# Directed Strongly Regular Cayley Graphs on Dihedral groups

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## Abstract

In this paper, we construct some directed strongly regular Cayley graphs on Dihedral groups, which generalizes some earlier constructions. We also characterize certain directed strongly regular Cayley graphs on Dihedral groups  $D_{p^{\alpha}}$ , where p is a prime and  $\alpha \ge 1$  is a positive integer.

*Keywords:* Directed strongly regular graph, Cayley graph, Dihedral group, Representation Theory, Fourier Transformation, Algebraic Number Theory

#### 1. Introduction

A directed strongly regular graph (DSRG) with parameters  $(n, k, \mu, \lambda, t)$  is a k-regular directed graph on n vertices such that every vertex is on t 2-cycles, and the number of paths of length two from a vertex x to a vertex y is  $\lambda$  if there is an edge directed from x to y and is  $\mu$  otherwise. A DSRG with t = k is an (undirected) strongly regular graph (SRG). Duval showed that DSRGs with t = 0 are the doubly regular tournaments. It is therefore usually assumed that 0 < t < k. The DSRGs which satisfy the condition 0 < t < k are called genuine DSRGs. The DSRGs appeared on this paper are all genuine.

Let *D* be a directed graph with *n* vertices. Let  $A = \mathbf{A}(D)$  denote the adjacency matrix of *D*, and let  $I = I_n$  and  $J = J_n$  denote the  $n \times n$  identity matrix and all-ones matrix, respectively. Then *D* is a directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$  if and only if (i) JA = AJ = kJ and (ii)  $A^2 = tI + \lambda A + \mu(J - I - A)$ .

The constructions of DSRGs is a significant problem and has long been concerned. There are many constructions of DSRGs. Some of the known constructions use quadratic residue [1], Kronecker product [1], block matrices [2, 1], combinatorial block designs [3], coherent algebras[3, 4, 5], Cayley digraph [3, 6, 5], generalized Cayley digraph [7], semidirect product [8], finite incidence structures [9, 10], finite geometries [3],  $1\frac{1}{2}$ -designs [11], generalized quadrangle [12], BIBD [12], partial geometry [12, 9], double Paley designs [3], group divisible [9], difference digraph, partial sum families [13] and equitable partition [14].

Let G be a finite (multiplicative) group and S be a subset of  $G \setminus \{e\}$ . The Cayley graph of G generated by S, denoted by  $\mathbf{Cay}(G, S)$ , is the digraph  $\Gamma$  such that  $V(\Gamma) = G$  and  $x \to y$  if and only if  $x^{-1}y \in S$ , for  $x, y \in G$ .

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Let  $C_n = \langle x \rangle$  be a cyclic multiplicative group of order n. The dihedral group  $D_n$  is the group of symmetries of a regular n-polygon, and it can be viewed as a semidirect product of two cyclic groups  $C_n = \langle x \rangle$  of order n and  $C_2 = \langle a \rangle$  of order 2. The presentation of  $D_n$  is  $D_n = C_n \rtimes C_2 = \langle x, a | x^n =$  $1, a^2 = 1, ax = x^{-1}a \rangle$ . The cyclic group  $C_n$  is a normal subgroup of  $D_n$  of index 2.

In this paper, we focus on the directed strongly regular Cayley graphs on dihedral groups. The *Cayley* graphs on dihedral groups are called *dihedrants*. A dihedrant which is a DSRG is called *directed strongly* regular dihedrant. We now give some known directed strongly regular dihedrants.

**Theorem 1.1.** ([5])Let n be odd and let  $X, Y \subset C_n$  satisfy the following conditions: (i)  $\overline{X} + \overline{X^{(-1)}} = \overline{C_n} - e$ , (ii)  $\overline{Y} \overline{Y^{(-1)}} - \overline{X} \overline{X^{(-1)}} = \varepsilon \overline{C_n}, \varepsilon \in \{0, 1\}.$ 

Then  $\operatorname{Cay}(D_n, X \cup aY)$  is a DSRG with parameters  $(2n, n-1+\varepsilon, \frac{n-1}{2}+\varepsilon, \frac{n-3}{2}+\varepsilon, \frac{n-1}{2}+\varepsilon)$ . In particular, if X satisfies (i) and Y = Xg or  $X^{(-1)}g$  for some  $g \in C_n$ , then  $\operatorname{Cay}(D_n, X \cup aY)$  is a DSRG with parameters  $(2n, n-1, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-1}{2})$ .

We can say more when n is an odd prime.

**Theorem 1.2.** ([5])Let n be an odd prime and let  $X, Y \subset C_n$  and  $b \in D_n \setminus C_n$ , Then the Cayley graph  $\mathbf{Cay}(D_n, X \cup bY)$  is a DSRG if and only if X, Y satisfy the conditions of Theorem 1.1.

**Theorem 1.3.** ([3])Let n be even,  $c \in C_n$  be an involution and let  $X, Y \subset C_n$  such that: (i)  $\overline{X} + \overline{X^{(-1)}} = \overline{C_n} - e - c$ , (ii)  $\overline{Y} = \overline{X}$  or  $\overline{Y} = \overline{X^{(-1)}}$ , (ii)  $\overline{Xc} = \overline{X^{(-1)}}$ . Let  $b \in D_n \setminus C_n$ , then the Cayley graph  $\mathbf{Cay}(D_n, X \cup bY)$  is a DSRG with parameters  $(2n, n - 1, \frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2})$ .

The following notataion will be used. Let A be a multiset together with a multiplicity function  $\Delta_A$ , where  $\Delta_A(a)$  counting how many times a occurs in the multiset A. We say x belongs to A (i.e.  $x \in A$ ) if  $\Delta_A(x) > 0$ . In the following, A and B are multisets, with multiplicity functions  $\Delta_A$  and  $\Delta_B$ .

- Union,  $A \uplus B$ : the union of multisets A and B, is defined by  $\Delta_{A \uplus B} = \Delta_A + \Delta_B$ ;
- Scalar multiplication,  $n \oplus A$ : the scalar multiplication of a multiset A by a natural number n by, is defined by  $\Delta_{n\oplus A} = n\Delta_A$ .
- Difference,  $A \setminus B$ : the difference of multisets A and B, is defined by  $\Delta_{A \setminus B}(x) = \max{\{\Delta_A(x) \Delta_B(x), 0\}}$  for any  $x \in A$ .

If A and B are usual sets, we use  $A \cup B$ ,  $A \cap B$  and  $A \setminus B$  denote the usual union, intersection and difference of A and B. For example, if  $A = \{1,2\}$  and  $B = \{1,3\}$ , then  $A \uplus B = \{1,1,2,3\}$ ,  $A \cup B = \{1,2,3\}$ ,  $2 \oplus A = \{1,1,2,2\}$ ,  $A \setminus B = \{2\}$  and  $\{1,1,2,2\} \setminus B = \{1,2,2\}$ .

Throughout this paper, n will denote a fixed positive integer,  $\mathbb{Z}_n$  is the modulo n residue class ring. For a positive divisor r of n let  $r\mathbb{Z}_n$  be the subgroup of the additive group of  $\mathbb{Z}_n$  of order  $\frac{n}{r}$ . Let v be a positive divisor of n and  $v\mathbb{Z}_n$  is a subgroup of  $\mathbb{Z}_n$ . Let  $\pi_v$  be the natural projection from  $\mathbb{Z}_n$  to the quotient group  $\mathbb{Z}_n/v\mathbb{Z}_n$ . It is clear that quotient group  $\mathbb{Z}_n/v\mathbb{Z}_n$  is isomorphic to the cyclic group  $\mathbb{Z}_v$ . Let  $\sigma_v$  be canonical isomorphism map between  $\mathbb{Z}_v$  and  $\mathbb{Z}_n/v\mathbb{Z}_n$ . Let  $\psi_v = \sigma_v \circ \pi_v$ . Then  $\psi_v$  is a canonical homomorphism

$$\psi_{n,v}: \mathbb{Z}_n \to \mathbb{Z}_v, \ i + n\mathbb{Z} \mapsto \sigma_v(\pi_v(i)) = i(mod \ v) + v\mathbb{Z}.$$

$$(1.1)$$

with ker  $\psi_{n,v} = v\mathbb{Z}_n$ . We also define  $\phi_v : \mathbb{Z}_v \to \frac{n}{v}\mathbb{Z}_n$ ,  $i + v\mathbb{Z} \to \frac{n}{v}i + n\mathbb{Z}$ . It is clearly that  $\phi_v$  gives an isomorphism between  $\mathbb{Z}_v$  and  $\frac{n}{v}\mathbb{Z}_n$ .

For a multisubset A of  $\mathbb{Z}_n$ , let

$$x^A = \biguplus_{i \in A} \Delta_A(i) \oplus \{x^i\},$$

then  $C_n = x^{\mathbb{Z}_n}$  and  $x^A$  is a multisubset of  $C_n$ . Let  $A_1, A_2$  be the multisubsets of  $\mathbb{Z}_n$  and  $i \in \mathbb{Z}_n$ , let  $iA_1 = \{ia_1|a_1 \in A_1\}, i+A_1 = \{i+a_1|a_1 \in A_1\}$ , and  $i-A_1 = \{i-a_1|a_1 \in A_1\}$ . The sum of multisubsets  $A_1$  and  $A_2$  is  $A_1 + A_2 = \{a_1 + a_2|a_1 \in A_1, a_2 \in A_2\}$ , and the multiplicity function of  $A_1 + A_2$  is

$$\Delta_{A_1+A_2}(c) = \sum_{a_1+a_2=c} \Delta_{A_1}(a_1) \Delta_{A_2}(a_2)$$

for any  $c \in \mathbb{Z}_n$ .

Hence each subset S of the dihedral group  $D_n = \langle x, a | x^n = 1, a^2 = 1, ax = x^{-1}a \rangle$  can be written in the form  $S = x^X \cup (x^Y a)$  for some subsets  $X, Y \subseteq \mathbb{Z}_n$ , we denote the Cayley graph  $\mathbf{Cay}(D_n, x^X \cup x^Y a)$ by Dih(n, X, Y). Let Dih(n, X, Y) be a directed strongly regular dihedrant. We list some properties in the following.

- Its complement  $Dih(n, \mathbb{Z}_n \setminus \{0\} \setminus X, \mathbb{Z}_n \setminus Y)$ , is also a DSRG.
- Observe that Dih(n, X, Y) is a digraph without loops, then  $0 \notin X, X \neq -X$ .
- The Lemma 2.5 in [15] asserts that the dihedrants Dih(n, bX, b' + bY) and Dih(n, X, Y) are isomorphic for any  $b \in \mathbb{Z}_n^*$ ,  $b' \in \mathbb{Z}_n$ . Hence, we can also assume  $0 \in Y$ .

For an odd prime p, the characterization of directed strongly regular dihedrant Dih(p; X, Y) has been achieved in [5], which was presented in Theorem 1.2. In this paper, we characterize some certain directed strongly regular dihedrants  $Dih(p^{\alpha}, X, Y)$  for prime power  $p^{\alpha}$ . We obtain the following results.

**Theorem 1.4.** For an odd prime p and a positive integer  $\alpha$ , then the dihedrant  $Dih(p^{\alpha}, X, X)$  is a DSRG if and only if  $X = \psi_{\gamma}^{-1}(H)$  for some  $1 \leq \gamma \leq \alpha$  and a subset  $H \subseteq \mathbb{Z}_{p^{\gamma}}$  satisfying the following conditions:

(i)  $H \cup (-H) = \mathbb{Z}_{p^{\gamma}} \setminus \{0\}.$ (ii)  $H \cap (-H) = \emptyset.$ 

**Theorem 1.5.** For a positive integer  $\alpha$ , the dihedrant  $Dih(2^{\alpha}, X, X)$  is a DSRG if and only if  $X = \psi_{\gamma}^{-1}(H)$  for some  $2 \leq \gamma \leq \alpha$  and a subset  $H \subseteq \mathbb{Z}_{p^{\gamma}}$  satisfying the following conditions: (i)  $H \cup (-H) = (\mathbb{Z}_{p^{\gamma}} \setminus \{0\}) \uplus \{2^{\gamma-1}\}.$ (ii)  $H \cap (2^{\gamma-1} + H) = \emptyset.$  **Theorem 1.6.** The dihedrant  $Dih(p^{\alpha}, X, Y)$  is a DSRG with  $X \subset Y$  and  $Y \setminus X$  is a union of some  $\mathbb{Z}_{p^{\alpha}}^{*}$ -orbits if and only if  $Y = \psi_{\gamma}^{-1}(H)$ ,  $X = Y \setminus \{0\}$  or  $X = Y \setminus p^{\gamma} \mathbb{Z}_{p^{\alpha}}$  for some  $1 \leq \gamma \leq \alpha$  and a subset  $H \subseteq \mathbb{Z}_{p^{\gamma}}$  such that  $H \uplus (-H) = \mathbb{Z}_{p^{\gamma}} \uplus \{0\}$ .

## 2. Preliminary

## 2.1. Properties of DSRG

Duval [1] developed necessary conditions on the parameters of  $(n, k, \mu, \lambda, t)$ -DSRG and calculated the spectrum of a DSRG.

**Proposition 2.1.** (see [1]) A DSRG with parameters  $(n, k, \mu, \lambda, t)$  with 0 < t < k satisfy

$$k(k + (\mu - \lambda)) = t + (n - 1)\mu,$$

$$d^{2} = (\mu - \lambda)^{2} + 4(t - \mu), d|2k - (\lambda - \mu)(n - 1),$$
(2.1)

$$\frac{2k - (\lambda - \mu)(n-1)}{d} \equiv n - 1(mod 2), \left| \frac{2k - (\lambda - \mu)(n-1)}{d} \right| \le n - 1,$$

where d is a positive integer, and

$$0 \leqslant \lambda < t < k, 0 < \mu \leqslant t < k, -2\left(k-t-1\right) \leq \mu - \lambda \leqslant 2\left(k-t\right)$$

**Remark 2.1.** If G is a DSRG with parameters  $(n, k, \mu, \lambda, t)$  with  $0 < t = \mu < k$ , then from the last inequality in the above proposition, we have  $\lambda - \mu < 0$ .

**Proposition 2.2.** (see [1]) A DSRG with parameters  $(n, k, \mu, \lambda, t)$  has three distinct integer eigenvalues

$$k > \rho = \frac{1}{2} \left( -(\mu - \lambda) + d \right) > \sigma = \frac{1}{2} \left( -(\mu - \lambda) - d \right)$$

The multiplicities are

1, 
$$m_{\rho} = -\frac{k + \sigma (n-1)}{\rho - \sigma}$$
 and  $m_{\sigma} = \frac{k + \rho (n-1)}{\rho - \sigma}$ ,

respectively.

**Proposition 2.3.** (see [1]) If G is a DSRG with parameters  $(n, k, \mu, \lambda, t)$ , then the complementary G' is also a DSRG with parameters  $(n', k', \mu', \lambda', t')$ , where k' = (n - 2k) + (k - 1),  $\lambda' = (n - 2k) + (\mu - 2)$ , t' = (n - 2k) + (t - 1),  $\mu' = (n - 2k) + \lambda$ .

#### **Definition 2.1.** (Group Ring)

For any finite (multiplicative) group G and ring R, the Group Ring R[G] is defined as the set of all formal sums of elements of G, with coefficients from R. i.e.,

$$R[G] = \left\{ \sum_{g \in G} r_g g | r_g \in R, \ r_g \neq 0 \text{ for finite g} \right\}.$$

The operations + and  $\cdot$  on R[G] are given by

$$\sum_{g \in G} r_g g + \sum_{g \in G} s_g g = \sum_{g \in G} (r_g + s_g)g,$$
$$\left(\sum_{g \in G} r_g g\right) \cdot \left(\sum_{g \in G} s_g g\right) = \left(\sum_{g \in G} t_g g\right), \ t_g = \sum_{g'g'' = g} r_{g'} s_{g''}.$$

For any multisubset X of G, Let  $\overline{X}$  denote the element of the group ring R[G] that is the sum of all elements of X, i.e.,

$$\overline{X} = \sum_{x \in X} \Delta_X(x) x.$$

The lemma below allows us to express a sufficient and necessary condition for a Cayley graph to be strongly regular in terms of group ring.

**Lemma 2.1.** The Cayley graph Cay(G, S) of G with respect to S is a DSRG with parameters  $(n, k, \mu, \lambda, t)$ if and only if |G| = n, |S| = k, and

$$\overline{S}^2 = te + \lambda \overline{S} + \mu (\overline{G} - e - \overline{S}).$$

A character  $\chi$  of a finite abelian group is a homomorphism from G to  $\mathbb{C}^*$ , the multiplicative group of  $\mathbb{C}$ . All characters of G form a group under the multiplication  $\chi\chi'(a) = \chi(a)\chi'(a)$  for any  $a \in G$ , which is denoted by  $\widehat{G}$  and it is called the *character group* of G. It is easy to see that  $\widehat{G}$  is isomorphic to G. Every character  $\chi \in \widehat{G}$  of G can be extended to a homomorphism from  $\mathbb{C}[G]$  to  $\mathbb{C}$  by

$$\chi(a) = \sum_{g \in G} a_g \chi(g), \text{ for } a = \sum_{g \in G} a_g g \in \mathbb{C}[G].$$

Let  $\omega$  be a fixed complex primitive *n*-th root of unity. Then  $\widehat{C}_n = \{\chi_j | j \in \mathbb{Z}_n\}$ , where  $\chi_j(x^i) = \omega^{ij}$  for  $0 \leq i, j \leq n-1$ .

### 2.2. Fourier Transformation

Throughout this section n will denote a fixed positive integer,  $\mathbb{Z}_n$  is the modulo n residue class ring. For a positive divisor r of n let  $r\mathbb{Z}_n$  be the subgroup of the additive group of  $\mathbb{Z}_n$  of order  $\frac{n}{r}$ .

The following statement and notations coincide with [15]. Let  $\mathbb{Z}_n^*$  be the multiplicative group of units in the ring  $\mathbb{Z}_n$ . Then  $\mathbb{Z}_n^*$  has an action on  $\mathbb{Z}_n$  by multiplication and hence  $\mathbb{Z}_n$  is a union of some  $\mathbb{Z}_n^*$ -orbits. Each  $\mathbb{Z}_n^*$ -orbit consists of all elements of a given order in the additive group  $\mathbb{Z}_n$ . If r is a positive divisor of n, we denote the  $\mathbb{Z}_n^*$ -orbit containing all elements of order r by  $\mathcal{O}_r$ . Thus

$$\mathcal{O}_r = \left\{ z \middle| z \in \mathbb{Z}_n, \ \frac{n}{(n,z)} = r \right\} = \left\{ c \frac{n}{r} \middle| 1 \leqslant c \leqslant r, (r,c) = 1 \right\}$$

and  $|\mathcal{O}_r| = \varphi(r)$ .

Let  $\zeta_n$  be a fixed primitive *n*-th root of unity and  $\mathbb{F} = \mathbb{Q}[\zeta_n]$  the *n*-th Cyclotomic Field over  $\mathbb{Q}$ . Further, let  $\mathbb{F}^{\mathbb{Z}_n}$  be the  $\mathbb{F}$ -vector space of all functions  $f : \mathbb{Z}_n \to \mathbb{F}$  mapping from the residue class ring  $\mathbb{Z}_n$  to the field  $\mathbb{F}$  (with the scalar multiplication and addition defined point-wise). The  $\mathbb{F}$ -algebra obtained from  $\mathbb{F}^{\mathbb{Z}_n}$  by defining the multiplication point-wise will be denoted by  $(\mathbb{F}^{\mathbb{Z}_n}, \cdot)$ . The  $\mathbb{F}$ -algebra obtained from  $\mathbb{F}^{\mathbb{Z}_n}$  by defining the multiplication as convolution will be denoted by  $(\mathbb{F}^{\mathbb{Z}_n}, *)$ , where the convolution is defined by:  $(f * g)(z) = \sum_{i \in \mathbb{Z}_n} f(i)g(z - i)$ .

We recall that  $cA = \{ca|a \in A\}, i + A = \{i + a|a \in A\}, i - A = \{i - a|a \in A\}, A + B = \{a + b|a \in A, b \in B\}$ . The Fourier transformation which is an isomorphism of  $\mathbb{F}$ -algebra  $(\mathbb{F}^{\mathbb{Z}_n}, *)$  and  $(\mathbb{F}^{\mathbb{Z}_n}, \cdot)$  is defined by

$$\mathcal{F}: (\mathbb{F}^{\mathbb{Z}_n}, *) \to (\mathbb{F}^{\mathbb{Z}_n}, \cdot), \quad (\mathcal{F}f)(z) = \sum_{i \in \mathbb{Z}_n} f(i)\zeta_n^{iz}.$$
(2.2)

Then,  $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$  for  $f, g \in \mathbb{F}^{\mathbb{Z}_n}$ . It also obeys the *inversion formula* 

$$\mathcal{F}(\mathcal{F}(f))(z) = nf(-z). \tag{2.3}$$

Let A and B be multisubsets of  $\mathbb{Z}_n$ , we have

$$\mathcal{F}\Delta_{(-A)} = \overline{\mathcal{F}\Delta_A}, \ \mathcal{F}\Delta_{A+B} = \mathcal{F}(\Delta_A * \Delta_B) = (\mathcal{F}\Delta_A)(\mathcal{F}\Delta_B).$$
(2.4)

Recall that  $\widehat{C}_n = \{\chi_j | j \in \mathbb{Z}_n\}$ , we also have

$$(\mathcal{F}\Delta_A)(z) = \chi_z(\overline{A}). \tag{2.5}$$

The Fourier transform of characteristic functions of additive subgroups in  $\mathbb{Z}_n$  can be easily computed. For a positive divisor r of n,

$$\mathcal{F}\Delta_{r\mathbb{Z}_n} = \frac{n}{r}\Delta_{\frac{n}{r}\mathbb{Z}_n}, \ \mathcal{F}\Delta_{\mathbb{Z}_n} = n\Delta_0, \ \text{and} \ \mathcal{F}\Delta_0 = \Delta_{\mathbb{Z}_n} = 1.$$
 (2.6)

The following lemma will be used in this paper.

**Lemma 2.2.** ([15])Suppose that  $f : \mathbb{Z}_n \to \mathbb{F}$  is a function such that  $\operatorname{Im}(f) \subseteq \mathbb{Q}$ . Then  $\operatorname{Im}(\mathcal{F}f) \subseteq \mathbb{Q}$  if and only if  $f = \sum_{r|n} \alpha_r \Delta_{\mathcal{O}_r}$  for some  $\alpha_r \in \mathbb{Q}$ .

The value of Fourier Transformation of the characteristic function of an orbit  $\mathcal{O}_r$  is also known as the Ramanujan's sum, i.e.,

$$(\mathcal{F}\Delta_{\mathcal{O}_r})(z) = \hat{\mu}\left(\frac{r}{(r,z)}\right) \frac{\varphi(r)}{\varphi\left(\frac{r}{(r,z)}\right)} \in \mathbb{Z}.$$
(2.7)

where  $\hat{\mu}$  is Möbius function.

Let r be a positive divisor of n, and H is a multisubset of  $\mathbb{Z}_v$ , then  $\psi_{n,v}^{-1}(H)$  is a multisubset of  $\mathbb{Z}_n$ . In other world,  $\psi_{n,v}^{-1}(H) = S + v\mathbb{Z}_n$  for some multisubset S of  $\{0, 1, \ldots, v-1\}$ . Let  $\Delta_H^{(v)}$  be multiplicity function of H in  $\mathbb{Z}_v$ , then for any  $z \in \mathbb{Z}_v$ ,

$$\left(\mathcal{F}\Delta_{H}^{(v)}\right)(z) = \frac{n}{v} \left(\mathcal{F}\Delta_{\psi_{n,v}^{-1}(H)}(\phi_{v}(z))\right).$$

$$(2.8)$$

## 3. Some lemmas

For  $z \in \mathbb{Z}_{p^{\alpha}}$ , define  $\nu_p(z)$  as the maximum power of the prime p that divides z. Note that all the  $\mathbb{Z}_{p^{\alpha}}^*$ -orbits are  $\mathcal{O}_1 = \mathcal{O}_{p^0}, \mathcal{O}_p, \mathcal{O}_{p^2}, \ldots$  and  $\mathcal{O}_{p^{\alpha}}$ , where

$$\mathcal{O}_{p^{z}} = \left\{ cp^{\alpha - z} | 1 \le c \le p^{z}, (p, c) = 1 \right\} = \left\{ i | i \in \mathbb{Z}_{p^{\alpha}}, \nu_{p}(i) = \alpha - z \right\}.$$

In particular,  $\mathcal{O}_1 = \{0\} = \{i | i \in \mathbb{Z}_{p^{\alpha}}, \nu_p(i) = \alpha\}$ . We denote  $\mathcal{O}_{p^i}$  with  $O_i$  for simplicity throughout this paper. Note that  $p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}$  is a subgroup of  $\mathbb{Z}_{p^{\alpha}}$  and

$$p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}} = \bigcup_{i=0}^{\beta} O_i.$$
(3.1)

for each  $0 \leq \beta \leq \alpha$ . Let  $\widehat{C_{p^{\alpha}}} = \{\chi_z | z \in \mathbb{Z}_{p^{\alpha}}\}$ , then the order of the character  $\chi_z$  in the character group  $\widehat{C_{p^{\alpha}}}$  is  $p^{\alpha-\nu_p(z)}$ .

In this section, let X be a subset of  $\mathbb{Z}_{p^{\alpha}}$ , let  $U_X = X \uplus (-X)$  and

$$\mathbf{q}(z) \stackrel{\text{def}}{=} (\mathcal{F}\Delta_{U_X})(z),$$

for  $z \in \mathbb{Z}_{p^{\alpha}}$ . For any  $\delta \in \mathbb{C}$ , define  $\Gamma_{\delta} = \{z \in \mathbb{Z}_{p^{\alpha}} | \mathbf{q}(z) = (\mathcal{F}\Delta_{U_X})(z) = \delta\}$ . We also define one condition of S by

$$0 \notin X, \ X \neq -X, \ \text{and} \ (\mathcal{F}\Delta_{X \uplus (-X)})(z) = (\mathcal{F}\Delta_X)(z) + \overline{(\mathcal{F}\Delta_X)(z)} \in \{0, -m\}, \ \forall \ 0 \neq z \in \mathbb{Z}_{p^{\alpha}},$$
(3.2)

where m is a positive integer. Therefore, if S satisfies the condition (3.2), we let  $\Gamma = \Gamma_{-m}$ , and we can obtain  $\mathbf{q}(z) \leq 0$  for any  $z \neq 0$ . Thus

$$\mathbf{q}(z) > 0 \text{ implies that } z = 0. \tag{3.3}$$

If for some  $z_1$  and  $z_2$ , we have

$$\mathbf{q}(z_1), \mathbf{q}(z_2) < 0, \text{ then } \mathbf{q}(z_1) = \mathbf{q}(z_2) = -m.$$
 (3.4)

**Lemma 3.1.** Let X be a subset of  $\mathbb{Z}_{p^{\alpha}}$  which satisfies 3.2. Then there are some integers  $1 \leq r_1 < r_2 < \ldots < r_s \leq \alpha$  and  $1 \leq r_{s+1} < r_{s+2} < \ldots < r_t \leq \alpha$  such that

$$U_X = (2 \otimes O_{r_1}) \uplus (2 \otimes O_{r_2}) \uplus \dots \uplus (2 \otimes O_{r_s}) \uplus (O_{r_{s+1}} \cup O_{r_{s+2}} \cup \dots \cup O_{r_t}), \tag{3.5}$$

*Proof.* Note that

$$\operatorname{Im}(\mathbf{q}) \in \mathbb{Q}$$

Therefore, we have

$$\Delta_{U_X} = \sum_{r|p^{\alpha}} \alpha_r \Delta_{\mathcal{O}_r} = \sum_{r=0}^{\alpha} \alpha_r \Delta_{O_r}$$

for some  $\alpha_r \in \{0, 1, 2\}$  by Lemma 2.1. Note that  $0 \notin X$ , so  $\alpha_0 = 0$ . This shows that there are some intergers  $1 \leq r_1 < r_2 < \ldots < r_s \leq \alpha$  and  $1 \leq r_{s+1} < r_{s+2} < \ldots < r_t \leq \alpha$  such that 3.5 holds.

Throughout this section, we let  $\mathcal{I}_1 = \{r_1, r_2, \dots, r_s\}$  and  $\mathcal{I}_2 = \{r_{s+1}, r_{s+2}, \dots, r_t\}$ , then  $\mathcal{I}_2 \neq \emptyset$  from  $X \neq -X$ . Thus  $|\mathcal{I}_1| = s$  and  $|\mathcal{I}_2| = t - s \ge 1$ .

**Lemma 3.2.** Let  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be sets defined above. Then  $\mathcal{I}_1$  and  $\mathcal{I}_2$  form a partition of  $\{\beta + 1, \beta + 2, ..., \alpha\}$ , where  $\beta = \min\{v_p(z) | z \in \Gamma\}$ . Moreover, we have

$$U_X = (O_{r_1} \cup O_{r_2} \cup \ldots \cup O_{r_s}) \uplus \left( \mathbb{Z}_{p^{\alpha}} \setminus p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}} \right).$$
(3.6)

*Proof.* By the inversion formula (2.3) of Fourier transformation, we have

$$p^{\alpha}\Delta_U(z) = (\mathcal{F}(\mathcal{F}\Delta_U))(-z) = -m\sum_{i\in\Gamma}\omega^{-iz} + |U|.$$

Note that  $0 \notin U$  and hence  $\Delta_U(0) = -m|\Gamma| + |U| = 0$ . Thus we can get that

$$z \in \mathbb{Z}_{p^{\alpha}} \setminus U \Leftrightarrow \Delta_{U}(z) = 0 \Leftrightarrow \Delta_{U}(z) - \Delta_{U}(0) = 0 \Leftrightarrow \left(\sum_{i \in \Gamma} \omega^{-iz} + |U|\right) - (|\Gamma| + |U|) = 0$$
$$\Leftrightarrow \sum_{i \in \Gamma} (1 - \omega^{-iz}) = 0 \Leftrightarrow \omega^{-iz} = 1, \forall i \in \Gamma \Leftrightarrow z \in \bigcap_{i \in \Gamma} \frac{p^{\alpha}}{(p^{\alpha}, i)} \mathbb{Z}_{p^{\alpha}}.$$

This gives that

$$\mathbb{Z}_{p^{\alpha}} \setminus U = \bigcap_{i \in \Gamma} \frac{p^{\alpha}}{(p^{\alpha}, i)} \mathbb{Z}_{p^{\alpha}} = p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}},$$

where  $\beta$  is defined as above. This completes the proof.

**Remark 3.1.** From the above lemma, we have  $t = \alpha - \beta$ .

We now prove the following lemma which asserts that  $\mathcal{I}_1 = \emptyset$  if p is a odd prime.

**Lemma 3.3.** Let  $X \subset \mathbb{Z}_{p^{\alpha}}$  which satisfies (3.2) and  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be sets defined above. If p be a odd prime, then  $\mathcal{I}_1 = \emptyset$  and

$$U_X = \mathbb{Z}_{p^{\alpha}} \setminus p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}}.$$
(3.7)

*Proof.* This lemma holds for  $\alpha = 1$  clearly. We assume that  $\alpha \ge 2$ . Suppose that  $\mathcal{I}_1 \ne \emptyset$ . Then  $s = |\mathcal{I}_1| > 0$ . Since  $X \ne (-X)$ , then  $\mathcal{I}_2 \ne \emptyset$  and hence  $\beta \le \alpha - 2$ . It follows from Lemma 3.2 and equations (2.6), (2.7) that

$$\mathbf{q}(z) = \sum_{i=1}^{s} (\mathcal{F}\Delta_{O_{r_i}})(z) + \mathcal{F}\Delta_{\mathbb{Z}_{p^{\alpha}}}(z) - (\mathcal{F}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}})(z)$$
$$= \sum_{i=1}^{s} \mu\left(\frac{p^{r_i}}{(p^{r_i}, z)}\right) \frac{\varphi(p^{r_i})}{\varphi\left(\frac{p^{r_i}}{(p^{r_i}, z)}\right)} + p^{\alpha}\Delta_0(z) - p^{\beta}(\Delta_{p^{\beta}\mathbb{Z}_{p^{\alpha}}})(z)$$

We now divide into two cases.

**Case 1**:  $s \ge 2$ . We assert that  $r_{i+1} = r_i + 1$  for  $1 \le i \le s - 1$ . Otherwise, there is an integer u with  $1 \le u \le s - 1$  and  $r_{u+1} > r_u + 1$ . Thus

$$\mathbf{q}(p^{r_u}) = \sum_{i=1}^u \mu\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_u})}\right) \frac{\varphi(p^{r_i})}{\varphi\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_u})}\right)} + p^{\alpha} \Delta_0(p^{r_u}) - p^{\beta} (\Delta_{p^{\beta} \mathbb{Z}_{p^{\alpha}}})(p^{r_u})$$
$$= \sum_{i=1}^u p^{r_i - 1}(p - 1) - p^{\beta} > p^{r_1 - 1}(p - 1) - p^{\beta} \ge p^{\beta}(p - 1) - p^{\beta} > 0,$$

contradicting (3.3). Thus  $\mathcal{I}_1 = \{r_1, r_2, \dots, r_s\} = \{\gamma, \gamma + 1, \dots, \gamma + s - 1\}$  for some  $\gamma \ge \beta + 1$ . Note that

$$\begin{aligned} \mathbf{q}(p^{\gamma+s-1}) &= \sum_{i=\gamma}^{\gamma+s-1} \mu\left(\frac{p^{i}}{(p^{i},p^{\gamma+s-1})}\right) \frac{\varphi(p^{i})}{\varphi\left(\frac{p^{i}}{(p^{i},p^{\gamma+s-1})}\right)} - p^{\beta} \\ &= \sum_{i=\gamma}^{\gamma+s-1} \varphi(p^{i}) - p^{\beta} = \sum_{i=\gamma}^{\gamma+s-1} p^{r_{i}-1}(p-1) - p^{\beta} \\ &= p^{\gamma+s-1} - p^{\gamma-1} - p^{\beta} = p^{\gamma-1}(p^{s}-1) - p^{\beta} \ge p^{\beta}(p^{s}-2) > 0, \end{aligned}$$

then  $p^{\gamma+s-1} = 0$  from (3.3) and hence  $\gamma + s - 1 = \alpha$ ,  $\mathcal{I}_1 = \{\alpha - s + 1, \alpha - s + 2, \dots, \alpha\}$  and  $\mathcal{I}_2 = \{\beta+1, \beta+2, \dots, \alpha-s\}$ . Note that  $\mathcal{I}_2 \neq \emptyset$  and hence  $\beta+1 \in \mathcal{I}_2$ , so  $\alpha-s+1 \ge \beta+2$ . If  $\alpha-s+1 > \beta+2$ , then

$$\mathbf{q}(p^{\beta+1}) = \mu\left(\frac{p^{\alpha-s+1}}{(p^{\alpha-s+1}, p^{\beta+1})}\right)\frac{\varphi(p^{\alpha-s+1})}{\varphi\left(\frac{p^{\alpha-s+1}}{(p^i, p^{\beta+1})}\right)} - p^{\beta} = -p^{\beta},$$

but

$$\mathbf{q}(p^{\alpha-s+1}) = -p^{\alpha-s} - p^{\beta} < -p^{\beta} = \mathbf{q}(p^{\beta+1}),$$

contradicting (3.4). This shows that  $\alpha - s + 1 = \beta + 2$ , so  $\mathcal{I}_1 = \{\beta + 2, \beta + 3, \dots, \alpha\}$  and  $\mathcal{I}_2 = \{\beta + 1\}$ . In this case, we can get  $\mathbf{q}(p^{\beta}) = -p^{\beta}$ , but note that

$$\mathbf{q}(p^{\beta+1}) = \mu\left(\frac{p^{\beta+2}}{(p^{\beta+2}, p^{\beta+1})}\right) \frac{\varphi(p^{\beta+2})}{\varphi\left(\frac{p^{\beta+2}}{(p^{\beta+2}, p^{\beta+1})}\right)} - p^{\beta} = -(1+p) \cdot p^{\beta} \neq \mathbf{q}(p^{\beta}),$$

contradicting (3.4).

**Case 2**: s = 1. Let  $\mathcal{I}_1 = \{\gamma\}$  with some  $\alpha \ge \gamma \ge \beta + 1$ . Note that

$$\mathbf{q}(p^{\gamma}) = p^{\gamma-1}(p-1) - p^{\beta} > 0$$

then  $q^{\gamma} = 0$ , then  $\mathcal{I}_1 = \{\alpha\}$ . But

$$\mathbf{q}(p^{\alpha-2}) = -p^{\beta} \neq \mathbf{q}(p^{\alpha-1}) = -p^{\beta} - p^{\alpha-1},$$

which also leads to a contradiction to (3.4).

Combining **Case 1** and **2**, we can obtain that  $\mathcal{I}_1 = \emptyset$  and

$$U_X = \bigcup_{i=\beta+1}^{\alpha} O_i = \mathbb{Z}_{p^{\alpha}} \setminus p^{\alpha-\beta} \mathbb{Z}_{p^{\alpha}}$$

from Lemma 3.2.

We now consider p = 2, and  $\alpha \ge 2$ , since we cannot find a subset X of  $\mathbb{Z}_2 \setminus \{0\} = \{1\}$  such that  $X \ne (-X)$ .

**Lemma 3.4.** Let  $X \subset \mathbb{Z}_{2^{\alpha}}$  satisfies condition (??) and  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be sets defined above. If  $\mathcal{I}_1 \neq \emptyset$ , then  $\mathcal{I}_1 = \{\beta + 1\}, \ \mathcal{I}_2 = \{\beta + 1, \beta + 2, \dots, \alpha\}$  and

$$U_X = O_{\beta+1} \uplus \left( \mathbb{Z}_{2^{\alpha}} \setminus 2^{\alpha-\beta} \mathbb{Z}_{2^{\alpha}} \right).$$
(3.8)

*Proof.* First, we show that  $\mathcal{I}_1$  is a singleton set. Otherwise, we assume  $s = |\mathcal{I}_1| \ge 2$ . Note that  $\mathcal{I}_2 \neq \emptyset$  and hence  $\beta \le \alpha - 3$ . From Lemma 3.2 and equations (2.6), (2.7), we have

$$\mathbf{q}(z) = \sum_{i=1}^{s} \mu\left(\frac{2^{r_i}}{(2^{r_i}, z)}\right) \frac{\varphi(2^{r_i})}{\varphi\left(\frac{2^{r_i}}{(2^{r_i}, z)}\right)} + 2^{\alpha} \Delta_0(z) - 2^{\beta} (\Delta_{2^{\beta} \mathbb{Z}_{2^{\alpha}}})(z)$$

We assert that  $\beta + 1 \in \mathcal{I}_1$ , i.e.,  $r_1 = \beta + 1$ . Otherwise we can assume  $r_1 \ge \beta + 2$  and hence  $\beta + 1 \in \mathcal{I}_2$ . If there is a u such that  $r_{u+1} > r_u + 1$  for some  $1 \le u \le s - 1$ , then

$$\mathbf{q}(2^{r_u}) = \sum_{i=1}^u 2^{r_i - 1} - 2^\beta \ge 2^{r_1 - 1} - 2^\beta \ge 2^{\beta + 1} - 2^\beta = 2^\beta > 0,$$

contradicting (3.3). So  $r_{i+1} = r_i + 1$  for  $1 \leq i \leq s-1$  and then  $\mathcal{I}_1 = \{\gamma, \gamma + 1, \dots, \gamma + s - 1\}$  for some  $\gamma \geq \beta + 2$ . Note that

$$\mathbf{q}(2^{\gamma+s-1}) = \sum_{i=\gamma}^{\gamma+s-1} \mu\left(\frac{2^{i}}{(2^{i},2^{\gamma+s-1})}\right) \frac{\varphi(2^{i})}{\varphi\left(\frac{2^{i}}{(2^{i},2^{\gamma+s-1})}\right)} - 2^{\beta}$$
$$= \sum_{i=\gamma}^{\gamma+s-1} \varphi(2^{i}) - 2^{\beta} = 2^{\gamma+s-1} - 2^{\gamma-1} - 2^{\beta} \ge 2^{\beta+1}(2^{s}-1) - 2^{\beta} > 0.$$

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Then from (3.3),  $2^{\gamma+s-1} = 0$ , so  $\gamma + s - 1 = \alpha$ ,  $\mathcal{I}_1 = \{\alpha - s + 1, \alpha - s + 2, \dots, \alpha\}$  and  $\mathcal{I}_2 = \{\beta + 1, \beta + 2, \dots, \alpha - s\}$ . Since  $\beta + 1 \in \mathcal{I}_2$ , we have  $\alpha - s + 1 \ge \beta + 2$ . If  $\alpha - s + 1 \ge \beta + 2$ , then

$$\mathbf{q}(2^{\beta+1}) = -2^{\beta} \neq \mathbf{q}(2^{\alpha-s+1}) = -2^{\alpha-s} - 2^{\beta},$$

contradicting (3.4) and hence  $\alpha - s + 1 = \beta + 2$ . So  $\mathcal{I}_1 = \{\beta + 2, \beta + 3, \dots, \alpha\}$  and  $\mathcal{I}_2 = \{\beta + 1\}$ . In this case,  $\mathbf{q}(2^{\beta}) = -2^{\beta}$ , but

$$\mathbf{q}(2^{\beta+1}) = \mu\left(\frac{2^{\beta+2}}{(2^{\beta+2}, 2^{\beta+1})}\right) \frac{\varphi(2^{\beta+2})}{\varphi\left(\frac{2^{\beta+2}}{(2^{\beta+2}, 2^{\beta+1})}\right)} - 2^{\beta} = -3 \cdot 2^{\beta} \neq \mathbf{q}(2^{\beta}),$$

contradicting (3.4). Therefore we have  $\beta + 1 \in \mathcal{I}_1$ .

**Case 1**. s = 2. In this case, we have

$$\mathcal{I}_1 = \{\beta + 1, \beta + \kappa\}, \ \mathcal{I}_2 = \{\beta + 2, \dots, \beta + \kappa - 1, \beta + \kappa + 1, \dots, \alpha\}.$$

for some  $\kappa$ . Note that

$$\mathbf{q}(2^{\beta+\kappa}) = 2^{\beta} + 2^{\beta+\kappa-1} - 2^{\beta} = 2^{\beta+\kappa-1} > 0,$$

then  $2^{\beta+\kappa} = 0$  by (3.3). Thus we have  $\beta + \kappa = \alpha$  and hence  $\mathcal{I}_1 = \{\beta + 1, \alpha\}$ . Therefore,  $\kappa = \alpha - \beta = t = s + (t-s) \ge 2 + 1 = 3$ . But  $\mathbf{q}(2^{\alpha-1}) = -2^{\alpha-1} \neq \mathbf{q}(2^{\beta}) = -2 \cdot 2^{\beta} = -2^{\beta+1}$ , which leads to a contradiction to (3.4).

**Case 2**. s > 2. Similar to the discussion of equation (??), we can also obtain

$$\mathcal{I}_1 = \{\beta + 1, \beta + \kappa, \dots, \beta + \kappa + s - 2\} \text{ and } \mathcal{I}_2 = \{\beta + 1, \dots, \alpha\} \setminus \mathcal{I}_1$$

for some  $\kappa \geq 2$ . Then we have

$$\mathbf{q}(2^{\beta+\kappa}) = 2^{\beta} + \varphi(2^{\beta+\kappa}) - 2^{\beta+\kappa} - 2^{\beta} = -2^{\beta+\kappa-1}.$$

Note that

$$\mathbf{q}(2^{\beta+\kappa+s-2}) = 2^{\beta} + \sum_{i=\beta+\kappa}^{\beta+\kappa+s-2} \varphi(2^{i}) - 2^{\beta} = 2^{\beta+\kappa-1}(2^{s-1}-1) > 0,$$

then  $2^{\beta+\kappa+s-2} = 0$  from (3.3), so  $\beta + \kappa + s - 2 = \alpha$  and  $\mathcal{I}_1 = \{\beta + 1, \beta + \kappa, \dots, \alpha\}$ . In this case,  $\mathcal{I}_2 = \{\beta + 2, \dots, \beta + \kappa - 1\} \neq \emptyset \Rightarrow \kappa > 2$ , but  $\mathbf{q}(2^{\beta}) = -2 \cdot 2^{\beta} = -2^{\beta+1} > -2^{\beta+\kappa-1}$ , this is impossible.

Combining **Case 1** and **2**, we can obtain that  $s = |\mathcal{I}_1| = 1$ . Let  $\mathcal{I}_1 = \{\gamma\}$  with some  $\alpha \leq \gamma \leq \beta + 1$ . If  $\mathcal{I}_1 = \{\alpha\}$ , then we have

$$\mathbf{q}(2^{\alpha-2}) = -2^{\beta} \neq \mathbf{q}(2^{\alpha-1}) = -2^{\beta} - 2^{\alpha-1}.$$

This is also a contradiction, so  $\gamma < \alpha$ , then

$$\mathbf{q}(2^{\gamma}) = 2^{\gamma - 1} - 2^{\beta} \leqslant 0 \Rightarrow \gamma - 1 \leqslant \beta, \gamma \leqslant \beta + 1.$$

This gives that  $\mathcal{I}_1 = \{\beta + 1\}$  and therefore

$$U_X = O_{\beta+1} \cup \left( \mathbb{Z}_{2^{\alpha}} \setminus 2^{\alpha-\beta} \mathbb{Z}_{2^{\alpha}} \right)$$

from Lemma 3.2.

**Lemma 3.5.** ([?], Proposition 1.2.12, Corollary 1.2.13.) Let p be a prime and let  $G = H \times P$  be an abelian group with a cyclic Sylow p-subgroup P of order  $p^s$ . Let  $P_i$  denote the unique subgroup of order  $p^i$  in G. If  $Y \in \mathbb{Z}[G]$  satisfies

$$\chi(Y) \equiv 0 \bmod p^a$$

for all character  $\chi$  of order divisible by  $p^{s-r}$ , where r is some fixed number  $r \leq s$ ,  $r \leq a$ , then there are elements  $X_0, X_1, \ldots, X_r, X$  in  $\mathbb{Z}[G]$  such that

$$Y = p^{a}X_{0} + p^{a-1}P_{1}X_{1} + \ldots + p^{a-r}P_{r}X_{r} + P_{r+1}X_{r}$$

If  $r = \min\{a, s\}$ , then

$$Y = p^{a}X_{0} + p^{a-1}P_{1}X_{1} + \ldots + p^{a-r}P_{r}X_{r}$$

Moreover, if Y has non-negative coefficients, then we can choose the  $X_i$  such that they have non-negative coefficients.

We now give another version of Lemma 3.5.

**Lemma 3.6.** Let p be a prime and X be a multisubset of  $\mathbb{Z}_{p^{\alpha}}$  such that the multiplicity of each element in X at most p-1, i.e.,  $\Delta_X(z) \leq p-1$  for all  $z \in X$ . If  $a \leq \alpha - 1$  and X satisfies

$$(\mathcal{F}\Delta_X)(z) \equiv 0 \mod p^a$$

for all z with  $\nu_p(z) \leq a$ , then there is a multiset  $S \subseteq \mathbb{Z}_{p^{\alpha}-a}$  such that

$$X = \psi_{\alpha-a}^{-1}(S),$$

*i.e.*, X is a collection of some cosets of the subgroup  $p^{\alpha-a}\mathbb{Z}_{p^{\alpha}}$  in  $\mathbb{Z}_{p^{\alpha}}$ .

*Proof.* We note that

$$\chi_z(\overline{x^X}) = (\mathcal{F}\Delta_X)(z).$$

Meawhile, the order of the character  $\chi_z$  is  $p^{\alpha-\nu_p(z)}$  and this order is divisible by  $p^{\alpha-a}$  for each z with  $\nu_p(z) \leq a$ . Thus the conditions given in this lemma can imply that  $\chi(\overline{x^X}) \equiv 0 \mod(p^a)$  for all characters  $\chi$  of order divisible by  $p^{\alpha-r}$ . From Lemma ?? with  $r = \min\{a, \alpha\} = a$ , there are elements  $X_0, X_1, \ldots, X_r$  with non-negative coefficients in  $\mathbb{Z}[C_{p^{\alpha}}]$  such that

$$\overline{x^X} = p^a X_0 + p^{a-1} \overline{P_1} X_1 + \ldots + p^{a-r} \overline{P_a} X_a = p^a X_0 + p^{a-1} \overline{P_1} X_1 + \ldots + p \overline{P_{a-1}} X_{a-1} + \overline{P_a} X_a.$$

It is clear that all the coefficients in  $p^{a-j}\overline{P_j}X_j$  are at least p provided  $X_j \neq 0$ , for  $0 \leq j \leq a-1$ . This gives that  $X_0 = X_1 = \ldots = X_{a-1} = 0$  since none of the coefficients in  $\overline{x^X}$  exceeds p-1. Note that  $X_a = \overline{x^S}$ for some multiset S of  $\mathbb{Z}_{p^{\alpha}}$  and hence  $\overline{x^X} = \overline{P_a}X_a = x^{p^{\alpha-a}\mathbb{Z}_{p^{\alpha}}}\overline{x^S} = x^{S+p^{\alpha-a}\mathbb{Z}_{p^{\alpha}}}$ , this claims that the multiset X is a collection of some cosets of the subgroup  $p^{\alpha-a}\mathbb{Z}_{p^{\alpha}}$  in  $\mathbb{Z}_{p^{\alpha}}$ . The result follows.

**Lemma 3.7.** Let X be a subset of  $\mathbb{Z}_{p^{\alpha}}$  and  $0 < a \leq \alpha$  be a positive integer. If X satisfies

$$(\mathcal{F}\Delta_X)(z) = 0$$

for all  $z \notin p^a \mathbb{Z}_{p^{\alpha}}$ , then there is a multiset  $S \subseteq \mathbb{Z}_{p^{\alpha}-a}$  such that

$$X = \psi_{\alpha-a}^{-1}(S).$$

*Proof.* For any  $z' \in p^{\alpha-a}\mathbb{Z}_{p^{\alpha}}$ , it follows from inverse formula 2.3 that

$$p^{\alpha}(\Delta_X(z+z') - \Delta_X(z))$$

$$= \sum_{i \in \mathbb{Z}_{p^{\alpha}}} (\mathcal{F}\Delta_X)(i)(\zeta_{p^{\alpha}}^{-i(z+z')} - \zeta_{p^{\alpha}}^{-iz})$$

$$= \left(\sum_{i \in p^{\alpha}\mathbb{Z}_{p^{\alpha}}} + \sum_{i \notin p^{\alpha}\mathbb{Z}_{p^{\alpha}}}\right) (\mathcal{F}\Delta_X)(i)(\zeta_{p^{\alpha}}^{-i(z+z')} - \zeta_{p^{\alpha}}^{-iz})$$

$$= 0.$$

This shows that X is a union of some cosets of  $p^{\alpha-a}\mathbb{Z}_{p^{\alpha}}$  in  $\mathbb{Z}_{p^{\alpha}}$ . This proves the lemma.

## 4. Directed strongly regular dihedrant

We now give a characterization of the dihedrant Dih(n, X, Y) to be directed strongly regular. Let  $\Delta_1 = \overline{x^X} + \overline{x^{-X}} = \overline{x^{U_X}}$  and  $\Delta_2 = \overline{x^Y} \overline{x^{-Y}} - \overline{x^X} \overline{x^{-X}}$ .

**Lemma 4.1.** The dihedrant Dih(n, X, Y) is a DSRG with parameters  $(2n, |X| + |Y|, \mu, \lambda, t)$  if and only if X and Y satisfy the following conditions:

(i) 
$$\overline{x^Y}\Delta_1 = (\lambda - \mu)\overline{x^Y} + \mu\overline{C_n};$$
 (4.1)

$$(ii) \ \overline{x^X} \Delta_1 + \Delta_2 = \overline{x^X}^2 + \overline{x^Y} \ \overline{x^{-Y}} = (t - \mu)e + (\lambda - \mu)\overline{x^X} + \mu \overline{C_n}.$$

$$(4.2)$$

*Proof.* Note that

$$\begin{aligned} (x^{\overline{X}} + \overline{x^{\overline{Y}}a})^2 &= \overline{x^{\overline{X}}} \ \overline{x^{\overline{X}}} + \overline{x^{\overline{X}}} \ \overline{x^{\overline{Y}}a} + \overline{x^{\overline{Y}}a} \ \overline{x^{\overline{X}}} + \overline{x^{\overline{Y}}a} \ \overline{x^{\overline{X}}} \\ &= \overline{x^{\overline{X}}} \ \overline{x^{\overline{X}}} + \overline{x^{\overline{Y}}} \ \overline{x^{-Y}} + (\overline{x^{\overline{X}}} \ \overline{x^{\overline{Y}}} + \overline{x^{\overline{Y}}} \ \overline{x^{-X}})a \\ &= \overline{x^{\overline{X}}} \Delta_1 + \Delta_2 + (\overline{x^{\overline{Y}}} \Delta_1)a. \end{aligned}$$

Thus, from Lemma 2.1, the dihedrant Dih(n, X, Y) is a DSRG with parameters  $(2n, |X| + |Y|, \mu, \lambda, t)$  if and only if

$$(\overline{x^X}\Delta_1 + \Delta_2) + (\overline{x^Y}\Delta_1)a = te + \lambda(\overline{x^X} + \overline{x^Y}a) + \mu(\overline{D_n} - (\overline{x^X} + \overline{x^Y}a) - e)$$
$$= (t - \mu)e + (\lambda - \mu)\overline{x^X} + \mu\overline{C_n} + ((\lambda - \mu)\overline{x^Y} + \mu\overline{C_n})a$$

This is equivalent to conditions (i) and (ii).

When Y = X, we have the following

**Lemma 4.2.** The dihedrant Dih(n, X, X) is a DSRG with parameters  $(2n, |X| + |Y|, \mu, \lambda, t)$  if and only if  $t = \mu$  and X satisfy the following conditions:

$$\overline{x^X}\Delta_1 = (\lambda - \mu)\overline{x^X} + \mu\overline{C_n}$$
(4.3)

We now define

$$\mathbf{r}(z) = (\mathcal{F}\Delta_X)(z) = \sum_{i \in X} \zeta_n^{iz} \text{ and } \mathbf{t}(z) = (\mathcal{F}\Delta_Y)(z) = \sum_{i \in Y} \zeta_n^{iz}.$$

Then  $\mathbf{r}(z) + \overline{\mathbf{r}}(z) = (\mathcal{F}\Delta_{U_X})(z)$ . The following lemma gives a characterization of the dihedrant Dih(n, X, Y) to be directed strongly regular by using  $\mathbf{r}(z)$  and  $\mathbf{t}(z)$ .

**Lemma 4.3.** The dihedrant Dih(n, X, Y) is a DSRG with parameters  $(2n, |X| + |Y|, \mu, \lambda, t)$  if and only if **r** and **t** satisfy the following conditions:

(i) 
$$\mathbf{t}(\mathbf{r} + \overline{\mathbf{r}}) = \mu n \Delta_0 + (\lambda - \mu) \mathbf{t};$$
 (4.4)

(*ii*) 
$$\mathbf{r}^2 + |\mathbf{t}|^2 = t - \mu + \mu n \Delta_0 + (\lambda - \mu) \mathbf{r}.$$
 (4.5)

*Proof.* The equations (4.1) and (4.2) given in Lemma 4.1 are equivalent to

$$(\Delta_Y * \Delta_{X \uplus (-X)})(i) = (\lambda - \mu) \Delta_Y(i) + \mu \Delta_{\mathbb{Z}_n}(i),$$

and  $(\Delta_X * \Delta_X)^2(i) + (\Delta_Y * \Delta_{-Y})(i) = (t - \mu)\Delta_0(i) + (\lambda - \mu)\Delta_X(i) + \mu\Delta_{\mathbb{Z}_n}(i)$  for any  $i \in \mathbb{Z}_n$ . Then by applying the Fourier transformation on these two equations, we can obtain that these two equations are equivalent to

$$\mathbf{t}(\mathbf{r}+\overline{\mathbf{r}}) = \mu n \Delta_0 + (\lambda - \mu) \mathbf{t}, \ \mathbf{r}^2 + |\mathbf{t}|^2 = t - \mu + \mu n \Delta_0 + (\lambda - \mu) \mathbf{r}.$$

Then the results follows.

When Y = X, we have the following results.

**Lemma 4.4.** The dihedrant Dih(n, X, X) is a DSRG with parameters  $(2n, |X| + |Y|, \mu, \lambda, t)$  if and only if  $t = \mu$  and the function **r** satisfies

$$\mathbf{r}(\mathbf{r} + \overline{\mathbf{r}}) = \mu n \Delta_0 + (\lambda - \mu) \mathbf{r}. \tag{4.6}$$

Recall in Section 3,  $\mathbf{q} \stackrel{\text{def}}{=} \mathbf{r} + \mathbf{\overline{r}} = \mathcal{F}\Delta_{U_X}$ . The next result will be needed quite often, as it gives a simple description of X if  $Dih(p^{\alpha}, X, X)$  is a directed strongly regular dihedrant.

**Lemma 4.5.** Let  $Dih(p^{\alpha}, X, X)$  be a directed strongly regular dihedrant with parameters  $(2p^{\alpha}, 2|X|, \mu, \lambda, t)$ . Then,

(a) The function  $\mathbf{q}$  satisfies

$$\mathbf{q}(z) = (\mathbf{r} + \overline{\mathbf{r}})(z) = (\mathcal{F}\Delta_{X \uplus (-X)})(z) \in \{0, \lambda - \mu\}$$
(4.7)

for any  $0 \neq z \in \mathbb{Z}_{p^{\alpha}}$ .

(b) There are some integers  $r_1, r_2, \cdots, r_s$  with  $\beta + 1 \leq r_1 < r_2 < \cdots < r_s \leq \alpha$  satisfying

$$U_X = (O_{r_1} \cup O_{r_2} \cup \dots \cup O_{r_s}) \uplus \left( \mathbb{Z}_{p^{\alpha}} \setminus p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}} \right),$$
(4.8)

for some  $\beta$  with  $0 \leq \beta \leq \alpha - 1$ .

(c) If p is an odd prime, then

$$U_X = \mathbb{Z}_{p^{\alpha}} \setminus p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}}.$$
(4.9)

(d) If p = 2 and  $X \cap (-X) \neq \emptyset$ , then

$$U_X = O_{\beta+1} \uplus \left( \mathbb{Z}_{2^{\alpha}} \setminus 2^{\alpha-\beta} \mathbb{Z}_{2^{\alpha}} \right).$$

$$(4.10)$$

Proof. We can get

$$\overline{\mathbf{r}}(\mathbf{r} + \overline{\mathbf{r}}) = \mu p^{\alpha} \Delta_0 + (\lambda - \mu) \overline{\mathbf{r}}$$
(4.11)

by taking conjugate on (4.6). Then the equations (4.6) and (4.11) gives

$$(\mathbf{r} + \overline{\mathbf{r}})^2 = 2\mu p^{\alpha} \Delta_0 + (\lambda - \mu)(\mathbf{r} + \overline{\mathbf{r}}),$$

and this implies

$$(\mathbf{r} + \overline{\mathbf{r}})(z) = \begin{cases} 2|X|, & z = 0; \\ 0 \text{ or } \lambda - \mu, & z \neq 0; \end{cases}$$

Then assertions (b), (c) and (d) follow from Remark 2.2, Lemmas 3.2, 3.3 and 3.4 respectively.

#### 5. Some constructions of directed strongly regular dihedrants

We now give some constructions of directed strongly regular dihedrants Dih(n; X, Y) with X = Yand  $X \subset Y$ . In the following constructions, v is a positive divisor of n and  $l = \frac{n}{v}$ .

The directed strongly regular dihedrant Dih(n, X, X) constructed in the following satisfies  $X \cap (-X) = \emptyset$ .

**Construction 5.1.** Let v be an odd positive divisor of n. Let H be a subset of  $\{1, \dots, v-1\} \subseteq \mathbb{Z}_n$ , and X be a subset of  $\mathbb{Z}_n$  satisfying the following conditions:

(i) 
$$X = H + v\mathbb{Z}_n$$
.

(*ii*) 
$$X \cup (-X) = \mathbb{Z}_n \setminus v\mathbb{Z}_n$$
.

Then Dih(n, X, X) is a DSRG with parameters  $\left(2n, n-l, \frac{n-l}{2}, \frac{n-l}{2}-l, \frac{n-l}{2}\right)$ .

*Proof.* Note that  $\overline{x^X}\Delta_1 = -l\overline{x^Y} + \frac{n-l}{2}\overline{C_n}$ . The result follows from Lemma ?? directly.

The directed strongly regular dihedrant Dih(n, X, X) constructed in the following satisfies  $X \cap (-X) \neq \emptyset$ .

**Construction 5.2.** Let v > 2 be an even positive divisor of n. Let H be a subset of  $\{1, \dots, v-1\} \subseteq \mathbb{Z}_n$ , and X be a subset of  $\mathbb{Z}_n$  satisfying the following conditions:

(i)  $X = H + v\mathbb{Z}_n$ . (ii)  $X \cup (-X) = (\mathbb{Z}_n \setminus v\mathbb{Z}_n) \uplus (\frac{v}{2} + v\mathbb{Z}_n)$ . (iii)  $X \cup (\frac{v}{2} + X) = \mathbb{Z}_n$ .

Then Dih(n, X, X) is a DSRG with parameters  $(2n, n, \frac{n}{2} + l, \frac{n}{2} - l, \frac{n}{2} + l)$ .

Proof. Note that  $|X| = \frac{n}{2}$  and  $\Delta_1 = \overline{C_n} - \overline{x^{v\mathbb{Z}_n}} + \overline{x^{\frac{n}{2}+v\mathbb{Z}_n}}$ . Thus  $\overline{x^X}\Delta_1 = -l\overline{x^X} + \frac{n}{2}\overline{C_n} + \overline{x^X} \ \overline{x^{\frac{n}{2}+v\mathbb{Z}_n}} = -l\overline{x^X} + \frac{n}{2}\overline{C_n} + l\overline{C_n} - l\overline{x^X} = (\frac{n}{2} + l)\overline{C_n} - 2l\overline{x^X}$ . The result follows from Lemma ?? directly.

**Remark 5.1.** For v = 2, the dihedrant Dih(n, X, X) is not a genuine DSRG which has parameters (2n, n, n, 0, n), we don't consider this case in this paper.

We now give some constructions of directed strongly regular dihedrants Dih(n; X, Y) with  $X \subset Y$ .

**Construction 5.3.** Let v be an odd positive divisor of n. Let H be a subset of  $\{0, 1, \dots, v-1\} \subseteq \mathbb{Z}_n$ with  $0 \in H$ , and  $X, Y \subseteq \mathbb{Z}_n$  satisfying the following conditions:

(i)  $Y = H + v\mathbb{Z}_n = X \cup \{0\}.$ 

 $(ii) Y \cup (-Y) = \mathbb{Z}_n \uplus v \mathbb{Z}_n.$ 

Then Dih(n; X, Y) is a DSRG with parameters  $(2n, n+l-1, \frac{n+l}{2}, \frac{n+3l}{2}-2, \frac{n+3l}{2}-1)$ .

*Proof.* We have  $|Y| = |X| + 1 = \frac{n+l}{2}$ ,  $\Delta_1 = \overline{C_n} + \overline{x^{v\mathbb{Z}_n}} - 2e$  and  $\Delta_2 = \Delta_1 + e = \overline{C_n} + \overline{x^{v\mathbb{Z}_n}} - e$ . Thus  $\overline{x^Y}\Delta_1 = (l-2)\overline{x^Y} + \frac{n+l}{2}\overline{C_n}$  and  $\overline{x^X}\Delta_1 + \Delta_2 = (l-1)e + (l-2)\overline{x^X} + \frac{n+l}{2}\overline{C_n}$ . The result follows from Lemma ?? directly.

**Construction 5.4.** Let v be an odd positive divisor of n. Let H be a subset of  $\{0, 1, \dots, v-1\} \subseteq \mathbb{Z}_n$ with  $0 \in H$ , and  $X, Y \subseteq \mathbb{Z}_n$  satisfying the following conditions: (i)  $Y = H + v\mathbb{Z}_n = X \cup v\mathbb{Z}_n$ .

(*ii*)  $Y \cup (-Y) = \mathbb{Z}_n \uplus v\mathbb{Z}_n$ .

Then Dih(n; X, Y) is a DSRG with parameters  $(2n, n, \frac{n+l}{2}, \frac{n-l}{2}, \frac{n+l}{2})$ .

*Proof.* We have  $|Y| = |X| + l = \frac{n+l}{2}$ ,  $\Delta_1 = \overline{C_n} - \overline{x^{v\mathbb{Z}_n}}$  and  $\Delta_2 = \overline{x^{v\mathbb{Z}_n}}\Delta_1 + l\overline{x^{v\mathbb{Z}_n}} = l\overline{C_n}$ . Thus  $\overline{x^Y}\Delta_1 = -l\overline{x^Y} + \frac{n+l}{2}\overline{C_n}$  and  $\overline{x^X}\Delta_1 + \Delta_2 = -l\overline{x^X} + \frac{n+l}{2}\overline{C_n}$ . The result follows from Lemma ?? directly.

**Remark 5.2.** The Lemma 2.4 in [15] asserts that the dihedrants Dih(n; X, Y) and Dih(n; bX, b' + bY) are isomorphic for any  $b \in \mathbb{Z}_n^*, b' \in \mathbb{Z}_n$ . Hence, indeed, we construct the directed strongly regular dihedrants Dih(n; bX, b' + bY) for any  $b \in \mathbb{Z}_n^*, b' \in \mathbb{Z}_n$ .

## 6. The characterization of directed strongly regular dihedrant $Dih(p^{\alpha}, X, X)$

We now can give a characterization of directed strongly regular dihedrant  $Dih(p^{\alpha}, X, X)$  with p > 2.

**Theorem 6.1.** For an odd prime p and a positive integer  $\alpha$ , then the dihedrant  $Dih(p^{\alpha}, X, X)$  is a DSRG if and only if  $X = \psi_{\gamma}^{-1}(H)$  for some  $1 \leq \gamma \leq \alpha$  and a subset  $H \subseteq \mathbb{Z}_{p^{\gamma}}$  satisfying the following conditions:

(i)  $H \cup (-H) = \mathbb{Z}_{p^{\gamma}} \setminus \{0\}.$ (ii)  $H \cap (-H) = \emptyset.$ 

*Proof.* It follows from Construction 5.1 that the dihedrant  $Dih(p^{\alpha}, X, X)$  with conditions (i) and (ii) is a DSRG (where  $n = p^{\alpha}$  and  $v = p^{\alpha-\beta}$ ). Conversely, suppose that the Cayley digraph  $Dih(p^{\alpha}, X, X)$  is a DSRG with parameters  $(2n, k = 2|X|, \mu, \lambda, t)$ . From Lemma 4.5 (c), we can get

$$X \uplus (-X) = \mathbb{Z}_{p^{\alpha}} \setminus p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}},$$

for some  $0 \leq \beta \leq \alpha$ , proving (*ii*). In this case,

$$\mathbf{r}(z) + \overline{\mathbf{r}}(z) = p^{\alpha} \Delta_0(z) - p^{\beta} \Delta_{p^{\beta} \mathbb{Z}_{p^{\alpha}}}(z)$$

and hence equation (4.6) becomes

$$p^{\alpha}\Delta_0(z)\mathbf{r}(z) - p^{\beta}\Delta_{p^{\beta}\mathbb{Z}_{p^{\alpha}}}(z)\mathbf{r}(z) = \mu n\Delta_0(z) + (\lambda - \mu)\mathbf{r}(z).$$

This implies that

$$\mathbf{r}(z) = 0, \ \forall z \notin p^{\beta} \mathbb{Z}_{p^{\alpha}}.$$

From lemma 3.7, there is a multiset  $H \subseteq \mathbb{Z}_{p^{\alpha-\beta}}$  such that

$$X = \psi_{\alpha-\beta}^{-1}(H)$$

Let  $\gamma = \alpha - \beta$ , then  $X \uplus (-X) = \mathbb{Z}_{p^{\alpha}} \setminus p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}}$  implies (i) and (ii).

We now focus on directed strongly regular dihedrant  $Dih(2^{\alpha}, X, X)$ . We now prove the non-existence of directed strongly regular dihedrant  $Dih(2^{\alpha}, X, X)$  with  $X \cap (-X) = \emptyset$  first.

**Lemma 6.2.** A DSRG cannot be a dihedrant  $Dih(2^{\alpha}, X, X)$  with  $X \cap (-X) = \emptyset$ .

*Proof.* Suppose  $X \cap (-X) = \emptyset$ . Then From Lemma 4.5 (b), we also have

$$X \uplus (-X) = O_{\beta+1} \cup O_{\beta+2} \cup \ldots \cup O_{\alpha} = \mathbb{Z}_{2^{\alpha}} \setminus 2^{\alpha-\beta} \mathbb{Z}_{2^{\alpha}},$$

Similar to the proof of Lemma 6.1, there is a multiset  $H \subseteq \mathbb{Z}_{2^{\alpha-\beta}}$  such that  $X = \psi_{\alpha-\beta}^{-1}(H)$ . This gives that  $2|X| = 2|H|2^{\beta} = 2^{\alpha} - 2^{\beta}$  and hence  $2|H| = 2^{\alpha-\beta} - 1$ , which is impossible.

**Theorem 6.3.** For a positive integer  $\alpha$ , the dihedrant  $Dih(2^{\alpha}, X, X)$  is a DSRG if and only if  $X = \psi_{\gamma}^{-1}(H)$  for some  $2 \leq \gamma \leq \alpha$  and a subset  $H \subseteq \mathbb{Z}_{p^{\gamma}}$  satisfying the following conditions: (i)  $H \uplus (-H) = (\mathbb{Z}_{2^{\gamma}} \setminus \{0\}) \uplus \{2^{\gamma-1}\}.$ (ii)  $H \cap (2^{\gamma-1} + H) = \mathbb{Z}_{2^{\gamma}}.$ 

*Proof.* It follows from Construction 5.2 that the dihedrant  $Dih(2^{\alpha}, X, X)$  with the conditions (i) and (ii) is a DSRG. Conversely, suppose that the dihedrant  $Dih(2^{\alpha}, X, X)$  is a DSRG with parameters  $(2n, k = 2|X|, \mu, \lambda, t)$ , then  $X \cap (-X) \neq \emptyset$ . Then from Lemma 4.5 (d) and equation (2.7), we can get

$$\begin{aligned} \mathbf{q}(z) &= (\mathbf{r} + \overline{\mathbf{r}})(z)(\mathcal{F}\Delta_{O_{\beta+1}})(z) + \sum_{i=\beta+1} (\mathcal{F}\Delta_{O_i})(z) \\ &= (\mathcal{F}\Delta_{O_{\beta+1}})(z) + (\mathcal{F}\Delta_{\mathbb{Z}_{2^{\alpha}}})(z) - (\mathcal{F}\Delta_{\mathbb{Z}_{2^{\alpha-\beta}2^{\alpha}}})(z) \\ &= \mu\left(\frac{2^{\beta+1}}{(2^{\beta+1},z)}\right) \frac{\varphi(2^{\beta+1})}{\varphi\left(\frac{2^{\beta+1}}{(2^{\beta+1},z)}\right)} + 2^{\alpha}\Delta_0(z) - 2^{\beta}\Delta_{2^{\beta}\mathbb{Z}_{2^{\alpha}}}(z) \in \{0,\lambda-\mu\},\end{aligned}$$

for some  $0 \leq \beta \leq \alpha - 1$ , and hence

$$k = \mathbf{q}(0) = 2^{\alpha}, \mathbf{q}(2^{\beta}) = \mu\left(\frac{2^{\beta+1}}{(2^{\beta+1}, 2^{\beta})}\right) \frac{\varphi(2^{\beta+1})}{\varphi\left(\frac{2^{\beta+1}}{(2^{\beta+1}, 2^{\beta})}\right)} - 2^{\beta} = -2^{\beta+1} = \lambda - \mu.$$

So from equation (2.1), we can get  $\mu = 2^{\alpha-1} + 2^{\beta}$ . Note that  $\mu < k$  and hence  $\beta \leq \alpha - 2$ . Thus from Lemma 4.2, equation (4.6) becomes

$$\mathbf{r}(\mathcal{F}\Delta_{O_{\beta+1}} + 2^{\alpha}\Delta_0 - 2^{\beta}\Delta_{2^{\beta}\mathbb{Z}_{2^{\alpha}}}) = \mu 2^{\alpha}\Delta_0 - 2^{\beta+1}\mathbf{r}.$$
(6.1)

Note that  $(\mathcal{F}\Delta_{O_{\beta+1}})(z) = \mu\left(\frac{2^{\beta+1}}{(2^{\beta+1},z)}\right) \frac{\varphi(2^{\beta+1})}{\varphi\left(\frac{2^{\beta+1}}{(2^{\beta+1},z^2)}\right)} = 0$  for  $z \notin 2^{\beta}\mathbb{Z}_{2^{\alpha}}$ . Then the above equation implies  $\mathbf{r}(z) = (\mathcal{F}\Delta_X)(z) = 0$  for  $z \notin 2^{\beta}\mathbb{Z}_{2^{\alpha}}$ . Thus from Lemma 3.7, there is a multiset  $H \subseteq \mathbb{Z}_{2^{\alpha-\beta}}$  such that  $X = \psi_{\alpha-\beta}^{-1}(H)$ . So the (2.8) and equation (6.1) implies that, for each  $z \in \mathbb{Z}_{2^{\alpha-\beta}}$ ,

$$(\mathcal{F}\Delta_H)(z)(\mathcal{F}\Delta_H(z) + \overline{\mathcal{F}\Delta_H(z)}) = 2^{\alpha-\beta}(1+2^{\alpha-\beta-1})\Delta_0(z) - 2(\mathcal{F}\Delta_H(z))$$
(6.2)

Let  $\gamma = \alpha - \beta$ , then  $\mathcal{F}\Delta_H(z) + \overline{\mathcal{F}\Delta_H(z)} \in \{0, -2\}$  for all  $0 \neq z \in \mathbb{Z}_{2^{\alpha-\beta}}$ . Hence, Lemma 3.4 gives that

$$H \uplus (-H) = O'_{\kappa+1} \cup \left(\mathbb{Z}_{2^{\gamma}} \setminus 2^{\gamma-\kappa}\mathbb{Z}_{2^{\gamma}}\right) \tag{6.3}$$

for some  $0 \leq \kappa \leq \gamma - 1$ . Thus we can get

$$\mathcal{F}\Delta_{H \uplus (-H)}(2^{\kappa}) = (\mathcal{F}\widetilde{\Delta}_{O'_{\kappa+1}}) + 2^{\gamma}\Delta_0(2^{\kappa}) - 2^{\kappa}\Delta_{2^{\kappa}\mathbb{Z}_{2^{\gamma}}}(2^{\kappa}) = -2^{\kappa+1} \in \{0, -2\},$$

which implies  $\kappa = 0$ . So  $H \uplus (-H) = (\mathbb{Z}_{2^{\gamma}} \uplus O'_1) \setminus \{0\} = (\mathbb{Z}_{2^{\gamma}} \uplus \{2^{\gamma-1}\}) \setminus \{0\}$ , proving (i). Note that  $|H| = 2^{\gamma-1}$  and (6.2) implies that

$$\overline{y^H}(\overline{y^{\mathbb{Z}_{2^{\gamma}}}} - e + y^{2^{\gamma-1}}) = (1 + 2^{\gamma-1})\overline{y^{\mathbb{Z}_{2^{\gamma}}}} - 2\overline{y^H}$$

which gives that  $\overline{y^H} + \overline{y^{2^{\gamma-1}+H}} = \overline{y^{\mathbb{Z}_{2^{\gamma}}}}$ , proving (*ii*).

## 7. Characterization of directed strongly regular dihedrant $Dih(p^{\alpha}, X, Y)$ with $X \subset Y$

Throughout this section, p is an odd prime. Let  $\mathbf{w} = \mathbf{r} - \mathbf{t}$ , then we have the following lemma. Recall that  $O_0, O_1, \ldots, O_{\alpha}$  are  $\mathbb{Z}_{p^{\alpha}}^*$ -orbits in  $\mathbb{Z}_{p^{\alpha}}$ .

**Lemma 7.1.** Suppose the dihedrant  $Dih(p^{\alpha}, X, Y)$  is a DSRG with  $X \subset Y$ . Then  $Im(\mathbf{w}) \subseteq \mathbb{R}$  if and only if  $Y \setminus X$  is a union of some  $\mathbb{Z}_{p^{\alpha}}^*$ -orbits. Forthermore, let  $Dih(p^{\alpha}, X, Y)$  be a DSRG with  $X \subset Y$  and  $Im(\mathbf{w}) \subseteq \mathbb{R}$ , then  $Im(\mathbf{w}) \subseteq \{\rho, \sigma\}$  and  $Y \setminus X = O_{r_1} \cup O_{r_1} \cup \ldots \cup O_{r_s}$  for some  $0 = r_1 < r_2 < \ldots < r_s \leq \alpha$ .

*Proof.* Let the dihedrant  $Dih(p^{\alpha}, X, Y)$  be a DSRG with  $X \subset Y$ . If  $Y \setminus X$  is a union of some  $\mathbb{Z}_{p^{\alpha}}^*$ orbits, then  $Im(\mathbf{w}) \subseteq \mathbb{R}$  clearly. If  $Im(\mathbf{w}) \subseteq \mathbb{R}$ , then from equations (4.4) and (4.5) in Lemma 4.3,  $\mathbf{w}$ satisfies

$$\mathbf{w}^2 = (t - \mu) + (\lambda - \mu)\mathbf{w}.$$

Note that the two eigenvalues  $\rho, \sigma$  of directed strongly regular dihedrant  $Dih(p^{\alpha}, X, Y)$  are two roots of the quadratic equation  $x^2 = (t - \mu) + (\lambda - \mu)x$ , so we can get  $Im(\mathbf{w}) \in \{\rho, \sigma\} \subseteq \mathbb{Z}$ . Thus, from Lemma 2.2, we have  $\Delta_X - \Delta_Y = \sum_{r=0}^{\alpha} \alpha_r \Delta_{O_r}$  for some  $\alpha_r \in \{0, -1\}$  and  $a_0 = -1$ . This implies that  $Y \setminus X = O_{r_1} \cup O_{r_1} \cup \ldots \cup O_{r_s}$  for some  $0 = r_1 < r_2 < \ldots < r_s \leq \alpha$ .

The following theorem characterize directed strongly regular dihedrant  $Dih(p^{\alpha}, X, Y)$  with  $X \subset Y$ and  $Y \setminus X$  is a union of some  $\mathbb{Z}_{p^{\alpha}}^*$ -orbits.

**Theorem 7.2.** The dihedrant  $Dih(p^{\alpha}, X, Y)$  is a DSRG with  $X \subset Y$  and  $Y \setminus X$  is a union of some  $\mathbb{Z}_{p^{\alpha}}^{*}$ -orbits if and only if  $Y = \psi_{\gamma}^{-1}(H)$ ,  $X = Y \setminus \{0\}$  or  $X = Y \setminus p^{\gamma} \mathbb{Z}_{p^{\alpha}}$  for some  $1 \leq \gamma \leq \alpha$  and a subset  $H \subseteq \mathbb{Z}_{p^{\gamma}}$  such that  $H \uplus (-H) = \mathbb{Z}_{p^{\gamma}} \uplus \{0\}$ .

*Proof.* It follows from Construction 5.3 that the dihedrant  $Dih(p^{\alpha}, X, Y)$  with condition (a) is a DSRG (where  $n = p^{\alpha}$  and  $v = p^{\gamma}$ ), and from Construction 5.4 that the dihedrant  $Dih(p^{\alpha}, X, Y)$  with condition (b) is a DSRG (where  $n = p^{\alpha}$  and  $v = p^{\beta}$ ).

Now suppose the dihedrant  $D_i(p^{\alpha}, X, Y)$  is a DSRG with  $X \subset Y$  and  $Y \setminus X$  is a union of some  $\mathbb{Z}_{p^{\alpha}}^*$ orbits, from Lemma 7.1, we have  $\operatorname{Im}(\mathbf{w}) \in \{\rho, \sigma\}$  and  $Y \setminus X = O_{r_1} \cup O_{r_2} \cup \ldots \cup O_{r_s}$ . So  $\mathbf{w} = -\sum_{i=1}^s \mathcal{F}\Delta_{O_i}$ .

We assert  $\{0 = r_1, r_2, \dots, r_s\} = \{0, 1, \dots, s-1\}$ . Assume s > 1 since this assertion hold for s = 1 trivially. In fact, if there is a integer u such that  $r_{u+1} > r_u + 1$  for some  $1 \le u \le s-1$ , then

$$\mathbf{w}(p^{r_u}) = -\sum_{i=1}^u \mu\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_u})}\right) \frac{\varphi(p^{r_i})}{\varphi\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_u})}\right)} = -\sum_{i=1}^u \varphi(p^{r_i}) < 0.$$

but

$$\mathbf{w}(p^{r_s}) = -\sum_{i=1}^s \mu\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_s})}\right) \frac{\varphi(p^{r_i})}{\varphi\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_s})}\right)} = -\sum_{i=1}^s \varphi(p^{r_i}) < -\sum_{i=1}^u \varphi(p^{r_i}) = \mathbf{w}(p^{r_u}) < 0,$$

a contradiction. This shows that  $r_{u+1} = r_u + 1$  for each  $1 \leq u \leq s - 1$ . Then

$$Y \setminus X = \bigcup_{i=0}^{s-1} O_i = p^{\beta} \mathbb{Z}_{p^{\alpha}},$$

where  $\beta = \alpha - s + 1$ . Therefore  $\mathbf{r} - \mathbf{t} = \mathbf{w} = -\mathcal{F}(\Delta_{p^{\beta}\mathbb{Z}_{p^{\alpha}}}) = -p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}}$  and so  $\mathbf{r} = \mathbf{t} - p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}}$ . Then from Lemma 4.3, equations (4.4) and (4.5) become

$$\mathbf{t}^{2} + |\mathbf{t}|^{2} - 2p^{\alpha-\beta}\mathbf{t}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}} = \mu p^{\alpha}\Delta_{0} + (\lambda-\mu)\mathbf{t},$$
(7.1)

$$(\mathbf{t} - p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}})^2 + |\mathbf{t}|^2 = t - \mu + \mu p^{\alpha}\Delta_0 + (\lambda - \mu)(\mathbf{t} - p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}}),$$
(7.2)

The difference of these two equations gives

$$p^{2(\alpha-\beta)}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}} = t - \mu + (\mu - \lambda)p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}}.$$
(7.3)

**Case 1**:  $\beta = \alpha$ . In this case,  $Y = X \cup \{0\}$ , hence  $t - \lambda = 1$ . Since the dihedrant  $Dih(p^{\alpha}, X, Y)$  is a DSRG with parameters  $(2n, |X| + |Y|, \mu, \lambda, t)$ , its complement  $Dih(p^{\alpha}, \mathbb{Z}_n \setminus X \setminus \{0\}, \mathbb{Z}_n \setminus Y)$  is also a DSRG. Note that  $\mathbb{Z}_n \setminus X \setminus \{0\} = \mathbb{Z}_n \setminus Y$  since  $Y = X \cup \{0\}$ . So from Theorem 6.1, we can get  $Y^c = \mathbb{Z}_{p^{\alpha}} \setminus Y = \psi_{\gamma}^{-1}(H')$  for some  $1 \leq \gamma \leq \alpha$  and a subset  $H' \subseteq \mathbb{Z}_{p^{\gamma}}$  such that  $H \uplus (-H) = \mathbb{Z}_{p^{\gamma}} \setminus \{0\}$ . This implies that  $Y = \psi_{\gamma}^{-1}(H)$ , where  $0 \in H = \mathbb{Z}_{p^{\gamma}} \setminus H'$ , and  $H \uplus (-H) = \mathbb{Z}_{p^{\gamma}} \uplus \{0\}$ .

**Case 2**:  $0 < \beta < \alpha$ . In this case, we have  $\mu - \lambda = p^{\alpha - \beta}$  and  $t = \mu$ . Thus (7.1) becomes

$$\mathbf{t}^{2} + |\mathbf{t}|^{2} = \mathbf{t}(\mathbf{t} + \overline{\mathbf{t}}) = \mu p^{\alpha} \Delta_{0} + p^{\alpha - \beta} (2\Delta_{p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}}} - 1)\mathbf{t}.$$
 (7.4)

Hence, we get

$$(\mathbf{t} + \overline{\mathbf{t}})^2 = 2\mu p^{\alpha} \Delta_0 + p^{\alpha - \beta} (2\Delta_{p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}}} - 1)(\mathbf{t} + \overline{\mathbf{t}}),$$

which implies

$$(\mathbf{t} + \overline{\mathbf{t}})(z) = \begin{cases} 0 \text{ or } p^{\alpha - \beta}, & z \in p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}} \setminus \{0\}; \\ 0 \text{ or } - p^{\alpha - \beta}, & z \notin p^{\alpha - \beta} \mathbb{Z}_{p^{\alpha}}. \end{cases}$$
(7.5)

Thus we have

$$(\mathbf{t} + \overline{\mathbf{t}})(z) = (\mathcal{F}\Delta_{Y \uplus (-Y)})(z) \equiv 0 \mod p^{\alpha - \beta}$$

for each  $z \neq 0$ . Therefore, from Lemma 3.8,

$$Y \uplus (-Y) = \psi_{\gamma}^{-1}(H'')$$

for some  $1 \leq \beta \leq \alpha$  and a subset  $H'' \subseteq \mathbb{Z}_{p^{\beta}}$ . We can also write  $Y \uplus (-Y) = S + p^{\beta} \mathbb{Z}_{p^{\alpha}}$  for some  $S \subseteq \{0, 1, \dots, p^{\beta} - 1\}$ , then from equation (2.4), we have

$$(\mathbf{t} + \overline{\mathbf{t}})(z) = p^{\alpha - \beta}(\mathcal{F}\Delta_S)(z)\Delta_{p^{\alpha - \beta}\mathbb{Z}_{p^{\alpha}}}(z).$$
(7.6)

Combining equations (7.4) and (7.6), we have

$$p^{\alpha-\beta}\mathbf{t}(\mathcal{F}\Delta_S)\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}} = \mu p^{\alpha}\Delta_0 + p^{\alpha-\beta}(2\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^{\alpha}}} - 1)\mathbf{t}.$$
(7.7)

Therefore (7.7) implies that

$$(\mathcal{F}\Delta_Y)(z) = \mathbf{t}(z) = 0.$$

for any  $z \notin p^{\alpha-\beta} \mathbb{Z}_{p^{\alpha}}$ . So from Lemma 3.8,

$$Y = \psi_{\beta}^{-1}(H)$$

for some subset  $H \subseteq \mathbb{Z}_{p^{\beta}}$ . Thus, (2.8), and (7.4) implies that for each  $z \in \mathbb{Z}_{2^{\beta}}$ ,

$$\mathcal{F}\Delta_H(z)(\mathcal{F}\Delta_H(z) + \overline{\mathcal{F}\Delta_H(z)}) = \mu p^{2\beta - \alpha} \Delta_0(z) + \mathcal{F}\Delta_H(z)$$

hold for all  $z \in \mathbb{Z}_{p^{\beta}}$ . Then  $(\mathcal{F}\Delta_H(z) + \overline{\mathcal{F}\Delta_H(z)}) \in \{0,1\}$  for all  $0 \neq z \in \mathbb{Z}_{p^{\beta}}$ . We assert that  $H \uplus (-H) = \mathbb{Z}_{p^{\beta}} \uplus \{0\}$ . Let  $H_1 = \mathbb{Z}_{p^{\alpha}} \setminus H$ , then

$$(\mathcal{F}\Delta_{H_1}(z) + \overline{\mathcal{F}\Delta_{H_1}(z)}) \in \{0, -1\}.$$

Let  $O'_1, O'_1, \ldots, O'_{\beta}$  are  $\mathbb{Z}^*_{p^{\beta}}$ -orbits in  $\mathbb{Z}_{p^{\beta}}$ . It follows from Lemma 3.3 that

$$H_1 \uplus (-H_1) = \mathbb{Z}_{p^\beta} \setminus p^{\beta - \kappa} \mathbb{Z}_{p^\beta}.$$
(7.8)

for some  $0 \leq \kappa \leq \beta - 1$ . Thus

$$(\mathcal{F}\Delta_{H_1 \uplus (-H_1)})(p^{\kappa}) = -p^{\kappa} = -1,$$

which implies that  $\kappa = 0$ . So  $H_1 \uplus (-H_1) = \mathbb{Z}_{p^\beta} \setminus \{0\}$  and hence  $H \uplus (-H) = \mathbb{Z}_{p^\beta} \uplus \{0\}$ . This completes the proof.

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