Graphs with the edge metric dimension smaller than the metric dimension

Martin Knor[1], Snježana Majstorović[2], Aoden Teo Masa Toshi[3], Riste Škrekovski[4,5] and Ismael G. Yero[6]

[1] *Slovak University of Technology in Bratislava, Bratislava, Slovakia*

[2] *University of Osijek, Department of Mathematics, Croatia*

[3] *Independent researcher, Singapore*

[4] *University of Ljubljana, FMF, 1000 Ljubljana, Slovenia*

[5] *Faculty of Information Studies, 8000 Novo Mesto, Slovenia*

[6] *Universidad de Cádiz, Departamento de Matemáticas, Algeciras, Spain*

June 23, 2020

Abstract

Given a connected graph G, the metric (resp. edge metric) dimension of G is the cardinality of the smallest ordered set of vertices that uniquely identifies every pair of distinct vertices (resp. edges) of G by means of distance vectors to such a set. In this work, we settle three open problems on (edge) metric dimension of graphs. Specifically, we show that for every $r, t \geq 2$ with $r \neq t$, there is n_0, such that for every $n \geq n_0$ there exists a graph G of order n with metric dimension r and edge metric dimension t, which among other consequences, shows the existence of infinitely many graph whose edge metric dimension is strictly smaller than its metric dimension. In addition, we also prove that it is not possible to bound the edge metric dimension of a graph G by some constant factor of the metric dimension of G.

Keywords: Edge metric dimension; metric dimension; unicyclic graphs.
AMS Subject Classification numbers: 05C12; 05C76

1 Introduction

Metric dimension is nowadays a well studied topic in graph theory and combinatorics, as well as in some computer science applications, and the theory involving it is indeed full of interesting results and open questions. One recent issue that has attracted the attention of several researchers concerns a variant of the standard metric dimension, in which it is required to uniquely recognize the edges of a graph, instead of its vertices, and by using vertices as the recognizing elements. This variant was introduced in [6], and since its appearance, a significant number of works have been published. In this sense, we

knor@math.sk, smajstor@mathos.hr, aodenteo@gmail.com, skrekovski@gmail.com, and ismael.gonzalez@uca.es

mention the most recent ones [3, 4, 9, 12, 13, 14]. Specifically, the edge metric dimension is studied in several situations as follows: [3] is dedicated to study several generalized Petersen graphs; in [4], a number of results about pattern avoidance in graphs with bounded edge metric dimension are given; [9] centers the attention on some product graphs (corona, join and lexicographic); [12] studies some convex polytopes and related graphs; in [13], a characterization of graphs with the largest possible edge metric dimension (order minus one) is given; and finally, in [14] the Cartesian product of any graph with a path is studied, as well as, it is proved to be not possible to bound the metric dimension of a graph G by some constant factor of the edge metric dimension of G. We should also remark some results from the seminal article [6]. There was proved for instance that computing the edge metric dimension of graphs is NP-hard, that can be approximated within a constant factor, and that becomes polynomial for the case of trees. Further, some bounds and closed formulas for several classes of graphs including trees, grid graphs and wheels (among others), were also deduced in [6].

We recall that the parameter edge metric dimension (from [6]) studied here is not the same as that one defined in [8], where the authors studied the metric dimension of the line graph of a graph (namely edges uniquely recognizing edges), and called such parameter as edge metric dimension, although it was further renamed as the edge version of metric dimension in [7].

In the next we recall the necessary terminology and notation. We consider only simple and connected graphs. Let G be a graph and let u, v be its vertices. By $d_G(u, v)$ (or by $d(u, v)$ when no confusion is likely) we denote the distance from u to v in G. Let z be a vertex of G. We say that z *identifies* (*resolves* or *determines*) a pair of vertices $u, v \in V(G)$, if $d_G(u, z) \neq d_G(v, z)$, An ordered set of vertices S is a *metric generator* for G if every two vertices $u, v \in V(G)$ are identified by a vertex of S. The *metric dimension* of G is then the cardinality of the smallest metric generator for G. Such cardinality is denoted by $\dim(G)$ and a metric generator of cardinality $\dim(G)$ is known as a *metric basis*. It is necessary to remark that the concepts above were first defined in [1] for a more general setting of metric spaces. The concepts were again independently rediscovered for the case of graphs in [5] and [11], where metric generators were called resolving sets and locating sets, respectively. Also, in [11], the metric dimension was called locating number. The terminology of metric generators was first used in [10].

Let G be a graph and let S be an ordered set of vertices of G. For every $v \in V(G)$ we can consider the vector $r(v|S)$ of distances from v to the vertices in S. If S is a metric generator, then all such vectors are pairwise different. The vector $r(v|S)$ is known as the *metric representation* of v with respect to S.

The concept of edge metric dimension was first described in [6], as a way to uniquely recognize the edges of a given graph G. A vertex $z \in V(G)$ *distinguishes* two edges $e, f \in E(G)$ if $d_G(e, z) \neq d_G(f, z)$, where $d_G(e, z) = d_G(uv, z) = \min\{d_G(u, z), d_G(v, z)\}$. A set of vertices $S \subset V(G)$ is an *edge metric generator* for G, if any two edges of G are distinguished by a vertex of S. The *edge metric dimension* of G is the cardinality of the smallest edge metric generator for G, and is denoted by $\mathrm{edim}(G)$. An edge metric generator of cardinality $\mathrm{edim}(G)$ is known as an *edge metric basis*. The edge metric representation is defined analogously as in the case of the metric dimension.

2 Edge metric dimension versus metric dimension

One would expect that $\dim(G)$ and $\operatorname{edim}(G)$ are related. The search for a relationship between these two invariants (in a shape of a bound for instance) was of interest in the seminal article [6], as well as in the subsequent works on the topic (see also for instance [14]). In this sense, in [6], several families of graphs for which $\dim(G) < \operatorname{edim}(G)$, or $\dim(G) = \operatorname{edim}(G)$, or $\dim(G) > \operatorname{edim}(G)$ were presented. For the last case, only one construction was given, namely the Cartesian product of two cycles $C_{4r}\Box C_{4t}$. It was shown in [6] that $\operatorname{edim}(C_{4r}\Box C_{4t}) = 3 < 4 = \dim(C_{4r}\Box C_{4t})$. In consequence, it was claimed in [6] that the metric dimension and the edge metric dimension of graphs seemed to be not comparable in general. This example above and the other results from [6] allowed the authors of that article to point out the following questions.

(i) For which integers $r, t, n \geq 1$, with $r, t \leq n-1$, can be constructed a graph G of order n with $\dim(G) = r$ and $\operatorname{edim}(G) = t$?

(ii) Is it possible that $\dim(G)$ or $\operatorname{edim}(G)$ would be bounded from above by some constant factor of $\operatorname{edim}(G)$ or $\dim(G)$, respectively?

(iii) Can you construct some other families of graphs for which $\dim(G) > \operatorname{edim}(G)$?

Note that the question (i) is precisely the realization of graphs G with prescribed values on its order, metric dimension and edge metric dimension, while the question (ii) is equivalent to ask whether the ratios $\frac{\operatorname{edim}(G)}{\dim(G)}$ and $\frac{\dim(G)}{\operatorname{edim}(G)}$ are bounded from above. The third question can be settled as a consequence of the other two. Realization results concerning the case in which $\dim(G) \leq \operatorname{edim}(G)$ were already studied in [6], although not completed. With respect to the ratios, it was proved in [14] that $\frac{\operatorname{edim}(G)}{\dim(G)}$ is not bounded from above. The other possibility has never been studied till now.

In this work we deal with these three problems mentioned above. That is, our results complete the unboundedness results given in [14], while studying the ratio $\frac{\dim(G)}{\operatorname{edim}(G)}$, and thus, the problem in (ii) is now completely settled. We also give positive answer to (iii), and moreover, we present an almost complete answer to (i), since we show that for every $r, t \geq 2$ with $r \neq t$, there is n_0, such that for every $n \geq n_0$ there exists an outerplanar graph G (a cactus graph indeed), of order n with $\dim(G) = r$ and $\operatorname{edim}(G) = t$. This result is in a sense best possible, because if $\operatorname{edim}(G) = 1$, then G is a path of length at least 2, and consequently $\dim(G) = 1$ as well. We remark that a similar result for $2 \leq r \leq t \leq 2r$ was proved in [6], where n_0 was shown to be at most $2r + 2$. So, our result complements the former one and a weaker version of this former result (with a weaker bound for n_0) can be proved also using a variant of our construction.

As a consequence of our results it is clear the existence of infinite families of graphs G for which the differences $\operatorname{edim}(G) - \dim(G)$ and $\dim(G) - \operatorname{edim}(G)$ are arbitrarily large. Proving that the difference $\operatorname{edim}(G) - \dim(G)$ is arbitrarily large was already presented in [6]. However, the other difference $\dim(G) - \operatorname{edim}(G)$ was only proved to be at most 1 in [6], and there was no more knowledge on this issue. Clearly, the unboundedness of the ratio $\frac{\dim(G)}{\operatorname{edim}(G)}$ gives as a consequence that $\dim(G) - \operatorname{edim}(G)$ can as large as possible.

While graphs for which $\dim(G) < \operatorname{edim}(G)$ are very common, and they are present in several investigations already published, the opposed version $\dim(G) < \operatorname{edim}(G)$ seemed to be more elusive till now. We have first observed that K_2 is the unique connected simple graph whose edge metric dimension is 0. Since $\dim(K_2) = 1$, K_2 is the smallest graph which has the edge metric dimension smaller than the metric dimension. For non-trivial examples one needs to consider graphs of order at least 10. By exhaustive computer search we found that the smallest possible graphs G (different from K_2) satisfying the inequality $\mathrm{edim}(G) < \dim(G)$, are the five graphs on 10 vertices depicted in Figure 4. Moreover, we have found 61 such graphs of order 11.

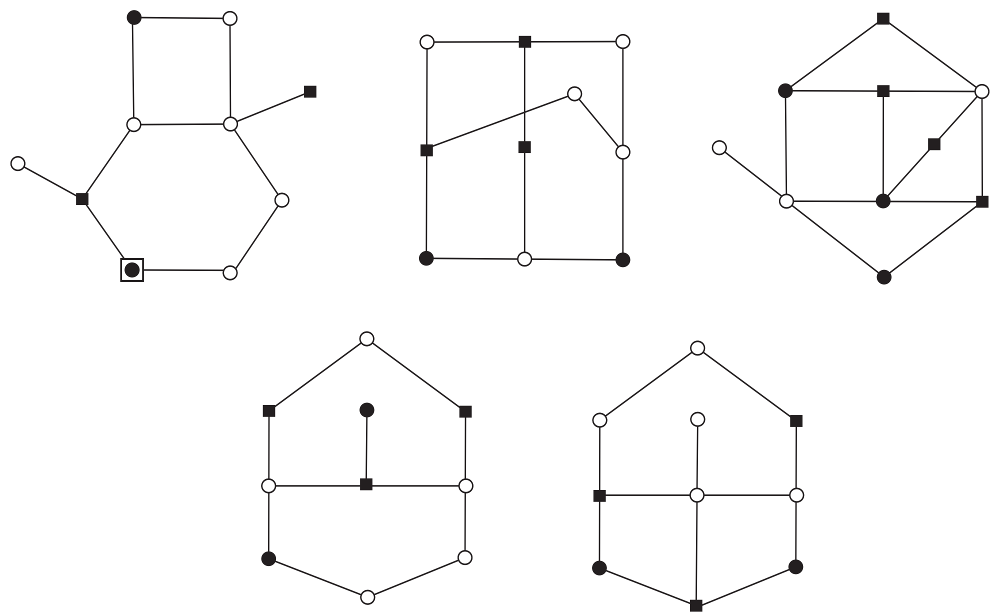

Figure 1: The smallest graphs G for which $\mathrm{edim}(G) < \dim(G)$, different from K_2. The squared vertices form a metric basis and the circled bolded vertices form an edge metric basis.

The main contributions of our work are as follows.

Theorem 1. *Let $k_1, k_2 \geq 2$ and $k_1 \neq k_2$. Then there is an integer n_0 such that for every $n \geq n_0$ there exists a graph on n vertices with $\dim(G) = k_1$ and $\mathrm{edim}(G) = k_2$.*

Theorem 2. *The ratio $\frac{\dim(G)}{\mathrm{edim}(G)}$ is not bounded from above.*

Proof. By Theorem 1, for arbitrarily large N, it is always possible to find a graph G such that $\dim(G) = Nk$ and $\mathrm{edim}(G) = k$. As such, $\frac{\dim(G)}{\mathrm{edim} G}$ can be made arbitrarily large. $\square$

Notice that by using a similar argument as the one in the proof above, although it is already known from [14], we can also prove that the ratio $\frac{\mathrm{edim}(G)}{\dim(G)}$ is not bounded from above.

3 Proof of Theorem 1

In order prove the main result of this work, we shall construct infinite families of graphs G for which $\mathrm{edim}(G) < \dim(G)$ as well as other ones where $\dim(G) < \mathrm{edim}(G)$. To this end, we need first some preliminary results. We first describe two graphs that will be used in this purpose. Take a cycle C on n_1 vertices, where $n_1 \geq 5$. We denote the vertices of C consecutively by $a_1, a_2, \ldots, a_{n_1}$. Further, take a path P on n_2 vertices denoted consecutively by $b_1, b_2, \ldots, b_{n_2}$, where $n_2 \geq 1$, and join P to C by the edge a_2b_1. Then take vertices c and i and connect them by edges to a_{n_1} and a_1, respectively. Finally, take n_3 vertices $j_1, j_2, \ldots, j_{n_3}$, where $n_3 \geq 2$, and join them by edges to the vertex i. We denote the resulting graph by G_{n_1,n_2,n_3}. A fairly representative example of a graph as described above in drawn in Figure 2.

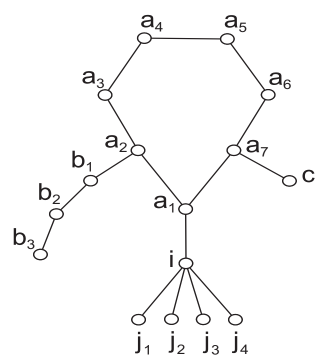

Figure 2: The graph $G_{7,3,4}$

In connection with the graphs G_{n_1,n_2,n_3}, we shall need a graph denoted by G_{n_1,n_2} that is obtained from G_{n_1,n_2,n_3} by removing the vertices of the set $\{i, j_1, j_2, \ldots, j_{n_3}\}$ and the edges incident with them. Thus, G_{n_1,n_2} is a proper subgraph of G_{n_1,n_2,n_3}.

3.1 Preliminaries of the proof

We now prove several auxiliary results about the graph G_{n_1,n_2,n_3}. We remark that we shall be using the graphs G_{n_1,n_2} in most of the described situations.

Observation 3. *Let $n_1 \geq 5$, $n_2 \geq 1$ and $n_3 \geq 2$. Then* $\dim(G_{n_1,n_2,n_3}) \geq n_3$ *and* $\mathrm{edim}(G_{n_1,n_2,n_3}) \geq n_3$.

Proof. Let S be a metric basis of G_{n_1,n_2,n_3}. If S does not contain a vertex of the subgraph G_{n_1,n_2}, then $r(a_2|S) = r(a_{n_1}|S)$. Thus, S contains at least one vertex of G_{n_1,n_2}. Moreover, if S does not contain two pendant vertices j_r and j_t attached to i, then $r(j_r|S) = r(j_t|S)$. Thus, S contains at least $n_3 - 1$ pendant vertices attached to i, and so, in conclusion we get $\dim(G_{n_1,n_2,n_3}) \geq n_3$.

Now let T be an edge metric basis. Here the situation is analogous since $r(a_1a_2|T) = r(a_1a_{n_1}|, T)$ if T does not contain a vertex of G_{n_1,n_2}, and $r(j_ri|T) = r(j_ti|T)$ if j_r and j_t are pendant vertices attached to i and not in T. $\square$

Lemma 4. *Let $n_1 \geq 5$, $n_2 \geq 1$ and $n_3 \geq 2$. Then* $\dim(G_{n_1,n_2,n_3}) \leq n_3 + 1$ *and* $\mathrm{edim}(G_{n_1,n_2,n_3}) \leq n_3 + 1$.

Proof. Let $S = \{j_1, j_2, \ldots, j_{n_3-1}, a_\alpha, a_\beta\}$, where $\alpha = \lfloor \frac{n_1+1}{2} \rfloor$ and $\beta = \lceil \frac{n_1+3}{2} \rceil$. Observe that $d(a_1, a_\alpha) = d(a_1, a_\beta)$ since $d(a_1, a_\alpha) = \lfloor \frac{n_1+1}{2} \rfloor - 1$ and $d(a_1, a_\beta) = n_1 + 1 - \lceil \frac{n_1+3}{2} \rceil$. Moreover,

$d(a_\alpha, a_\beta) = 1$ if n_1 is odd and $d(a_\alpha, a_\beta) = 2$ otherwise. Obviously, S contains n_3+1 vertices. And since $n_1 \geq 5$, we have $\beta < n_1$. Denote $\gamma = d(a_\alpha, a_1) = \lfloor \frac{n_1-1}{2} \rfloor$.

First we show that S is a metric generator of G_{n_1,n_2,n_3}. Since $n_3 \geq 2$, there is a vertex of S outside the subgraph G_{n_1,n_2}. This vertex distinguishes those vertices of G_{n_1,n_2} which are at different distances from a_1, but not those which are at the same distance from a_1. According to the distance from a_1, the vertices of G_{n_1,n_2} are partitioned into nontrivial sets $\{a_2, a_{n_1}\}$, $\{a_3, a_{n_1-1}, b_1, c\}$, $\{a_4, a_{n_1-2}, b_2\}$, etc. Some of these sets are probably smaller since they do not need to contain vertices of both C and P, and n_1 could be even.

Let $\delta = \lfloor \frac{n_1}{2} \rfloor$. For every $v \in V(G_{n_1,n_2})$, denote $\tilde{r}(v) = (d(v, a_\alpha), d(v, a_\beta))$. Then the vertices in the first set of the partition have $\tilde{r}(a_2) = (\gamma - 1, \delta)$ and $\tilde{r}(a_{n_1}) = (\delta, \gamma - 1)$. Since $\gamma \leq \delta$, these pairs are different. Further, the pairs $\tilde{r}$ for vertices in the second set of the partition are $(\gamma-2, \delta-1)$, $(\delta-1, \gamma-2)$, $(\gamma, \delta+1)$, $(\delta+1, \gamma)$, and they are pairwise distinct too. Since the distances to vertices of C decrease while the distances to vertices of P increase, it is obvious that the pair a_α, a_β distinguishes the vertices in the sets of the partition. Hence, S identifies the vertices of G_{n_1,n_2}.

Since a pendant vertex in S is identified trivially, it remains to check the vertices i and j_{n_3}. Obviously, j_{n_3} is the only vertex at distance 2 from the pendant vertices in S and at distance $\gamma + 2$ from a_α. On the other hand, i is the only vertex at distance 1 from the pendant vertices in S. Thus, S is a metric generator of G_{n_1,n_2,n_3}.

Now we show that S is an edge metric generator of G_{n_1,n_2,n_3}. We proceed analogously as in the case for the metric generator. According to the distance from a_1, the vertices of S outside G_{n_1,n_2} partition the edges of G into sets $\{a_1a_2, a_1a_{n_1}\}$, $\{a_2a_3, a_{n_1}a_{n_1-1}, a_2b_1, a_{n_1}c\}$, $\{a_3a_4, a_{n_1-1}a_{n_1-2}, b_1b_2\}$, etc. Again, some of these sets are probably smaller since they do not need to contain the edges of both C and P, and n_1 may be odd. For every $e \in E(G_{n_1,n_2,n_3})$, denote $\tilde{r}(v) = (d(e, a_\alpha), d(e, a_\beta))$. Then the pairs $\tilde{r}$ for edges in the first set of the partition are $(\gamma - 1, \gamma)$ and $(\gamma, \gamma - 1)$. Further, the pairs $\tilde{r}$ for edges in the second set of the partition are $(\gamma-2, \delta-1)$, $(\delta-1, \gamma-2)$, $(\gamma-1, \delta)$, $(\delta, \gamma-1)$ and they are pairwise distinct too. Since the distances to edges of C decrease while the distances to edges of P increase, it is obvious that the pair a_α, a_β distinguishes the edges in the sets of the partition. Hence, S identifies the edges of G_{n_1,n_2}.

Since a pendant edge ij_t is identified trivially if $j_t \in S$, it remains to check the edges ia_0 and ij_{n_3}. Obviously, ij_{n_3} is the only edge at distance 1 from the pendant vertices in S and at distance $\gamma + 1$ from a_α. On the other hand, ia_1 is the only edge at distance 1 from the pendant vertices in S and at distance γ from a_α. Therefore, S is a metric generator of G_{n_1,n_2,n_3} and the proof is completed. □

The next two propositions show that $\dim(G_{n_1,n_2,n_3})$ and $\mathrm{edim}(G_{n_1,n_2,n_3})$ depend on the parity of n_1.

Lemma 5. *Let $n_1 \geq 5$, $n_2 \geq 1$ and $n_3 \geq 2$. Then $\dim(G_{n_1,n_2,n_3}) = n_3$ if n_1 is odd, and $\dim(G_{n_1,n_2,n_3}) = n_3 + 1$ if n_1 is even.*

Proof. First assume that n_1 is odd. Analogously as in the proof of Lemma 4, let $\alpha = \lfloor \frac{n_1+1}{2} \rfloor$, $\gamma = \lfloor \frac{n_1-1}{2} \rfloor$ and $\delta = \lfloor \frac{n_1}{2} \rfloor$. Since n_1 is odd, $\gamma = \delta$. As shown in Observation 3, every metric basis of G_{n_1,n_2,n_3} contains a vertex outside G_{n_1,n_2}. By using the distances, this vertex partitions $V(G_{n_1,n_2})$ into nontrivial sets $P_1 = \{a_2, a_{n_1}\}$, $P_2 = \{a_3, a_{n_1-1}, b_1, c\}$,

$P_3 \subseteq \{a_4, a_{n_1-2}, b_2\}$, etc. For every $v \in V(G_{n_1,n_2})$, we denote $\tilde{r}(v) = d(a_\alpha, v)$. Then the values of $\tilde{r}$ for the vertices in P_1 are $\gamma - 1$ and δ, and they are different. The values of $\tilde{r}$ for vertices in P_2 are $\gamma - 2$, $\delta - 1$, γ and $\delta + 1$, and they are different too. The set P_3 contains vertices with values of $\tilde{r}$ being $\gamma - 3$, $\delta - 2$ and $\gamma + 1$, etc. Hence, to identify the vertices of G_{n_1,n_2} it suffices (and is necessary by the proof of Observation 3) that a metric generator of G_{n_1,n_2,n_3} will contain only one vertex of G_{n_1,n_2}, namely a_α. Hence, $\{j_1, j-2, \ldots, j_{n_3-1}, a_\alpha\}$ is a metric generator of G_{n_1,n_2,n_3}, see the proof of Lemma 4. By Observation 3, we then get $\dim(G_{n_1,n_2,n_3}) = n_3$.

We now assume n_1 is even and proceed analogously as above. Vertices outside G_{n_1,n_2} partition $V(G_{n_1,n_2,n_3})$ into nontrivial sets $P_1 = \{a_2, a_{n_1}\}$, $P_2 = \{a_3, a_{n_1-1}, b_1, c\}$, $P_3 \subseteq \{a_4, a_{n_1-2}, b_2\}$, etc. We denote this partition by $\mathcal{P}$. (Though it is not obvious, this partition is slightly different from the partition for the case when n_1 is odd.) We show that there is not a unique vertex in G_{n_1,n_2} which distinguishes all the vertices inside the sets of $\mathcal{P}$. Let $v \in V(G_{n_1,n_2,n_3})$. By symmetry, it suffices to distinguish four cases:
Case 1: $v \in \{a_1, a_{\frac{n_1+2}{2}}\}$. Then $d(v, a_2) = d(v, a_{n_1})$ and $a_2, a_{n_1} \in P_1$.
Case 2: $v \in \{a_2, b_1, b_2, \ldots, b_{n_2}\}$. Then $d(v, c) = d(v, a_{n_1-1})$ and $a_{n_1-1}, c \in P_2$.
Case 3: $v \in \{a_3, a_4, \ldots, a_{\frac{n_1-2}{2}}\}$. (This set is empty if $n_1 = 6$.) Then again $d(v, c) = d(v, a_{n_1-1})$.
Case 4: $v = a_{\frac{n_1}{2}}$. Then $d(v, b_1) = d(v, a_{n_1-1})$ and $a_{n_1-1}, b_1 \in P_2$.

Since the other possibilities are symmetric, every metric basis of G_{n_1,n_2,n_3} contains at least two vertices of G_{n_1,n_2}. And since every metric basis of G_{n_1,n_2,n_3} contains at least $n_3 - 1$ pendant vertices attached to i, we have $\dim(G_{n_1,n_2,n_3}) \geq n_3 + 1$. Thus, by Lemma 4, $\dim(G_{n_1,n_2,n_3}) = n_3 + 1$. □

We now consider the counterpart of Lemma 5 for the edge metric dimension case.

Lemma 6. *Let $n_1 \geq 5$, $n_2 \geq 1$ and $n_3 \geq 2$. Then $\mathrm{edim}(G_{n_1,n_2,n_3}) = n_3 + 1$ if n_1 is odd, and $\mathrm{edim}(G_{n_1,n_2,n_3}) = n_3$ if n_1 is even.*

Proof. First assume that n_1 is odd. Since $n_3 \geq 2$, every edge metric basis contains a vertex outside of G_{n_1,n_2}, and by using distances, this vertex partitions $E(G_{n_1,n_2,n_3})$ into sets $P_1 = \{a_1a_2, a_1a_{n_1}\}$, $P_2 = \{a_2a_3, a_{n_1}a_{n_1-1}, a_2b_1, a_{n_1}c\}$, $P_3 \subseteq \{a_3a_4, a_{n_1-1}a_{n_1-2}, b_1b_2\}$, etc. We show that there is no vertex in G_{n_1,n_2,n_3} which distinguishes edges inside these sets. Let $v \in V(G_{n_1,n_2,n_3})$. By symmetry, it suffices to distinguish the following four situations.
Case 1: $v = a_1$. Then $d(v, a_1a_2) = d(v, a_1a_{n_1})$ and $a_1a_2, a_1a_{n_1} \in P_1$.
Case 2: $v \in \{a_2, b_1, b_2, \ldots, b_{n_2}\}$. Then $d(v, a_{n_1}a_{n_1-1}) = d(v, a_{n_1}c)$ and $a_{n_1}a_{n_1-1}, a_{n_1}c \in P_2$.
Case 3: $v \in \{a_3, a_4, \ldots, a_{\frac{n_1-1}{2}}\}$. (This set is empty if $n_1 = 5$.) Then again $d(v, a_{n_1}a_{n_1-1}) = d(v, a_{n_1}c)$.
Case 4: $v = a_{\frac{n_1+1}{2}}$. Then $d(v, a_{n_1}a_{n_1-1}) = d(v, a_2b_1)$ and $a_{n_1}a_{n_1-1}, a_2b_1 \in P_2$.

Since the other possibilities are symmetric, every edge metric basis of G_{n_1,n_2,n_3} contains at least two vertices of G_{n_1,n_2}. And since also every edge metric basis of G_{n_1,n_2,n_3} contains at least $n_3 - 1$ pendant vertices attached to i, we have $\mathrm{edim}(G_{n_1,n_2,n_3}) \geq n_3 + 1$. By Lemma 4, we then have $\mathrm{edim}(G_{n_1,n_2,n_3}) = n_3 + 1$.

Now assume that n_1 is even. Analogously to the proof of Lemma 4, let $\alpha = \lfloor \frac{n_1+1}{2} \rfloor$, $\gamma = \lfloor \frac{n_1-1}{2} \rfloor$ and $\delta = \lfloor \frac{n_1}{2} \rfloor$. Since n_1 is even, $\gamma = \delta - 1$. The vertices of an edge metric basis outside G_{n_1,n_2} partition the edges of G_{n_1,n_2,n_3} into sets $P_1 = \{a_1a_2, a_1a_{n_1}\}$, $P_2 = \{a_2a_3, a_{n_1}a_{n_1-1}, a_2b_1, a_{n_1}c\}$, $P_3 \subseteq \{a_3a_4, a_{n_1-1}a_{n_1-2}, b_1b_2\}$, etc. For every $e \in E(G_{n_1,n_2,n_3})$, let us denote $\tilde{r}(e) = d(a_\alpha, e)$. Then the values of $\tilde{r}$ for the edges in P_1 are $\gamma - 1$ and γ and they are different. The values of $\tilde{r}$ for edges in P_2 are $\gamma - 2$, $\delta - 1$, $\gamma - 1$ and δ and they are different too. The values of $\tilde{r}$ for edges in P_3 are $\gamma - 3$, $\delta - 2$ and γ, etc. Hence, in order to identify the edges of G_{n_1,n_2,n_3}, it suffices (and it is indeed necessary by the proof of Observation 3) that an edge metric generator will contain only one vertex of G_{n_1,n_2}, namely a_α. Hence, the set $\{j_1, j_2, \ldots, j_{n_3-1}, a_\alpha\}$ is an edge metric generator of G_{n_1,n_2,n_3}, see the proof of Lemma 4. Therefore, by Observation 3, we get $\mathrm{edim}(G_{n_1,n_2,n_3}) = n_3$. □

3.2 Core of the proof

To obtain the required graphs we are searching for, we will connect several copies of the graph G_{n_1,n_2,n_3} by adding a few edges. To this end, we need the following powerful tool.

Lemma 7. *Let G_1 and G_2 be two graphs which are not paths, such that for any $i \in \{1, 2\}$, the graph G_i contains a metric basis S_i and an edge metric basis T_i satisfying the following conditions.*

(1) *There is $v_1 \in S_1 \cap T_1$.*

(2) *There are $v_2, u_2 \in S_2 \cap T_2$ such that $d_{G_2}(u_2, v_2) \geq d_{G_2}(u_2, z)$ for every $z \in V(G_2)$.*

Let G be a graph obtained by adding the edge v_1v_2 to the disjoint union of the graphs G_1 and G_2. Then, $\dim(G) = \dim(G_1) + \dim(G_2) - 2$ *and* $\mathrm{edim}(G) = \mathrm{edim}(G_1) + \mathrm{edim}(G_2) - 2$. *Moreover, $S = S_1 \cup S_2 - \{v_1, v_2\}$ is a metric basis of G and $T = T_1 \cup T_2 - \{v_1, v_2\}$ is an edge metric basis of G.*

Proof. First observe that for every $z_2 \in V(G_2)$, the set $\big(S_1 - \{v_1\}\big) \cup \{z_2\}$ identifies the vertices of G_1. Analogously, if $z_1 \in V(G_1)$, then the set $\big(S_2 - \{v_2\}\big) \cup \{z_1\}$ identifies the vertices of G_2. Since G_1 and G_2 are not paths, $|S_1| \geq 2$ and $|S_2| \geq 2$. Hence, the set $S = S_1 \cup S_2 - \{v_1, v_2\}$ identifies the vertices of G_1 and it also identifies the vertices of G_2. Thus, to conclude that S is a metric generator, we just may need to consider a pair of vertices x, y with $x \in V(G_1)$ and $y \in V(G_2)$.

By (2), $d_{G_2}(u_2, v_2) \geq d_{G_2}(u_2, z)$ for every $z \in V(G_2)$. Hence, for $y \in V(G_2)$, we have $d_G(u_2, v_2) \geq d_G(u_2, y)$, while for $x \in V(G_1)$ it follows $d_G(u_2, x) \geq d_G(u_2, v_2) + 1$. Thus, clearly $d_G(u_2, x) > d_G(u_2, y)$. Since $u_2 \in S$, we conclude that S identifies all the vertices of G, and so it is a metric generator for G.

Now suppose that S' is a metric generator for G such that $|S'| < |S|$. Then either $|S' \cap V(G_1)| < |S_1| - 1$ or $|S' \cap V(G_2)| < |S_2| - 1$. In the first case $(S' \cap V(G_1)) \cup \{v_1\}$ identifies G_1 which contradicts $\dim(G_1) = |S_1|$, while in the second case $(S' \cap V(G_2)) \cup \{v_2\}$ identifies G_2 which contradicts $\dim(G_2) = |S_2|$. Consequently, such a set S' does not exist, and we obtain that S is a metric basis for G, which means $\dim(G) = \dim(G_1) + \dim(G_2) - 2$.

The situation for the edge metric basis is analogous. The only difference comes by noticing that we may need to consider the edge metric representation of the edge v_1v_2, that is $r(v_1v_2|S)$, but this is unique in G. Observe that if $z_1 \in T_1 - \{v_1\}$ and $z_2 \in T_2 - \{v_2\}$, then $d_G(z_1, v_1v_2) < d_G(z_1, e_2)$ and $d_G(z_2, v_1v_2) < d_G(z_2, e_1)$ for every $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$. □

3.3 Conclusion of the proof

In order to complete the proof of Theorem 1, we make the following construction, that uses Lemma 7.

For some positive integer $\ell \geq 2$, we consider ℓ graphs given as follows. Let $G_1 = G_{n_1,n_2,n_3}$, and let $G_2 = G_3 = \cdots = G_\ell = G_{n_1,1,2}$, with $n_1 \geq 5$, $n_2 \geq 1$ and $n_3 \geq 2$. To distinguish vertices in distinct copies of G_i, if x is a vertex in G_k, $1 \leq k \leq \ell$, then we denote it by x^k. Let $\alpha = \lfloor \frac{n_1+1}{2} \rfloor$. Then $L^\ell_{n_1,n_2,n_3}$ is a graph obtained from the disjoint union $G_1 \cup G_2 \cup \cdots \cup G_\ell$ by adding the edges $a^1_\alpha j^2_1$, $a^2_\alpha j^3_1$, $\ldots$, $a^{\ell-1}_\alpha j^\ell_1$. See Figure 3 for an example.

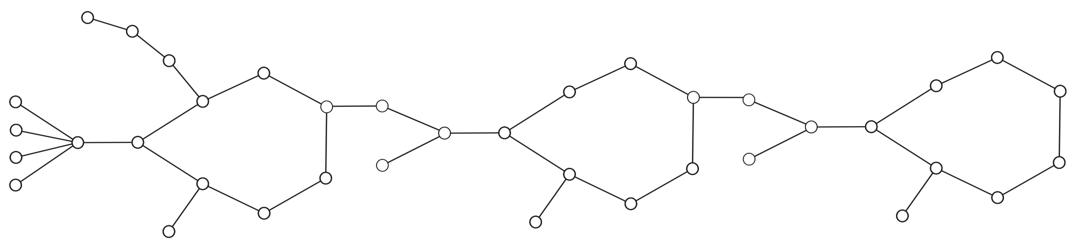

Figure 3: The graph $L^3_{7,3,4}$. Note that G_1 is the graph of Figure 2.

Obviously, $L^\ell_{n_1,n_2,n_3}$ is a connected graph, and if $\ell = 1$, then $L^\ell_{n_1,n_2,n_3}$ is just G_{n_1,n_2,n_3}. We next give some results concerning the metric and edge metric dimensions of such graphs.

Lemma 8. *Let $n_1 \geq 5$, $n_2 \geq 1$, $n_3 \geq 2$ and $\ell \geq 1$. Then the following holds.*

(1) *If n_1 is odd, then* $\dim(L^\ell_{n_1,n_2,n_3}) = n_3$ *and* $\operatorname{edim}(L^\ell_{n_1,n_2,n_3}) = n_3 + \ell$.

(2) *If n_1 is even, then* $\dim(L^\ell_{n_1,n_2,n_3}) = n_3 + \ell$ *and* $\operatorname{edim}(L^\ell_{n_1,n_2,n_3}) = n_3$.

Proof. We only prove the result for the case when n_1 is odd, since the proof for the case when n_1 is even is in fact the same.

Let $\beta = \lceil \frac{n_1+3}{2} \rceil$. As shown in the proof of Lemma 5, $\dim(G_{n_1,n_2,n_3}) = n_3$ and $S = \{j_1, j_2, \ldots, j_{n_3-1}, a_\alpha\}$ is a metric basis of G_{n_1,n_2,n_3}. By Lemma 6, $\operatorname{edim}(G_{n_1,n_2,n_3}) = n_3 + 1$ and $T = \{j_1, j_2, \ldots, j_{n_3-1}, a_\alpha, a_\beta\}$ is an edge metric basis of G_{n_1,n_2,n_3}. We first consider the graphs G_1 and G_2, in concordance with the construction of $L^\ell_{n_1,n_2,n_3}$. Now, for $i \in \{1, 2\}$, denote the metric basis S and an edge metric basis T in G_i by S_i and T_i, respectively. Then $a^1_\alpha \in S_1 \cap T_1$ and $j^2_1, a^2_\alpha \in S_2 \cap T_2$, and moreover, $d_{G_2}(a^2_\alpha, j_1) \geq d_{G_2}(a^2_\alpha, z)$ for every $z \in V(G_2)$. Hence, the graphs G_1 and G_2 satisfy the assumptions of Lemma 7. Thus, $L^2_{n_1,n_2,n_3}$ has metric dimension n_3 and edge metric dimension $n_3 + 2$. Moreover, $S = S_1 \cup S_2 - \{a^1_\alpha, j^2_1\}$ is a metric basis of $L^2_{n_1,n_2,n_3}$ and $T = T_1 \cup T_2 - \{a^1_\alpha, j^2_1\}$ is an edge metric basis of $L^2_{n_1,n_2,n_3}$, for which $a^2_\alpha \in S \cap T$.

Since $L^2_{n_1,n_2,n_3}$ and G_3 satisfy the assumptions of Lemma 7, we can then proceed with $L^2_{n_1,n_2,n_3}$ and G_3 instead of G_1 and G_2, and continue with this process until we reach the largest value of ℓ. This concludes the proof. □

By using the exposition of results above, we are then able to complete the proof of Theorem 1. That is, if $k_1 < k_2$, then $\dim(L^{k_2-k_1}_{5,n_2,k_1}) = k_1$ and $\operatorname{edim}(L^{k_2-k_1}_{5,n_2,k_1}) = k_2$, by Lemma 8. Hence, if $n_0 = |V(L^{k_2-k_1}_{5,1,k_1})|$, then for every $n \geq n_0$ the graph $L^{k_2-k_1}_{5,1+n-n_0,k_1}$ has the required properties.

On the other hand, if $k_1 > k_2$, then $\dim(L^{k_1-k_2}_{6,n_2,k_1}) = k_1$ and $\mathrm{edim}(L^{k_1-k_2}_{6,n_2,k_2}) = k_2$, by Lemma 8. Hence, if $n_0 = |V(L^{k_1-k_2}_{6,1,k_2})|$, then for every $n \geq n_0$ the graph $L^{k_1-k_2}_{6,1+n-n_0,k_2}$ has the required properties. ■

4 Further work

The graphs from Figure 2, together with the graphs $L^{\ell}_{n_1,n_2,n_3}$ defined above, for n_1 even, allow to think that characterizing the whole class of graphs G for which $\mathrm{edim}(G) < \dim(G)$ is a highly challenging problem, since the structures that such graphs can have is rather wide. In this concern, observe also for instance the graphs of Figure 4, which have order 11, and other examples are the already mentioned torus graphs $C_{4r}\Box C_{4t}$.

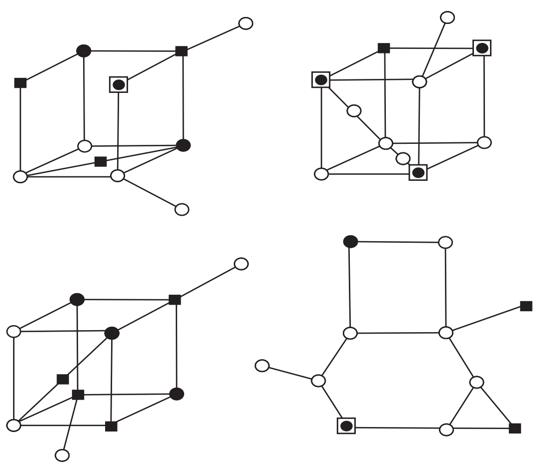

Figure 4: Some graphs G with 11 vertices for which $\dim(G) > \mathrm{edim}(G)$. As in Figure 4, the squared vertices form a metric basis and the circled bolded vertices form an edge metric basis.

Notwithstanding, one could think into characterizing some special families of graphs achieving this property. Thus, some open problems that would be of interest from our point of view are the following ones.

- Characterize the class of unicyclic graphs G for which $\mathrm{edim}(G) < \dim(G)$.
- Characterize all the graphs (or maybe only the unicyclic ones) G for which $\mathrm{edim}(G) = \dim(G) - 1$.
- Characterize all the graphs G for which ($\mathrm{edim}(G) = 2$ and $\dim(G) = 3$) or ($\mathrm{edim}(G) = 3$ and $\dim(G) = 4$).
- Find some necessary and/or sufficient conditions for a connected graph G to satisfy that $\mathrm{edim}(G) < \dim(G)$.

Acknowledgements. The authors acknowledge partial support by Slovak research grants VEGA 1/0142/17, VEGA 1/0238/19, APVV–15–0220, APVV–17–0428, Slovenian research agency ARRS program P1–0383 and project J1-1692.

References

[1] L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford University Press, Oxford (1953).

[2] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood, On the metric dimension of Cartesian products of graphs, *SIAM J. Discrete Math.* **21** (2) (2007) 423–441.

[3] V. Filipović, A. Kartelj, and J. Kratica, Edge metric dimension of some generalized Petersen graphs, *Results Math.* 74 (4) (2019) article # 182.

[4] J. Geneson, Metric dimension and pattern avoidance in graphs, *Discrete Appl. Math.* (2020). In press. DOI: 10.1016/j.dam.2020.03.001

[5] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combin.* **2** (1976) 191–195.

[6] A. Kelenc, N. Tratnik, and I. G. Yero, Uniquely identifying the edges of a graph: the edge metric dimension, *Discrete Appl. Math.* **251** (2018) 204–220.

[7] J. B. Liu, Z. Zahid, R. Nasir, and W. Nazeer, Edge version of metric dimension and doubly resolving sets of the necklace graph, *Mathematics* **6** (11) (2018) article # 243.

[8] R. Nasir, S. Zafar, and Z. Zahid, Edge metric dimension of graphs, *Ars Combin.* In press.

[9] I. Peterin and I. G. Yero, Edge metric dimension of some graph operations, *Bull. Malays. Math. Sci. Soc.* **43** (2020) 2465–2477.

[10] A. Sebő and E. Tannier, On metric generators of graphs, *Math. Oper. Res.* **29** (2) (2004) 383–393.

[11] P. J. Slater, Leaves of trees, *Congr. Numer.* **14** (1975) 549–559.

[12] Y. Zhang and S. Gao, On the edge metric dimension of convex polytopes and its related graphs, *J. Comb. Optim.* **39** (2) (2020) 334-350.

[13] E. Zhu, A. Taranenko, Z. Shao, and J. Xu, On graphs with the maximum edge metric dimension, *Discrete Appl. Math.* **257** (2019) 317–324.

[14] N. Zubrilina, On the edge dimension of a graph, *Discrete Math.* **341** (7) (2018) 2083–2088.