}^{k-1} \lambda_{k,s}^{(j+1)} \left( \mathcal{L}_s^{(j)} + \gamma_{s(p+1)+j+2} \mathcal{L}_{s+1}^{(j)} \right) = \\ &= \sum_{s=0}^{k-1} \lambda_{k,s}^{(j+1)} \mathcal{L}_s^{(j)} + \sum_{s=1}^k \gamma_{(s-1)(p+1)+j+2} \lambda_{k,s-1}^{(j+1)} \mathcal{L}_s^{(j)} = \\ &= \lambda_{k,0}^{(j+1)} \mathcal{L}_0^{(j)} + \sum_{s=1}^{k-1} \left( \lambda_{k,s}^{(j+1)} + \gamma_{(s-1)(p+1)+j+2} \lambda_{k,s-1}^{(j+1)} \right) \mathcal{L}_s^{(j)} + \gamma_{(k-1)(p+1)+j+2} \lambda_{k,k-1}^{(j+1)} \mathcal{L}_k^{(j)}. \end{split}$$

Comparing with (54),

$$\lambda_{k+1,s}^{(j)} = \begin{cases} \lambda_{k,0}^{(j+1)}, & s = 0, \\ \lambda_{k,s}^{(j+1)} + \gamma_{(s-1)(p+1)+j+2}\lambda_{k,s-1}^{(j+1)}, & s = 1, \dots, k-1, \\ \gamma_{(k-1)(p+1)+j+2}\lambda_{k,k-1}^{(j+1)}, & s = k. \end{cases}$$
(55)

Similarly to what was done in (45)-(47), (55) can be rewritten as

$$L_{k+1}^{(j+1)} \begin{pmatrix} \lambda_{k,0}^{(j+1)} \\ \lambda_{k,1}^{(j+1)} \\ \vdots \\ \lambda_{k,k-1}^{(j+1)} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_{k+1,0}^{(j)} \\ \lambda_{k+1,1}^{(j)} \\ \vdots \\ \lambda_{k+1,k-1}^{(j)} \\ \lambda_{k+1,k}^{(j)} \end{pmatrix}, \ k = 1, \dots, p-j-1,$$
(56)

or, what is the same,

$$L_{k}^{(j+1)} \begin{pmatrix} \lambda_{k,0}^{(j+1)} \\ \lambda_{k,1}^{(j+1)} \\ \vdots \\ \lambda_{k,k-1}^{(j+1)} \end{pmatrix} = \begin{pmatrix} \lambda_{k+1,0}^{(j)} \\ \lambda_{k+1,1}^{(j)} \\ \vdots \\ \lambda_{k+1,k-1}^{(j)} \end{pmatrix}, \ k = 1, \dots, p - j - 1,$$
(57)

with the additional condition

$$\lambda_{k+1,k}^{(j)} = \gamma_{(k-1)(p+1)+j+2} \lambda_{k,k-1}^{(j+1)}, \quad k = 1, \dots, p-j-1.$$
(58)

Therefore, finding

$$\lambda_{k,0}^{(j+1)},\ldots,\lambda_{k,k-1}^{(j+1)}$$

as in (53) comes down to choose the entries

$$\gamma_{j+2},\ldots,\gamma_{(p-j-1)p}$$

of  $L_{p-j}^{(j+1)}$  with the aim of (57)-(58) take place. We note that (57) defines the coefficients  $\lambda_{k,s}^{(j+1)}$ ,  $s = 0, \ldots, k-1$ , because  $L_{p-j}^{(j+1)}$  is an invertible matrix. Then, (58) is equivalent to the fact that the last row of  $\left(L_{k+1}^{(j+1)}\right)^{-1}$  multiplied by  $\left(\lambda_{k+1,0}^{(j)}, \ldots, \lambda_{k+1,k}^{(j)}\right)$  vanishes. This is,

$$\lambda_{k+1,k}^{(j)} - \lambda_{k+1,k-1}^{(j)}\gamma_{(k-1)(p+1)+j+2} + \lambda_{k+1,k-2}^{(j)}\gamma_{(k-2)(p+1)+j+2}\gamma_{(k-1)(p+1)+j+2}$$
(59)  
-... +  $(-1)^k \lambda_{k+1,0}^{(j)}\gamma_{j+2}\gamma_{(p+1)+j+2} \dots \gamma_{(k-1)(p+1)+j+2} = 0, \quad k = 1, 2, \dots, p-j.$ 

Now we show that in the above conditions the following matrix equality is verified for

 $j = 1, 2, \dots, p - 1$  and  $s = 1, 2, \dots, j$ 

$$L_{s}^{(1)} \dots L_{s}^{(j)} \begin{pmatrix} \lambda_{2,0}^{(j)} & \lambda_{3,0}^{(j)} & \cdots & \cdots & \lambda_{s+1,0}^{(j)} \\ \lambda_{2,1}^{(j)} & \lambda_{3,1}^{(j)} & \cdots & \cdots & \lambda_{s+1,1}^{(j)} \\ 0 & \lambda_{3,2}^{(j)} & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & 0 & \cdots & 0 & \lambda_{s,s-1}^{(j)} & \lambda_{s+1,s-1}^{(j)} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_{j+2,0} & \lambda_{j+3,0} & \cdots & \cdots & \lambda_{s,0} & \cdots & \lambda_{j+s+1,0} \\ \vdots & \vdots & & \vdots & & \vdots \\ \lambda_{j+2,j+1} & \lambda_{j+3,j+1} & \cdots & \cdots & \lambda_{s,j+1} & \cdots & \lambda_{j+s+1,j+1} \\ 0 & \lambda_{j+3,j+2} & \ddots & \vdots & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \cdots & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{s,s-1} & \cdots & \lambda_{j+s+1,s-1} \end{pmatrix}.$$
(60)

In fact, as in (56), it is easy to see

$$L_{k+1}^{(r)} \begin{pmatrix} \lambda_{k,0}^{(r)} \\ \lambda_{k,1}^{(r)} \\ \vdots \\ \lambda_{k,k-1}^{(r)} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_{k+1,0}^{(r-1)} \\ \lambda_{k+1,1}^{(r-1)} \\ \vdots \\ \lambda_{k+1,k-1}^{(r-1)} \\ \lambda_{k+1,k}^{(r-1)} \end{pmatrix}, \ k = 1, \dots, p-r, \quad r = 1, 2, \dots, j+1.$$

Then, considering each infinite matrix  $L^{(r)}$ ,

$$L^{(r)}\begin{pmatrix}\lambda_{k,0}^{(r)}\\\lambda_{k,1}^{(r)}\\\vdots\\\lambda_{k,k-1}^{(r)}\\0\\0\\\vdots\end{pmatrix} = \begin{pmatrix}\lambda_{k+1,0}^{(r-1)}\\\lambda_{k+1,1}^{(r-1)}\\\vdots\\\lambda_{k+1,k-1}^{(r-1)}\\\lambda_{k+1,k}^{(r-1)}\\0\\\vdots\end{pmatrix}, k = 1, \dots, p-r, \quad r = 1, 2, \dots, j+1.$$
(61)

Applying iteratively  $L^{(r)}, \ldots, L^{(1)}$  to (61) and then taking r = j we arrive to

.

$$L^{(1)}\cdots L^{(j)}\begin{pmatrix}\lambda_{k,0}^{(j)}\\\vdots\\\lambda_{k,k-1}^{(rj)}\\0\\\vdots\end{pmatrix} = \begin{pmatrix}\lambda_{j+k,0}\\\vdots\\\vdots\\\lambda_{j+k,j+k-1}\\0\\\vdots\end{pmatrix}, k = 1, \dots, p-j.$$

From this, for  $s \in \mathbb{N}$ , taking  $k = 2, 3, \ldots, s + 1$ ,

$$L^{(1)} \cdots L^{(j)} \begin{pmatrix} \lambda_{2,0}^{(j)} & \lambda_{3,0}^{(j)} & \cdots & \lambda_{s+1,0}^{(j)} \\ \lambda_{2,1}^{(j)} & \lambda_{3,1}^{(j)} & \cdots & \lambda_{s+1,1}^{(j)} \\ 0 & \lambda_{3,2}^{(j)} & \ddots & \vdots \\ \vdots & 0 & \cdots & \vdots \\ \vdots & \ddots & \lambda_{s+1,s}^{(j)} \\ & \vdots & \ddots & \lambda_{s+1,s}^{(j)} \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} \lambda_{j+2,0} & \lambda_{j+3,0} & \cdots & \lambda_{j+s+1,0} \\ \vdots & \vdots & \ddots & \lambda_{j+s+1,j+1} \\ 0 & \lambda_{j+3,j+2} & \ddots & \vdots \\ \vdots & 0 & \cdots & \vdots \\ \vdots & 0 & \cdots & \vdots \\ & \vdots & \ddots & \lambda_{j+s+1,j+s} \\ & & & & 0 \\ & & & & & \vdots \end{pmatrix}$$

Therefore, due to  $L^{(1)} \cdots L^{(j)}$  is an infinite lower triangular matrix, we have

$$\left(L^{(1)}\cdots L^{(j)}\right)_s = L_s^{(1)}\cdots L_s^{(j)}$$

and we arrive to (60).

As a consequence of (60),

$$\Delta_{s}^{(j)} = \begin{vmatrix} \lambda_{2,0}^{(j)} & \lambda_{3,0}^{(j)} & \cdots & \cdots & \lambda_{s+1,0}^{(j)} \\ \lambda_{2,1}^{(j)} & \lambda_{3,1}^{(j)} & \cdots & \cdots & \lambda_{s+1,1}^{(j)} \\ 0 & \lambda_{3,2}^{(j)} & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{s,s-1}^{(j)} & \lambda_{s+1,s-1}^{(j)} \end{vmatrix}, \quad j = 1, \dots, p-1, \quad s \in \mathbb{N}.$$

Taking k = 1 in (59), we have  $\lambda_{2,1}^{(j)} - \lambda_{2,0}^{(j)} \gamma_{j+2} = 0$ . Because we know that  $\lambda_{2,1}^{(j)} \neq 0$  (see (4)) and  $\Delta_1^{(j)} = \lambda_{2,0}^{(j)} \neq 0$ , it is possible to take

$$\gamma_{j+2} = \lambda_{2,1}^{(j)} \frac{\Delta_0^{(j)}}{\Delta_1^{(j)}}$$

(where we define  $\Delta_0^{(j)} := 1$ ). Iterating the procedure, assuming

$$\gamma_{m(p+1)+j+2} = \lambda_{m+2,m+1}^{(j)} \frac{\Delta_m^{(j)}}{\Delta_{m+1}^{(j)}}, \quad m = 0, 1, \dots, s,$$
(62)

with s and taking <math>k = s + 2 in (59),

$$\lambda_{s+3,s+2}^{(j)} - \gamma_{(s+1)(p+1)+j+2} \Big[ \lambda_{s+3,s+1}^{(j)} - \lambda_{s+3,s}^{(j)} \gamma_{s(p+1)+j+2} + \dots + (-1)^{s+1} \lambda_{s+3,0}^{(j)} \gamma_{j+2} \dots \gamma_{s(p+1)+j+2} \Big] = 0.$$

where  $\gamma_{(s+1)(p+1)+j+2}$  can be defined as

$$\gamma_{(s+1)(p+1)+j+2} = \lambda_{s+3,s+2}^{(j)} \frac{\Delta_{s+1}^{(j)}}{\Delta_{s+1}^{(j)} \left[\lambda_{s+3,s+1}^{(j)} - \dots + (-1)^{s+1}\lambda_{s+3,0}^{(j)}\gamma_{j+2} \dots \gamma_{s(p+1)+j+2}\right]}.$$
(63)

Further, from (62) it is easy to check that the denominator in (63) is the development of the determinant  $\Delta_{s+2}^{(j)}$  by its last column. Thus  $\gamma_{m(p+1)+j+2}$  can be defined as in (62) for  $m = 0, 1, \ldots, p - j - 2$ .

Finally, we study the entries  $\nu_k^{(j+1)}$ ,  $k = p - j, \ldots, p$  of  $\nu^{(j+1)}$ . We want to prove

$$(z-C)\nu_{j+k+1-p}\left[z^{s}P_{n}^{(j+1)}\right] = 0, \quad n \ge sp+k$$
 (64)

$$(z-C)\nu_{j+k+1-p}\left[z^{s}P_{sp+k-1}^{(j+1)}\right] \neq 0,$$
(65)

because this means, from (2), that  $(z - C)\nu_{j+k+1-p}$  is the entry k of a vector of porthogonality for  $\{P_n^{(j+1)}\}$ .

Lemma 4 implies

$$(z - C)\nu_{j+k+1-p} \left[ z^{s} P_{m}^{(p)} \right] = 0, \qquad m \ge sp + j + k + 1 - p$$

$$(z - C)\nu_{j+k+1-p} \left[ z^{s} P_{sp+j+k-p}^{(p)} \right] \ne 0.$$
(66)

In addition, since (31), for any Darboux factorization,

$$\begin{pmatrix} P_0^{(j+1)} \\ P_1^{(j+1)} \\ \vdots \end{pmatrix} = L^{(j+2)} \begin{pmatrix} P_0^{(j+2)} \\ P_1^{(j+2)} \\ \vdots \end{pmatrix} = \dots = L^{(j+2)} L^{(j+3)} \dots L^{(p)} \begin{pmatrix} P_0^{(p)} \\ P_1^{(p)} \\ \vdots \end{pmatrix}.$$

Thus,  $P_n^{(j+1)}$  can be expressed in terms of the entries in the (n+1)-th row of  $L^{(j+2)}L^{(j+3)} \dots L^{(p)}$ and the sequence  $\{P_n^{(p)}\}$ . From this, taking into account that  $L^{(j+2)}L^{(j+3)} \dots L^{(p)}$  is a lower triangular (p-j)-banded matrix, we see

$$P_n^{(j+1)} = \sum_{r=n+j-p+1}^n \alpha_r P_r^{(p)}, \quad n \ge 0 \quad (\text{with } \alpha_n = 1).$$

Hence,

$$(z-C)\nu_{j+k+1-p}\left[z^{s}P_{n}^{(j+1)}\right] = \sum_{r=n+j-p+1}^{n} \alpha_{r}(z-C)\nu_{j+k+1-p}\left[z^{s}P_{r}^{(p)}\right]$$
(67)

and, using (66), we see that each term on the right hand side of (67) vanishes when  $n \ge sp + k$ . Thus (64) is verified.

To see (65), if n = sp + k - 1 in (67), then using (66) on the right hand side of (67) we have

$$\sum_{r=sp+k+j-p}^{sp+k-1} \alpha_r(z-C)\nu_{j+k+1-p} \left[ z^s P_r^{(p)} \right] = \alpha_{sp+k+j-p}(z-C)\nu_{j+k+1-p} \left[ z^s P_{sp+j+k-p}^{(p)} \right] \neq 0.$$

**Remark 3** Note that, if  $\Delta_s^{(j)} \neq 0$  for s = 0, 1, ..., p - m - 1 and j = 1, ..., m - 1, then there exist m bidiagonal matrices  $L^{(1)}, ..., L^{(m)}$  such that

$$J - CI = L^{(1)} \cdots L^{(m)} \tilde{L}U,$$

where  $\tilde{L}$  is a lower triangular matrix (non bidiagonal, in general). However, if  $\Delta_s^{(m)} = 0$ for some  $s \in \{0, \ldots, p - m - 2\}$  then we can not assure the existence of a Darboux factorization (11) of J - CI such that  $\nu^{(m+1)}$  is defined as in (17).

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