On even-cycle-free subgraphs of the doubled Johnson graphs

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Abstract

The generalized Turán number $\mathrm{ex}(G, H)$ is the maximum number of edges in an H-free subgraph of a graph G. It is an important extension of the classical Turán number $\mathrm{ex}(n, H)$, which is the maximum number of edges in a graph with n vertices that does not contain H as a subgraph. In this paper, we consider the maximum number of edges in an even-cycle-free subgraph of the doubled Johnson graphs $J(n; k, k+1)$, which are bipartite subgraphs of hypercube graphs. We give an upper bound for $\mathrm{ex}(J(n; k, k+1), C_{2r})$ with any fixed $k \in \mathbb{Z}^+$ and any $n \in \mathbb{Z}^+$ with $n \geq 2k+1$. We also give an upper bound for $\mathrm{ex}(J(2k+1; k, k+1), C_{2r})$ with any $k \in \mathbb{Z}^+$, where $J(2k+1; k, k+1)$ is known as doubled Odd graph $\widetilde{O}_{k+1}$. This bound induces that the number of edges in any C_{2r}-free subgraph of $\widetilde{O}_{k+1}$ is $o(e(\widetilde{O}_{k+1}))$ for $r \geq 6$, which also implies a Ramsey-type result.

Key words Turán number, even-cycle-free subgraph, doubled Johnson graph, doubled Odd graph, Ramsey-type problem

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1 Introduction

Throughout this paper, all graphs are finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $v(G)$ and $e(G)$ to denote the number of vertices and the number of edges in G, respectively. For any two distinct vertices $x, y \in V(G)$, a *path* of length r from x to y in G is a finite sequence of $r+1$ distinct vertices $(x = w_0, w_1, \ldots, w_r = y)$ such that $\{w_{t-1}, w_t\} \in E(G)$ for $t = 1, 2, \ldots, r$. If there is a path between any two vertices of a graph G, then G is *connected*. A cycle is a connected graph where any vertex in the graph has exactly two neighbours. A *cycle* is called to be an *l-cycle* or a *cycle of length l* if the number of edges in the cycle is l, denoted by C_l. The phrase "a cycle in a graph G" refers to a subgraph of G which is a cycle. Two graphs G and G' are *isomorphic* if there is a bijection σ from $V(G)$ to $V(G')$ such that $\{x, y\} \in E(G)$ if and only if $\{\sigma(x), \sigma(y)\} \in E(G')$.

Let G and H be two graphs. We call that G is *H-free* if there does not exist a subgraph of G which is isomorphic to H. The *generalized Turán number* $\mathrm{ex}(G, H)$ is the maximum number of edges in a H-free subgraph of G. When $G = K_n$ is the complete graph of n vertices, $\mathrm{ex}(G, H)$ is usually denoted by $\mathrm{ex}(n, H)$, specifying the maximum possible number of edges in an H-free graph on n vertices. There are a huge amount of literatures investigating this function, starting with the theorems of Mantel [17] and Turán [19] that determine it for $H = K_r$. It is showed in [12] that $\mathrm{ex}(n, H)$ is related to the chromatic

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number of H. But when H is bipartite one can only deduce that $\mathrm{ex}(n,H) = o(n^2)$. In general, it is also a major open problem to determine the generalized Turán number $\mathrm{ex}(G,H)$ when H is a bipartite graph, especially for even cycles. In this aspect, there are two widely studied functions $\mathrm{ex}(K_{m,n}, K_{s,t})$ and $\mathrm{ex}(Q_n, C_{2l})$, where $K_{m,n}$ is a complete bipartite graph and Q_n is a hypercube graph.

The former function $\mathrm{ex}(K_{m,n}, K_{s,t})$, known as the *problem of Zarankiewicz* raised in 1951 ([20]), is the analogue of Turán's problem in bipartite graphs. We refer the reader to [15] for the details about this problem. The latter function $\mathrm{ex}(Q_n, C_{2l})$, started with a problem raised by Erdős, which is "How many edges can a subgraph of Q_n have that contains no 4-cycles?" In [9], Erdős conjectured that the upper bound would be $(\frac{1}{2} + o(1))e(Q_n)$, and also asked whether $o(e(Q_n))$ edges of Q_n would ensure the existence of a cycle C_{2l} for $l \geq 3$. The best upper bound for $\mathrm{ex}(Q_n, C_4)$ is obtained by Balogn et al. ([3]), which is $(0.6068+o(1))e(Q_n)$, slightly improving the upper bounds given by Chung ([6]) and Thomason Wagner ([18]). The problem of deciding the values of C_6 and C_{10} is still open. In [6], Chung showed that $\frac{1}{4}e(Q_n) \leq \mathrm{ex}(Q_n, C_6) \leq (\sqrt{2} - 1 + o(1))e(Q_n)$, and negatively answered the question of Erdős for C_6. Conder ([7]) found a 3-colouring with the same property. This implies that $\mathrm{ex}(Q_n, C_6) \geq \frac{1}{3}e(Q_n)$. The best upper bound is given by Balogn et al. ([3]). For some progress about $\mathrm{ex}(Q_n, C_{10})$, we refer the reader to [1, 2]. For $l \geq 2$, the upper bounds for $\mathrm{ex}(Q_n, C_{4l})$ and $\mathrm{ex}(Q_n, C_{4l+6})$ were obtained by Chung ([6]) and Füredi and Özkahya ([14]), respectively, which imply that $\mathrm{ex}(Q_n, C_{2l'}) = o(e(Q_n))$ for $l' \geq 6$ or $l' = 4$. In [8], Conlon unified these results by showing $\mathrm{ex}(Q_n, H) = o(e(Q_n))$ for all H that admit a k-partite representation, which holds for each $H = C_{2l}$ except $l \in \{2, 3, 5\}$.

Now we consider another noteworthy family of bipartite graphs, which are called doubled Johnson graphs. Let n and k be two positive integers with $n \geq k+1$. Let $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{k}$ be the set of all k-subsets of $[n]$. The *doubled Johnson graph* $J(n; k, k+1)$ is a bipartite graph with vertex set $\binom{[n]}{k} \cup \binom{[n]}{k+1}$, where two distinct vertices u and v are adjacent if and only if $u \subset v$ or $v \subset u$. Recall that doubled Johnson graphs with $n = 2k+1$ are usually called *doubled Odd graphs*, which are distance-transitive graphs ([5]). We usually use $\widetilde{O}_{k+1}$ to denote the doubled Odd graph $J(2k+1; k, k+1)$. Notice that $J(n; k, k+1)$ is a subgraph of the hypercube Q_n, and the halved graphs of $J(n; k, k+1)$ are the Johnson graphs $J(n, k)$ and $J(n, k+1)$. By the definition, in the graph $J(n; k, k+1)$, the degree of each vertex in $\binom{[n]}{k}$ is $n-k$ and the degree of each vertex in $\binom{[n]}{k+1}$ is $k+1$. Therefore, $e(J(n; k, k+1)) = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$. Since the graphs $J(n; k, k+1)$ and $J(n; n-k-1, n-k)$ are isomorphic, in the following, we only consider the case when $n \geq 2k+1$.

In this paper, we study the generalized Turán number $\mathrm{ex}(J(n; k, k+1), C_{2l})$. For each vertex x_2 in $\binom{[n]}{k+1}$, choose an edge which is incident with x_2. Let E be the set of those edges and K be the graph with vertex set $\binom{[n]}{k} \cup \binom{[n]}{k+1}$ and edge set E. Notice that the degree of each vertex from $\binom{[n]}{k+1}$ in K is 1, which implies that K is cycle-free. Hence we have $\mathrm{ex}(J(n; k, k+1), C_{2l}) \geq \binom{n}{k+1} = \frac{1}{k+1}e(J(n; k, k+1))$. In the following, we consider the upper bound of $\mathrm{ex}(J(n; k, k+1), C_{2l})$ and obtain the following theorems.

Theorem 1.1 *Let k and l be any fixed positive integers. For any $n \in \mathbb{Z}^+$ with $n \geq 2k+1$, the following hold.*

(i) *For $l \geq 2$, there exists constant c_l such that*

$$\mathrm{ex}(J(n; k, k+1), C_{4l}) \leq \left(c_l (n-k)^{-\frac{1}{2}+\frac{1}{2l}} + \frac{1}{\sqrt{k+1}}\right) e(J(n; k, k+1)).$$

(ii) *For $l \geq 1$, we have*

$$\mathrm{ex}(J(n; k, k+1), C_{4l+2}) \leq \left(\frac{1}{2(k+1)} + \frac{\sqrt{2}}{2} + o(1)\right) e(J(n; k, k+1)),$$

where $o(1)$ is a function $f_k(n)$ of the variable n such that $\lim_{n\to+\infty} f_k(n) = 0$.

Theorem 1.2 *Let l be a any fixed positive integer. For any $k \in \mathbb{Z}^+$, the following hold.*

(i) *For $l \geq 3$, we have* $\mathrm{ex}(\widetilde{O}_{k+1}, C_{4l}) = O(k^{-\frac{1}{2}+\frac{1}{l}})e(\widetilde{O}_{k+1})$.

(ii) *For $l \geq 3$, we have*

$$\mathrm{ex}(\widetilde{O}_{k+1}, C_{4l+2}) = \begin{cases} O(k^{-\frac{1}{2l+1}})e(\widetilde{O}_{k+1}), & \text{if } l = 3,5,7,9, \\ O(k^{-\frac{1}{16}+\frac{1}{8(l-1)}})e(\widetilde{O}_{k+1}), & \text{otherwise}. \end{cases}$$

(iii) $\mathrm{ex}(\widetilde{O}_{k+1}, C_6) \leq \frac{5}{6}e(\widetilde{O}_{k+1})$; $\mathrm{ex}(\widetilde{O}_{k+1}, C_8) \leq (\frac{2}{3} + o(1))e(\widetilde{O}_{k+1})$; $\mathrm{ex}(\widetilde{O}_{k+1}, C_{10}) \leq (\frac{2}{3} + o(1))e(\widetilde{O}_{k+1})$.

From Theorem 1.2, we have $\mathrm{ex}(\widetilde{O}_{k+1}, C_{2l}) = o(e(\widetilde{O}_{k+1}))$ for $l \geq 6$, which leads to the following Ramsey-type result:

Theorem 1.3 *Let t and l be positive integers with $l \geq 6$. If $\widetilde{O}_{k+1}$ is edge-partitioned into t subgraphs, then one of the subgraphs must contain the even cycle C_{2l}, provided that k is sufficiently large (depending only on t and l).*

This paper is organized as follows. In Section 2, we introduce some properties of the doubled Johnson graphs. In Section 3, we give an upper bound for $\mathrm{ex}(\widetilde{O}_{k+1}, C_{2l})$ with $l = 3, 4, 5$. In Section 4, we give an upper bound for the number of edges in C_{4l}-free subgraphs of $J(n; k, k+1)$ with $l \geq 2$. In Section 5, we give an upper bound for the number of edges in C_{4l+2}-free subgraphs of $J(n; k, k+1)$ with $l \geq 1$.

2 Preliminary

In this section, we will give some important properties of the doubled Johnson graphs. It is obvious that each cycle in $J(n; k, k+1)$ has even length since it is a bipartite graph.

Suppose Γ is a graph. For any $x \in V(\Gamma)$, let $N_\Gamma(x)$ and $d_\Gamma(x)$ denote the set of neighbours of x and the degree of x in Γ, respectively. For any two vertices $x, y \in V(\Gamma)$, let $\partial_\Gamma(x, y)$ denote the distance between x and y in Γ. Given a doubled Johnson graph $J(n; k, k+1)$, in the following, we usually use V_1 and V_2 to denote the set $\binom{[n]}{k}$ and $\binom{[n]}{k+1}$, respectively, which are two parts of this bipartite graph. Set $v_1 := |V_1|$ and $v_2 := |V_2|$. Observe that $v_1 = \binom{n}{k}$ and $v_2 = \binom{n}{k+1}$, and $v_1 = v_2 = \binom{2k+1}{k}$ if $n = 2k+1$. For any two vertices x and y in $J(n; k, k+1)$, from [16], we have $\partial_\Gamma(x, y) = |x| + |y| - 2|x \cap y|$.

Proposition 2.1 *Let $U = (u_0, u_1, \ldots, u_i)$ be any path in $J(n; k, k+1)$. The following hold.*

(i) *If $i = 3$, there exists a unique cycle of length 6 containing U in $J(n; k, k+1)$.*

(ii) *If $i = 2$ and $u_2 \in V_1$, there exist $n-k-1$ cycles of length 6 containing U in $J(n; k, k+1)$.*

(iii) *If $i = 2$ and $u_2 \in V_2$, there exist k cycles of length 6 containing U in $J(n; k, k+1)$.*

(iv) *If $i = 1$, there exist $k(n-k-1)$ cycles of length 6 containing U in $J(n; k, k+1)$.*

Proof. (i) If $i = 3$, then $u_0 \in V_1$ or $u_3 \in V_1$. Without loss of generality, suppose $u_0 \in V_1$, $u_0 \cap u_2 = F$, $u_0 = F \cup \{x\}$ and $u_2 = F \cup \{y\}$. Then $u_1 = F \cup \{x, y\}$. Assume that $u_3 = F \cup \{y, z\}$, where $z \notin u_1$. Let

$w = (u_0, u_1, u_2, u_3, w_4, w_5)$ be any cycle of length 6. Since $u_0 \subseteq w_5 \neq u_1$ and $|w_5 \cap u_3| = k$, we have $w_5 = F \cup \{x, y\}$ and $w_4 = u_3 \cap w_5$. Hence, w is unique and (i) holds.

(ii) and (iii) By (i), it suffices to count the number of the paths (u_0, u_1, u_2, w_3). If $u_2 \in V_1$, then $u_2 \subseteq w_3 \neq u_1$ and there are $n-k-1$ choices for w_3. If $u_2 \in V_2$, then $u_1 \neq w_3 \subseteq u_2$ and there are k choices for w_3. Hence (ii) and (iii) hold.

(iv) Without loss of generality, suppose $u_1 \in V_1$. There exist $n-k-1$ vertices w_2 such that u_0, u_1, w_2 is a path. By (ii), the desired result follows. □

Corollary 2.2 *The following hold.*

(i) *The length of the shortest cycle in $J(n; k, k+1)$ is* 6.

(ii) *The number of* 6*-cycles in $J(n; k, k+1)$ is $n(C_6) = \frac{1}{6}k(n-k-1)e(J(n; k, k+1)) = \frac{1}{6}\binom{n}{k}(n-k)k(n-k-1)$.*

Proof. (i) It suffices to prove that there does not exist a 4-cycle in $J(n; k, k+1)$. Suppose (v_1, v_2, v_3, v_4) is a 4-cycle in $J(n; k, k+1)$ such that $v_1, v_3 \in V_1$ and $v_2, v_4 \in V_2$. Then $v_2 = v_1 \cup v_3 = v_4$, a contradiction.

(ii) Since $e(J(n; k, k+1)) = \binom{n}{k}(n-k)$ and every edge is contained in $k(n-k-1)$ cycles of length 6 by Proposition 2.1, we have $n(C_6) = \frac{1}{6}\binom{n}{k}(n-k)k(n-k-1)$. □

In the following, we consider the number of 2-paths in a spanning subgraph G of $J(n; k, k+1)$. For any 2-path (x, w, y) in G, note that $x, y \in N_G(w)$. Hence, the number of 2-paths in G whose middle vertex is in V_i is

$$\sum_{w \in V_i} \binom{d_G(w)}{2} = \frac{1}{2} \sum_{w \in V_i} d_G(w)^2 - \frac{1}{2} e(G) \tag{1}$$

for $i = 1, 2$. Observe that the total number of 2-paths in $J(n; k, k+1)$ is $\frac{n-1}{2} \cdot e(J(n; k, k+1))$.

By Cauchy-Schwarz inequality, for $i \in \{1, 2\}$, we have

$$\sum_{w \in V_i} d_G(w)^2 \geq \left(\sum_{w \in V_i} d_G(w) \right)^2 / v_i = e(G)^2 / v_i, \tag{2}$$

which implies that

$$\sum_{w \in V_i} \binom{d_G(w)}{2} \geq \frac{1}{2v_i} e(G)^2 - \frac{1}{2} e(G).$$

3 Upper bounds for $ex(\widetilde{O}_{k+1}, C_{2l})$ with $l = 3, 4, 5$

Let $\mathscr{C}_6$ be the set of all 6-cycles in $\widetilde{O}_{k+1}$ and G be any spanning subgraph of $\widetilde{O}_{k+1}$. For any subgraphs H and L of $\widetilde{O}_{k+1}$, let $G \cap H$ be the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$, and $H \setminus E(L)$ be the graph with vertex set $V(H)$ and edge set $E(G) \setminus E(L)$. Notice that for any 6-cycle $H \in \mathscr{C}_6$, $G \cap H$ is isomorphic to one of the graphs in Figure 1. Let χ_0, χ_1, χ_2^1, χ_2^2, χ_3^1, χ_3^2, χ_3^3, χ_4^1, χ_4^2, χ_4^3, χ_5, χ_6 denote the ratio of the number of 6-cycles H satisfying that $G \cap H$ is isomorphic to the graphs $(1) - (12)$ in Figure 1 to the total number of 6-cycles in $\widetilde{O}_{k+1}$, respectively.

Then we have

$$\chi_0 + \chi_1 + \chi_2^1 + \chi_2^2 + \chi_3^1 + \chi_3^2 + \chi_3^3 + \chi_4^1 + \chi_4^2 + \chi_4^3 + \chi_5 + \chi_6 = 1. \tag{3}$$

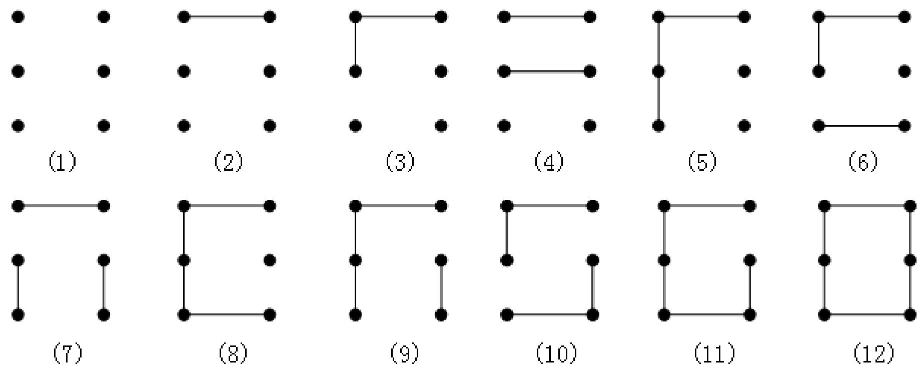

Figure 1: Subgraphs of C_6

For any two distinct $H_1, H_2 \in \mathscr{C}_6$, since the least length of a cycle in $\widetilde{O}_{k+1}$ is 6, we have $V(H_1) \neq V(H_2)$, which implies that $G \cap H_1 \neq G \cap H_2$. For any $e \in E(\widetilde{O}_{k+1})$, let $(\mathscr{C}_6)_e$ denote the set of all 6-cycles in $\mathscr{C}_6$ which contain e. By computing the size of the set $\{(e, G \cap H) \mid H \in \mathscr{C}_6,\ e \in E(G \cap H)\}$ in two ways, we obtain

$$\sum_{H \in \mathscr{C}_6} e(G \cap H) = \sum_{e \in E(G)} |(\mathscr{C}_6)_e|.$$

By Proposition 2.1 and Corollary 2.2, we get

$$\begin{aligned}\frac{e(G)}{e(\widetilde{O}_{k+1})} &= \frac{1}{6n(C_6)} \sum_{H \in \mathscr{C}_6} e(G \cap H) \\ &= \frac{1}{6}\Big(\chi_1 + 2(\chi_2^1 + \chi_2^2) + 3(\chi_3^1 + \chi_3^2 + \chi_3^3) + 4(\chi_4^1 + \chi_4^2 + \chi_4^3) + 5\chi_5 + 6\chi_6\Big). \qquad (4)\end{aligned}$$

Proof of Theorem 1.2 (iii). (a) Suppose G is C_6-free. Then $\chi_6 = 0$, which implies that $\mathrm{ex}(\widetilde{O}_{k+1}, C_6) \leq \frac{5}{6}e(\widetilde{O}_{k+1})$ by (3) and (4).

(b) Suppose G is C_8-free. For any 2-path L in $\widetilde{O}_{k+1}$, we claim that there is at most one H in $\mathscr{C}_6$ such that $(G \cap H) \setminus E(L)$ is isomorphic to the graph (8) in Figure 1. Assume that $L = (u_1, u_2, u_3)$ is a 2-path, and $H_1 = (u_1, u_2, u_3, u_4, u_5, u_6)$ and $H_2 = (u_1, u_2, u_3, w_4, w_5, w_6)$ are two cycles in $\mathscr{C}_6$ such that $(G \cap H_i) \setminus E(L)$ is isomorphic to the graph (8) in Figure 1 for $i \in \{1, 2\}$. Since G is C_8-free, $\{u_4, u_5, u_6\} \cap \{w_4, w_5, w_6\} \neq \emptyset$. If $u_4 = w_4$ or $u_6 = w_6$, then H_1 and H_2 contain a same 3-path, which is impossible by Proposition 2.1. If $u_4 \neq w_4$ and $u_6 \neq w_6$, then one can construct a cycle with length less than 6 from H_1 and H_2. That is impossible because the least length of cycles in $\widetilde{O}_{k+1}$ is 6. Hence, the claim holds. It is easy to see that if there is an H in $\mathscr{C}_6$ such that $(G \cap H) \setminus E(L)$ is isomorphic to the graph (8) in Figure 1, $G \cap H$ must be isomorphic to one of the graphs (8), (11) and (12) in Figure 1. Conversely, for any $H \in \mathscr{C}_6$, if $G \cap H$ is isomorphic to the graph (8), (11) or (12) in Figure 1, the number of 2-paths L in $\widetilde{O}_{k+1}$ such that $(G \cap H) \setminus E(L)$ is isomorphic to the graph (8) in Figure 1 is 1, 2 or 6, respectively.

By counting in two ways the pairs (L, H) where L is a 2-path in $\widetilde{O}_{k+1}$, $H \in \mathscr{C}_6$ such that $(G \cap H) \setminus E(L)$ is isomorphic to the graph (8) in Figure 1, we have

$$k \cdot e(\widetilde{O}_{k+1}) \geq (6\chi_6 + 2\chi_5 + \chi_4^1)n(C_6),$$

which implies that $\chi_4^1 \leq \frac{6}{k} = o(1)$, $\chi_5 \leq \frac{3}{k} = o(1)$ and $\chi_6 \leq \frac{1}{k} = o(1)$. By (3) and (4), we have

$$\frac{e(G)}{e(\widetilde{O}_{k+1})} \leq \frac{1}{6}(4 + o(1)),$$

and hence $\mathrm{ex}(\widetilde{O}_{k+1}, C_8) \leq (\frac{2}{3} + o(1))e(\widetilde{O}_{k+1})$.

(c) Suppose G is C_{10}-free. For any $e \in E(\widetilde{O}_{k+1})$, we claim that there are at most $2k-1$ cycles H in $\mathscr{C}_6$ such that $(G \cap H) \setminus \{e\}$ is isomorphic to the graph (11) in Figure 1. Assume that $e = (u_1, u_2)$ is an edge, and $H_1 = (u_1, u_2, u_3, u_4, u_5, u_6)$ is a cycle in $\mathscr{C}_6$ such that $(G \cap H_1) \setminus \{e\}$ is isomorphic to the graph (11) in Figure 1. Let $H_2 = (u_1, u_2, w_3, w_4, w_5, w_6)$ be any other cycle in $\mathscr{C}_6$ such that $(G \cap H_2) \setminus \{e\}$ is isomorphic to the graph (11). since G is C_{10}-free, $\{u_3, u_4, u_5, u_6\} \cap \{w_3, w_4, w_5, w_6\} \neq \emptyset$. If $u_3 \neq w_3$ and $u_6 \neq w_6$, then one can construct a cycle with length less than 6 from H_1 and H_2. That is impossible because the least length of cycles in $\widetilde{O}_{k+1}$ is 6. Then we have $u_4 = w_4$ or $u_6 = w_6$. By Proposition 2.1, note that there are at most $2k-1$ cycles in $\mathscr{C}_6$ containing the 2-path (u_1, u_2, u_3) or (u_6, u_1, u_2). Hence, the claim holds. It is easy to see that if there is an H in $\mathscr{C}_6$ such that $(G \cap H) \setminus \{e\}$ is isomorphic to the graph (11) in Figure 1, $G \cap H$ must be isomorphic to one of the graphs (11) and (12) in Figure 1. Conversely, for any $H \in \mathscr{C}_6$, if $G \cap H$ is isomorphic to the graph (11) or (12) in Figure 1, the number of e in $\widetilde{O}_{k+1}$ such that $(G \cap H) \setminus \{e\}$ is isomorphic to the graph (11) in Figure 1 is 1 or 6, respectively.

By counting in two ways the pairs (e, H) where $e \in E(\widetilde{O}_{k+1})$, $H \in \mathscr{C}_6$ such that $(G \cap H) \setminus \{e\}$ is isomorphic to the graph (11) in Figure 1, we have

$$(2k-1) \cdot e(\widetilde{O}_{k+1}) \geq (6\chi_6 + \chi_5) n(C_6),$$

which implies that $\chi_5 \leq \frac{6(2k-1)}{k^2} = o(1)$ and $\chi_6 \leq \frac{2k-1}{k^2} = o(1)$. By (3) and (4), we have

$$\frac{e(G)}{e(\widetilde{O}_{k+1})} \leq \frac{1}{6}(4 + o(1)),$$

and hence $\mathrm{ex}(\widetilde{O}_{k+1}, C_{10}) \leq (\frac{2}{3} + o(1))e(\widetilde{O}_{k+1})$. □

4 C_{4l}-free subgraphs of $J(n; k, k+1)$ with $l \geq 2$

Let l be an integer with $l \geq 2$. Suppose G is a maximal spanning C_{4l}-free subgraph of $J := J(n; k, k+1)$. Notice that $V(G) = V(J)$ and $d_G(x) \geq 1$ for any $x \in V(G)$.

Firstly, we define an auxiliary graph $H_x := H_x(G)$ for each vertex $x \in V(J)$. We note that the H_x in this form is similar to but different from the auxiliary graph which was used by Chung [6] and Füredi et al. [14]. The vertex set of H_x consists of the vertices which have distance 2 from x in J. In H_x, for any two distinct vertices y and z, they are adjacent if and only if there exists a vertex $w \notin N_J(x)$ such that (y, w, z) is a 2-path in G. Notice that $|V(H_x)| = k(n-k)$ if $x \in V_1$, and $|V(H_x)| = (k+1)(n-k-1)$ if $x \in V_2$.

If $\{y, z\} \in E(H_x)$, then $\partial_G(y, z) = 2$. Hence, there exits a unique 6-cycle containing x, y and z in J, and there exists a unique vertex w such that (y, w, z) is a 2-path in G. Conversely, for any two distinct vertices $y, z \in V(J)$ such that $\partial_G(y, z) = 2$, by Proposition 2.1 (ii) and (iii), there are $n-k-1$ (resp. k) vertices x in $V(J)$ such $\{y, z\} \in E(H_x)$ if $y, z \in V_1$ (resp. $y, z \in V_2$). Let

$$\mathcal{F}_i = \{(x, \{y, z\}) \mid x \in V_i,\ \{y, z\} \in E(H_x)\}$$

for $i = \{1, 2\}$. By counting in two ways the elements in $\mathcal{F}_i$, from (1), observe that

$$\sum_{x \in V_1} e(H_x) = (n-k-1) \sum_{w \in V_2} \binom{d_G(w)}{2} = \frac{1}{2}(n-k-1) \sum_{w \in V_2} d_G(w)^2 - \frac{1}{2}(n-k-1)e(G), \tag{5}$$

$$\sum_{x \in V_2} e(H_x) = k \sum_{w \in V_1} \binom{d_G(w)}{2} = \frac{k}{2} \sum_{w \in V_1} d_G(w)^2 - \frac{k}{2}e(G). \tag{6}$$

Since G is C_{4l}-free, we have H_x is C_{2l}-free. If not, suppose $(y_0, y_1, \ldots, y_{2l-1})$ is a cycle in H_x. By the definition of H_x, assume that $w_0, w_1, \ldots, w_{2l-1} \notin N_J(x)$ are the vertices such that (y_i, w_i, y_{i+1}) is a 2-path in G for any $i \in \{0, 1, \ldots, 2l-1\}$, where $y_{2l} = y_0$. We claim that $w_0, w_1, \ldots, w_{2l-1}$ are pair-wise distinct. Suppose $w_i = w_j$ with $i \neq j$. Then (x, u_i, y_i, w_i), $(x, u_{i+1}, y_{i+1}, w_i)$, (x, u_j, y_j, w_j) are three 3-paths in J, That is impossible since the least length of a cycle in J is 6 and there exists a unique cycle of length 6 containing (x, u_i, y_i, w_i) in J. Hence $(y_0, w_0, y_1, w_1, \ldots, y_{2l-2}, w_{2l-2}, y_{2l-1}, w_{2l-1})$ is a cycle with length $4l$ in G, which is a contradiction. Thus, by the consequence of Bondy and Simonovits [4], H_x can have at most $c_l'(v(H_x))^{1+1/l}$ edges, where c_l' is a constant. Therefore, we have

$$\sum_{x \in V_1} e(H_x) \leq v_l c_1'(k(n-k))^{1+1/l}. \tag{7}$$

Proof of Theorem 1.1 (i). Firstly, we give a lower bound of $\sum_{x \in V_1} e(H_x)$. Since $d_G(w) \leq n-k$ for any $w \in V_1$, we get

$$\sum_{w \in V_1} d_G(w)^2 \leq (n-k) \sum_{w \in V_1} d_G(w) = (n-k)e(G),$$

which implies that

$$\sum_{x \in V_2} e(H_x) \leq \frac{k}{2}(n-k-1)e(G) \tag{8}$$

from (6). Since $d_G(w)^2 - 2d_G(w) \geq -1$ for any $w \in V_2$, by (5) and (8), we have

$$\begin{aligned}
\sum_{x \in V_1} e(H_x) - \frac{1}{k} \sum_{x \in V_2} e(H_x) &\geq \frac{1}{2}(n-k-1) \sum_{x \in V_2} d_G(w)^2 - (n-k-1)e(G) \\
&= \frac{1}{2}(n-k-1)\left(\sum_{x \in V_2} d_G(w)^2 - 2\sum_{x \in V_2} d_G(w)\right) \\
&\geq -\frac{1}{2}(n-k-1)v_2. \qquad (9)
\end{aligned}$$

Therefore, by (2), (6) and (9), we have

$$\begin{aligned}
\sum_{x \in V_1} e(H_x) &= \left(\sum_{x \in V_1} e(H_x) - \frac{1}{k}\sum_{x \in V_2} e(H_x)\right) + \frac{1}{k}\sum_{x \in V_2} e(H_x) \\
&\geq -\frac{1}{2}(n-k-1)v_2 + \frac{1}{2}\sum_{w \in V_1} d_G(w)^2 - \frac{1}{2}e(G) \\
&\geq -\frac{1}{2}(n-k-1)v_2 + \frac{1}{2}\frac{e(G)^2}{v_1} - \frac{1}{2}e(G) \\
&= -\frac{1}{2}nv_2 + \frac{1}{2}\frac{e(G)^2}{v_1}. \qquad (10)
\end{aligned}$$

By (7) and (10), we get

$$e(G)^2 \leq 2v_1^2 c_l'(k(n-k))^{1+1/l} + nv_1v_2,$$

which implies that

$$\begin{aligned}
\frac{e(G)^2}{e(J)^2} &\leq 2c_l' k^{1+1/l}(n-k)^{-1+1/l} + n(n-k)^{-1}(k+1)^{-1} \\
&= (2c_l' k^{1+1/l} + k(k+1)^{-1}(n-k)^{-1/l})(n-k)^{-1+1/l} + (k+1)^{-1}
\end{aligned}$$

from $e(J) = v_1(n-k) = v_2(k+1)$. Since $\lim_{n\to+\infty}(n-k)^{-1/l} = 0$, there exists constant c_l such that

$$e(G) \le (c_l(n-k)^{-\frac{1}{2}+\frac{1}{2l}} + (k+1)^{-\frac{1}{2}})e(J).$$

Therefore, Theorem 1.1 (i) holds. $\square$

Proof of Theorem 1.2 (i). By (2), (5) and (7), we have

$$v_1 c_l'(k(k+1))^{1+1/l} \ge \frac{ke(G)^2}{2v_2} - \frac{ke(G)}{2},$$

which implies that

$$e(G)^2 \le 2v_1 v_2 c_l' k^{1/l}(k+1)^{1+1/l} + v_2 e(G).$$

Since $e(\widetilde{O}_{k+1}) = v_1(k+1) = v_2(k+1)$, observe that

$$\begin{aligned}\frac{e(G)^2}{e(\widetilde{O}_{k+1})^2} &\le 2c_l' k^{1/l}(k+1)^{-1+1/l} + e(G)(k+1)^{-1}e(\widetilde{O}_{k+1})^{-1}\\ &\le (2c_l' + e(G)e(\widetilde{O}_{k+1})^{-1}(k+1)^{-2/l})(k+1)^{-1+2/l}.\end{aligned}$$

Thus, there exists a constant c_l such that

$$e(G) \le c_l(k+1)^{-\frac{1}{2}+\frac{1}{l}}e(\widetilde{O}_{k+1}),$$

and Theorem 1.2 (i) holds. $\square$

5 C_{4l+2}-free subgraphs of $J(n;k,k+1)$ with $l \ge 1$

We update the auxiliary graph used in Section 4. Let G be a spanning subgraph of $J(n;k,k+1)$ and $\Omega = \binom{[n]}{k-1}$. For any $\gamma \in \Omega$, we define a new auxiliary graph $H_\gamma = H_\gamma(G)$ as follows. The vertex set of H_γ consists of all the k-subsets of $[n]$ which contain γ. For any two vertices x and y in $V(H_\gamma)$, x and y are adjacent if and only if there exists a 2-path between x and y in G.

Note that $|V(H_\gamma)| = n-k+1$ for any $\gamma \in \Omega$. For any two distinct elements x and y in V_1, if there exists a 2-path between x and y in G, then the 2-path is unique in G, and there exists a unique $\gamma \in \Omega$ such that $\{x,y\} \in E(H_\gamma)$. Therefore, the number of edges in $\cup_{\gamma\in\Omega}E(H_\gamma)$ equals the number of 2-paths in G whose endpoints are in V_1, that is

$$\sum_{\gamma\in\Omega} e(H_\gamma) = \sum_{w\in V_2}\binom{d_G(w)}{2} = \frac{1}{2}\sum_{w\in V_2} d_G(w)^2 - \frac{1}{2}e(G). \tag{11}$$

Proposition 5.1 *If there exists an m-cycle in H_γ for some $\gamma \in \Omega$, then there exists a $2m$-cycle in G.*

Proof. Suppose $(y_0, y_1, \ldots, y_m)$ is a cycle in H_γ. By the definition of H_γ, assume that $w_0, w_1, \ldots, w_m$ are the vertices such that (y_i, w_i, y_{i+1}) is a 2-path in G for any $i \in \{0,1,\ldots,m\}$, where $y_{m+1} = y_0$. We claim that $w_0, w_1, \ldots, w_m$ are pair-wise distinct. Suppose $w_i = w_j$ with $i \ne j$. Then $y_i \cup y_{i+1} = y_j \cup y_{j+1}$, which is impossible since y_i, y_{i+1}, y_j and y_{j+1} are four distinct k-subsets of $[n]$ which contain γ. Hence $(y_0, w_0, y_1, w_1, \ldots, y_{m-2}, w_{m-2}, y_{m-1}, w_{m-1})$ is a cycle of length $2m$ in G. $\square$

5.1 Upper bound for $\mathrm{ex}(J(n;k,k+1), C_{4l+2})$ with $l \geq 1$

Proof of Theorem 1.1 (ii). To get an upper bound for $\mathrm{ex}(J(n;k,k+1), C_{4l+2})$, we will apply the Erdős-Stone-Simonovits Theorem [11, 13], that if F is a graph with $\chi(F) = t$ and $\chi(F \setminus \{e\}) < t$ for some edge e of F, then

$$\mathrm{ex}(m, F) = \left(1 - \frac{1}{t-1} + o(1)\right)\binom{m}{2},$$

where $\chi(F)$ is the chromatic number of the graph F.

Suppose G is C_{4l+2}-free. By Proposition 5.1, we have H_γ is C_{2l+1}-free. Therefore, for $l \geq 1$, according to the Erdős-Stone-Simonovits Theorem, H_γ has at most $(\frac{1}{2} + o(1))\binom{n-k+1}{2}$ edges. By (2) and (11), we have

$$\binom{n}{k-1}\left(\frac{1}{2} + o(1)\right)\binom{n-k+1}{2} \geq \frac{e(G)^2}{2v_2} - \frac{e(G)}{2},$$

which implies that

$$e(G)^2 \leq v_2 v_1 k(n-k)\left(\frac{1}{2} + o(1)\right) + v_2 e(G).$$

Since $e(J) = v_1(n-k) = v_2(k+1)$, we obtain

$$\frac{e(G)^2}{e(J)^2} \leq \frac{k}{k+1}\left(\frac{1}{2} + o(1)\right) + \frac{1}{k+1}\frac{e(G)}{e(J)},$$

which implies that

$$\begin{aligned}
\frac{e(G)}{e(J)} &\leq \frac{1}{2}\left(\frac{1}{k+1} + \sqrt{\frac{1}{(k+1)^2} + \frac{4k}{k+1}\left(\frac{1}{2} + o(1)\right)}\right) \\
&\leq \frac{1}{2(k+1)} + \frac{\sqrt{1 + 2k(k+1)}}{2(k+1)} + o(1) \\
&\leq \frac{1}{2(k+1)} + \frac{\sqrt{2}}{2} + o(1).
\end{aligned}$$

Therefore, Theorem 1.1 (ii) holds. □

5.2 C_{4l+2}-free subgraphs of $\widetilde{O}_{k+1}$ with $l \geq 3$

In this subsection, let G be a C_{4l+2}-free spanning subgraph of $\widetilde{O}_{k+1}$ with $l \geq 3$. Let a and b be two integers such that $4a + 4b = 4l + 4$ and $a, b \geq 2$. Notice that a cycle of length $4a$ can not intersect a cycle of length $4b$ at a single edge, otherwise their union contains a cycle of length $4l+2$. For any graph H, define $N(G, H)$ to be the number of subgraphs of G that are isomorphic to H. Firstly, we provide an upper bound on $N(G, C_{4a})$. Secondly, a lower bound on $N(G, C_{4a})$ is obtained via a lower bound on the number of C_{2a}'s in the auxiliary graphs constructed from G. Last of all, we obtain an upper bound of $\mathrm{ex}(\widetilde{O}_{k+1}, C_{4l+2})$ and slightly improve our bound in a specific situation.

5.2.1 An upper bound on $N(G, C_{4a})$

Definition 5.2 *The direction of an edge $\{u, v\}$ in $E(J)$, denote by $d(uv)$, to be the single number in $u \Delta v$, where Δ is symmetric difference.*

Let $D(F) := \{d(e) \mid e \in E(F)\}$, where F is any subgraph of $\widetilde{O}_{k+1}$. Notice that for any path $P = (u_1, u_2, \ldots, u_s)$, we have $u_1 \Delta u_s \subseteq D(P)$.

Lemma 5.3 *For any cycle C of length $2r$ in $\widetilde{O}_{k+1}$, we have $|D(C)| \leq r$.*

Proof. It suffices to prove that for any $x \in D(C)$ there exist at least two edges in C whose direction is x. Assume that there exists $x' \in D(C)$ such that the number of edges in C with direction x' is 1. Without loss of generality, suppose that $C = (u_1, u_2, \ldots, u_{2r})$, $d(u_{2r}u_1) = x'$ and $x' \in u_{2r}$. Since $x' \notin u_1$ and $x' \notin u_i \Delta u_{i+1}$ for $i \in \{1, 2, \ldots, 2r-1\}$, we have $x' \notin u_{2r}$, a contradiction. Hence, the desired result follows. □

Lemma 5.4 *Let C and C' be cycles of length $4a$ and $4b$ of G, respectively. If $E(C) \cap E(C') \neq \emptyset$, then $|D(C) \cap D(C')| \geq 2$.*

Proof. Suppose $\{u_1, u_2\} \in E(C) \cap E(C')$. Since G has no cycles of length $4a + 4b - 2$, there exists $u_3 \notin \{u_1, u_2\}$ such that $u_3 \in V(C) \cap V(C')$. Since $|u_1 \Delta u_2| = 1$ and $u_3 \neq u_2$, we have $u_1 \Delta u_2 \neq u_1 \Delta u_3$. Notice that $(u_1 \Delta u_2) \cup (u_1 \Delta u_3) \subseteq D(C) \cap D(C')$, which implies that $|D(C) \cap D(C')| \geq 2$. □

Lemma 5.5 *We have*

$$N(G, C_{4a}) = O(k^{2a-2})e(G) + O(v_1 k^{2a-\frac{1}{2}+\frac{1}{b}}).$$

Moreover, if $a = b$, then $N(G, C_{4a}) = O(k^{2a-2})e(G)$.

Proof. Let $\mathscr{C}$ denote the set of cycles of length $4a$ in G and $\mathscr{C}_e$ denote the set of cycles in $\mathscr{C}$ which contain the edge e. Note that $|\mathscr{C}| = N(G, C_{4a})$. Let $\mathscr{E} = \cup_{C \in \mathscr{C}} E(C)$ and $\mathscr{E} := \mathscr{E}_1 \cup \mathscr{E}_2$, where $\mathscr{E}_1$ is the collection of edges that are contained in a cycle of length $4b$ in G, and $\mathscr{E}_2 := \mathscr{E} \setminus \mathscr{E}_1$. By counting the size of $\{(H, e) \mid H \in \mathscr{C},\ e \in \mathscr{E} \text{ and } e \in E(H)\}$ in two ways, we have

$$4aN(G, C_{4a}) = \sum_{e_1 \in \mathscr{E}_1} |\mathscr{C}_{e_1}| + \sum_{e_2 \in \mathscr{E}_2} |\mathscr{C}_{e_2}|. \tag{12}$$

Since every $4a$-cycle containing a fixed edge e is determined by a sequence of directions, for each $B \in \{D(C^*) \mid C^* \in \mathscr{C}_e\}$, there are at most $|B|^{4a-1}$ $4a$-cycles C'_{4a} such that $D(C'_{4a}) = B$ and $e \in E(C'_{4a})$.

For each $e_1 \in \mathscr{E}_1$ (if $\mathscr{E}_1 \neq \emptyset$), let C' be a fixed $4b$-cycle with $e_1 \in E(C')$. For any $4a$-cycle $C^* \in \mathscr{C}_{e_1}$, we have $d(e) \in D(C^*)$ and $|D(C^*) \cap D(C')| \geq 2$ from Lemma 5.4. Hence, by Lemma 5.3, we have

$$|\{D(C^*) \mid C^* \in \mathscr{C}_{e_1}\}| \leq \sum_{i=1}^{2a-1} \binom{|D(C')| - 1}{i} \sum_{j=0}^{2a-1-i} \binom{2k + 1 - |D(C')|}{j},$$

which implies that

$$|\mathscr{C}_{e_1}| \leq \sum_{i=1}^{2a-1} \binom{D(C') - 1}{i} \sum_{j=0}^{2a-1-i} \binom{2k + 1 - |D(C')|}{j} (i + 1 + j)^{4a-1} = O(k^{2a-2}). \tag{13}$$

For each $e_2 \in \mathscr{E}_2$ (if $\mathscr{E}_2 \neq \emptyset$), by Lemma 5.3 again, we have

$$|\{D(C^*) \mid C^* \in \mathscr{C}_{e_2}\}| \leq \sum_{i=0}^{2a-1} \binom{2k}{i},$$

which implies that

$$|\mathscr{C}_{e_2}| \leq \sum_{i=0}^{2a-1} \binom{2k}{i} (i+1)^{4a-1} = O(k^{2a-1}). \tag{14}$$

Notice that $|\mathscr{E}_1| \leq e(G)$ and $|\mathscr{E}_2| \leq \mathrm{ex}(\widetilde{O}_{k+1}, C_{4b})$ because the subgraph induced by $\mathscr{E}_2$ is C_{4b}-free. By (12), (13), (14) and Theorem 1.2 (i), we obtain

$$N(G, C_{4a}) = \frac{1}{4a} \left(\sum_{e \in \mathscr{E}_1} O(k^{2a-2}) + \sum_{e \in \mathscr{E}_2} O(k^{2a-1}) \right) \leq O(k^{2a-2}) e(G) + O(v_1 k^{2a-\frac{1}{2}+\frac{1}{b}}).$$

In particular, if $a = b$, then $|\mathscr{E}_2| = 0$. Hence,

$$N(G, C_{4a}) = \frac{1}{4a} \sum_{e \in \mathscr{E}_1} O(k^{2a-2}) \leq O(k^{2a-2}) e(G).$$

We complete the proof of this lemma and obtain an upper bound of $N(G, C_{4a})$. □

5.2.2 A lower bound on $N(G, C_{4a})$

In this part, we use the auxiliary graphs defined in the beginning of this section to get a lower bound of $N(G, C_{4a})$ via a lower bound on the number of $2a$-cycles in these auxiliary graphs.

By the definition of the auxiliary graph and the proof of Proposition 5.1, we get

$$N(G, C_{4a}) \geq \sum_{\gamma \in \Omega} N(H_\gamma, C_{2a}). \tag{15}$$

Lemma 5.6 (Erdős, Simonovits [12]) *Let L be a bipartite graph, where there exist vertices x and y such that $L \setminus \{x, y\}$ is a tree. Then for a graph H with n vertices and e edges, there exist constants $c_1,\ c_2 > 0$ such that if H contains more than $c_1 n^{\frac{3}{2}}$ edges, then*

$$N(H, L) \geq c_2 \frac{e^{n(L)}}{n^{2e(L)-n(L)}},$$

where $n(L)$ and $e(L)$ are the number of vertices and edges in L, respectively. □

Lemma 5.7 *We have $N(G, C_{4a}) \geq c v_1 \frac{\bar{d}^{4a}}{k^{2a}} - O(v_1 k^a)$, where $\bar{d} = e(G)/v_1 = e(G)/v_2$.*

Proof. We use Lemma 5.6 with $L = C_{2a}$ in the following form so that the condition on the minimum number of edges is incorporated. Since $n(L) = e(L) = 2a$, we have

$$N(H_\gamma, C_{2a}) \geq c_2 \left(\frac{e(H_\gamma)^{2a}}{v(H_\gamma)^{2a}} - \frac{(c_1 v(H_\gamma)^{3/2})^{2a}}{v(H_\gamma)^{2a}} \right),$$

which implies that

$$N(G, C_{4a}) \geq \sum_{\gamma \in \Omega} c_2 \left(\frac{e(H_\gamma)^{2a}}{v(H_\gamma)^{2a}} - \frac{(c_1 v(H_\gamma)^{3/2})^{2a}}{v(H_\gamma)^{2a}} \right) \tag{16}$$

by (15). By Hölder inequality, (2) and (11), we have

$$\sum_{\gamma\in\Omega} e(H_\gamma)^{2a} \geq \left(\sum_{\gamma\in\Omega} e(H_\gamma)\right)^{2a} \cdot |\Omega|^{-2a+1} = \left(\sum_{w\in V_2} \binom{d_G(w)}{2}\right)^{2a} \cdot |\Omega|^{-2a+1}$$
$$= \left(v_2\binom{\bar{d}}{2}\right)^{2a} \cdot |\Omega|^{-2a+1} = \left(\frac{k+2}{k}\binom{\bar{d}}{2}\right)^{2a} \cdot \frac{kv_1}{k+2}.$$

Since $\binom{\bar{d}}{2}/\bar{d}^2 \leq \frac{1}{2}$, by (16), we get

$$N(G, C_{4a}) \geq c_2 v_1 \frac{k}{k+2} \frac{\binom{\bar{d}}{2}^{2a}}{k^{2a}} - O\left(\frac{k}{k+2} v_1 (k+2)^a\right) \geq c v_1 \frac{\bar{d}^{4a}}{k^{2a}} - O(v_1 k^a).$$

Therefore, the desired result follows. □

5.2.3 Proof of Theorem 1.2 (ii)

Since $e(G)/v_1 \leq k+1$, by Lemmas 5.5 and 5.7, we have

$$\bar{d}^{4a} \leq O(k^{3a}) + \bar{d} O(k^{4a-2}) + O(k^{4a-\frac{1}{2}+\frac{1}{b}}).$$

Hence,

$$\bar{d}^{4a} = \max\left\{\bar{d} O(k^{4a-2}),\ O(k^{4a-\frac{1}{2}+\frac{1}{b}})\right\},$$

which implies that $\bar{d} = \max\left\{O(k^{1-\frac{1}{4a-1}}),\ O(k^{1-\frac{1}{4a}(\frac{1}{2}-\frac{1}{b})})\right\}$.

This bound is minimized when $a = 2$ and $b = l-1$ and we get $\bar{d} = O(k^{1-\frac{1}{16}+\frac{1}{8(l-1)}})$, which implies that

$$e(G) = O(k^{1-\frac{1}{16}+\frac{1}{8(l-1)}}) v_1 = O(k^{-\frac{1}{16}+\frac{1}{8(l-1)}}) e(\widetilde{O}_{k+1}), \tag{17}$$

where $e(\widetilde{O}_{k+1}) = (k+1)v_1$.

Finally, we consider the case $a = b = (l+1)/2$ when l is odd. By Lemmas 5.5 and 5.7, we have

$$\bar{d}^{4a} \leq O(k^{3a}) + \bar{d} O(k^{4a-2}),$$

which implies that $\bar{d} = O(n^{1-\frac{1}{4a-1}})$. Since $a = b = (l+1)/2$, we immediately get $\bar{d} = O(k^{1-\frac{1}{2l+1}})$, which implies that

$$e(G) = O(k^{1-\frac{1}{2l+1}}) v_1 = O(k^{-\frac{1}{2l+1}}) e(\widetilde{O}_{k+1}). \tag{18}$$

By comparing (17) and (18) when l is odd, observe that $k^{-\frac{1}{2l+1}} \leq k^{-\frac{1}{16}+\frac{1}{8(l-1)}}$ if and only if $0 < l < 9.8$. Since $l \geq 3$, (18) improves (17) for $l = 3, 5, 7, 9$. We compete the proof of Theorem 1.2 (ii). □

Remark 5.8 *Our proof also implies that* $\mathrm{ex}(\widetilde{O}_{k+1}, \Theta_{4a-1,1,4b-1})$ *is* $o(e(\widetilde{O}_{k+1}))$ *for* $a, b \geq 2$ *and* $k \geq 1$*, where* $\Theta_{u,v,w}$ *is a theta-graph consisting of three paths of lengths u, v and w having the same endpoints and distinct inner vertices. Our result also naturally implies that* C_{2l} *is Ramsey for* $l \geq 6$*, i.e., there is a monochromatic copy of* C_{2l} *in any t-edge-coloring of* $\widetilde{O}_{k+1}$ *when* $k > k(t, l)$ *(Theorem 1.3).*

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