

Proper connection and proper-walk connection of digraphs

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Abstract

An arc-colored digraph D is properly (properly-walk) connected if, for any ordered pair of vertices (u, v) , the digraph D contains a directed path (a directed walk) from u to v such that arcs adjacent on that path (on that walk) have distinct colors. The proper connection number $\vec{pc}(D)$ (the proper-walk connection number $\vec{wpc}(D)$) of a digraph D is the minimum number of colours to make D properly connected (properly-walk connected). We prove that $\vec{pc}(C_n(S)) \leq 2$ for every circulant digraph $C_n(S)$ with $S \subseteq \{1, \dots, n-1\}$, $|S| \geq 2$ and $1 \in S$. Furthermore, we give some sufficient conditions for a Hamiltonian digraph D to satisfy $\vec{pc}(D) = \vec{wpc}(D) = 2$.

Keywords: proper connection; digraph; arc colouring

1 Introduction

In an edge-coloured graph G , a path P is *rainbow* if no two edges in P have the same colour. An edge-coloured graph G is *rainbow connected* if every two vertices of G are connected by a rainbow path, and its colouring is said to be a *rainbow connected colouring*. The *rainbow connection number* of G is the smallest possible number of colours in a rainbow connected colouring of G . The rainbow connection number of graphs was introduced by Chartrand, Johns, McKeon and Zhang in [4].

A weaker version of the rainbow connection number –the proper connection number– was introduced by Borozan *et al.* in [3]. An edge-coloured graph is said to be *properly coloured* if no two adjacent edges share the same colour. A connected edge-coloured graph G is *properly connected* if there exists a properly coloured path between every two vertices in G . The

proper connection number of a connected graph G is the minimum number of colours needed to colour the edges of G to make it properly connected.

The concepts of rainbow connected graphs and properly connected graphs have attracted much attention during the last decade. For more details, the reader can refer to surveys [13, 12] (for rainbow connection) and [10, 11] (for proper connection). Melville and Goddard considered in [16, 17] the analogous concept for walks and trails. For a connected graph G , the *proper-walk (proper-trail) connection number* is the minimum number of colours that one needs in order to get a properly coloured walk (trail) between every two vertices in G .

The rainbow connection, the proper connection and the proper-walk connection numbers of graphs readily extend to digraphs, using arc-colourings instead of edge-colourings and directed paths (directed walks, respectively) instead of the paths (walks, respectively). The study of rainbow connections of digraphs was initiated by Dorbec *et al.* in [5]. Then the rainbow connection number of some digraph classes was determined and different notions similar to the rainbow connection were introduced, such as the strong rainbow connection, the rainbow vertex connection and the rainbow total connection (see [1, 8, 9, 18, 19]).

The directed version of the proper connection was introduced by Magnant *et al.* in [15], and the directed version of the proper-walk connection by Melville and Goddard in [16]. In [15] and in [14] the strong and the vertex version of directed proper connection were considered. In this paper we study the proper connection and the proper-walk connection of digraphs.

An arc-coloured directed path (directed walk) is *properly coloured* if it does not contain two adjacent arcs with the same colour. An arc-coloured digraph D is *properly connected* if, between every ordered pair of vertices, there is a directed properly coloured path. In that case, we say that the corresponding arc-colouring is a properly connected arc-colouring of D . The *proper connection number* of D , denoted by $\vec{pc}(D)$, is the minimum number of colours needed to colour the arcs of D so that D is properly connected.

An arc-coloured digraph D is *properly-walk connected* if, between every ordered pair of vertices, there is a properly coloured directed walk. Again, we say that the corresponding arc-colouring is a properly-walk connected arc-colouring of D . Clearly, every properly connected digraph is also properly-walk connected. The *proper-walk connection number* of D , denoted by $\vec{pw}(D)$, is the minimum number of colours needed to colour the arcs of D so that D is properly-walk connected. Note that in order to admit an arc-colouring which makes it properly (properly-walk) connected, a digraph must be strongly connected.

Magnant *et al.* [15] proved that the proper connection number of every strong digraph is at most 3. This result suggests the problem of characterizing the digraphs whose proper connection number is at most 2. Ducoff *et al.* [6] proved that determining whether $\vec{pc}(D) \leq 2$ is NP-complete for any given digraph D . Gu *et al.* [7] considered the proper connection number of random digraphs. They proved that if the probability p is at least $(\log n + \log \log n + \lambda(n))/n$, then the random digraph $D(n, p)$ satisfies $\vec{pc}(D(n, p)) \leq 2$ with high probability.

The following observation is important for our study. Given two digraphs D_1 and D_2 such that D_1 is a spanning subdigraph of D_2 , if D_1 is properly connected under some arc-colouring then, by using that arc-colouring for the arcs of D_2 , we obtain a partial arc-

colouring which makes D_2 properly connected. Since every even cycle has an arc-colouring with two colours that makes it properly connected, every Hamiltonian digraph of even order has proper connection number at most 2. We will thus focus on Hamiltonian digraphs of odd order.

The motivation for this paper was the theorem proved by Magnant *et al.* in [15] which states that the proper connection number of a strong tournament of order at least 4 is 2. On one other hand, the proper connection number of any odd directed cycle is 3. We can thus try to determine the maximum number of arcs we can remove from an odd tournament while keeping its proper connection number equal to 2. Since a digraph D must be strongly connected to be properly connected, we must assume that, when removing arcs from D , the resultant digraph is still strongly connected. In this paper we will assume that the resulting digraph has a directed cycle going through all its vertices, i.e., that the resultant digraph is Hamiltonian.

Our paper is organised as follows. We introduce definitions and notation in Section 2 and give some preliminary results in Section 3. In Section 4, we prove that if D is a Hamiltonian digraph such that either (i) for every vertex v there is an even chord with a tail in v , or (ii) for every vertex v there is an even chord with a head in v , then the proper connection number of D is at most 2. We also prove that $\vec{pc}(C_n(S)) \leq 2$ for every circulant digraph $C_n(S)$ with $S \subseteq \{1, \dots, n-1\}$, $|S| \geq 2$ and $1 \in S$. This result implies a theorem proved in [14], which states that $\vec{pc}(C_n([k])) = 2$ whenever $k \neq n-1$ and $k \neq 1$. In Section 5, we give some sufficient conditions for a Hamiltonian digraph to have proper-walk connection number equal to 2. We conclude the paper with some open questions in Section 6.

2 Definitions and notation

All digraphs in this paper are simple in the following sense: they are loopless, they do not contain parallel arcs, but opposite arcs are allowed. For a given digraph D , we denote by $V(D)$ and $A(D)$ its set of vertices and its set of arcs, respectively. Two arcs xy and zt in D are said to be *consecutive* if $y = z$. Given an arc xy in D , we say that y is an *out-neighbour* of x , while x is an *in-neighbour* of y . Moreover, x is the *tail* of xy and y the *head* of xy . The *out-degree* $d_D^+(x)$ of x in D is the number of arcs with the tail in x and the *in-degree* $d_D^-(x)$ of x in D is the number of arcs with the head in x . The *degree* $d_D(x)$ of x in D is the number of arcs incident with x , $d_D(x) = d_D^+(x) + d_D^-(x)$.

Let D be a digraph. For an arc xy in $A(D)$, we denote by $D - xy$ the digraph $D - xy = (V(D), A(D) \setminus \{xy\})$. For a vertex u in $V(D)$, we denote by $D - u$ the digraph $D - u = (V(D) \setminus \{u\}, (A(D) \setminus (\{u\} \times V(D)) \setminus (V(D) \times \{u\}))$. For a digraph D' , we denote by $D \cup D'$ the digraph $D \cup D' = (V(D) \cup V(D'), A(D) \cup A(D'))$.

For a digraph D and a set of vertices $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by S , that is, $D[S] = (S, A(D) \cap (S \times S))$.

A *walk* of length $k \geq 1$ in a digraph D is a sequence $x_0 \dots x_k$ of vertices such that $x_i x_{i+1} \in A(D)$ for every i , $0 \leq i \leq k-1$. A *path* is a walk in which no vertex appears twice. Such a path (a walk), going from x_0 to x_k , is referred to as an $x_0 x_k$ -*path* (an $x_0 x_k$ -*walk*). Let

$P = x_0 \dots x_k$ be an x_0x_k -path and $xx_0 \in A(D)$ ($x_kx' \in A(D)$). By xx_0P (Px_kx') we denote the path $xx_0 \dots x_k$ ($x_0 \dots x_kx'$).

An *ear* in a digraph D is an xy -path Q such that $d_D(x) > 2$, $d_D(y) > 2$ and $d_D(z) = 2$ for every internal vertex z of Q .

A digraph D is *strongly connected* (*strong*, for short) if, for every ordered pair of vertices (u, v) , there exists a uv -path in D .

A *cycle* of length $k \geq 1$ in a digraph D is a sequence $x_0 \dots x_kx_0$ of vertices such that $x_i x_{i+1} \in A(D)$ for every i , $0 \leq i \leq k$ (subscripts are taken modulo k), and $x_i \neq x_j$ for every i, j , $0 \leq i < j \leq k$. Since cycles are denoted similarly all along the paper, it is taken for granted that subscripts are always taken modulo the length of the cycle.

For a path P , we denote by $|P|$ the length of P , i.e., $|P| = |A(P)|$. For a cycle C , we denote by $C[x_i, x_j]$ the $x_i x_j$ -path contained in C (that is, $C[x_i, x_j] = x_i x_{i+1} \dots x_j$).

The *distance* from a vertex x to a vertex y in a digraph D , denoted by $\text{dist}_D(x, y)$, is the length of a shortest xy -path in D (if there is no such path, we let $\text{dist}_D(x, y) = \infty$).

Let $C = x_0 \dots x_{n-1}x_0$ be a cycle of length n . A *chord* of C is an arc $x_p x_q$, $0 \leq p, q \leq n-1$, with $x_q \neq x_{p+1}$. The *length* of the chord $x_p x_q$ is $|C[x_p, x_q]|$. The chord $x_p x_q$ is *even* if $|C[x_p, x_q]|$ is even, otherwise, the chord $x_p x_q$ is *odd*.

Let D be a Hamiltonian digraph, C be a Hamiltonian cycle of D , and u, v be two adjacent vertices of D . If uv is an arc of C then we write $u \rightarrow v$; if uv is a chord of C , then we write $u \curvearrowright v$. For a path $x_0 x_1 \dots x_{k-1} x_k$, we write $x_0 \rightsquigarrow x_1 \rightsquigarrow \dots \rightsquigarrow x_{k-1} \rightsquigarrow x_k$, where \rightsquigarrow stands either for \rightarrow or \curvearrowright , depending on the type of the corresponding arc.

Given a graph $G = (V, E)$, the *biorientation* of G is the symmetric digraph \overleftrightarrow{G} obtained from G by replacing each edge uv of G by the pair of symmetric arcs uv and vu .

For an integer $n \geq 3$ and a set $S \subseteq \{1, 2, \dots, n-1\}$, the *circulant digraph* $C_n(S)$ is the digraph with vertex set $V(C_n(S)) = \{v_0, v_1, \dots, v_{n-1}\}$ and arc set $A(C_n(S)) = \{v_i v_j : j - i \equiv s \pmod{n}, s \in S\}$.

3 Preliminaries

For every digraph D , we denote by D^{-1} the *reversed digraph* of D , that is, $V(D^{-1}) = V(D)$ and $xy \in A(D^{-1})$ if and only if $yx \in A(D)$. It directly follows from the definitions that if an arc-colouring λ makes D properly connected (properly-walk connected), then the colouring λ' defined by $\lambda'(xy) = \lambda(yx)$ for every arc $xy \in A(D^{-1})$ makes D^{-1} properly connected (properly-walk connected). Hence, we have the following observation.

Observation 1 *For every digraph D , $\overrightarrow{pc}(D^{-1}) = \overrightarrow{pc}(D)$ and $\overleftarrow{pc}(D^{-1}) = \overleftarrow{pc}(D)$.*

Magnant *et al.* [15] proved the following theorem.

Theorem 2 [15] *If D is a strong digraph, then $\overrightarrow{pc}(D) \leq 3$.*

The only digraphs with proper connection number equal to 1 are biorientations of complete graphs \overleftrightarrow{K}_n . Since it is NP-complete to decide whether a strong digraph has proper

connection number at most 2 [6], it seems to be interesting to find some sufficient conditions for a digraph to have this property.

One can see that every bipartite strong digraph, except \overleftrightarrow{K}_2 , has proper connection number equal to 2. Indeed, let $D = (X \cup Y, A)$ be a strong bipartite digraph. If we colour all arcs with tail in X with colour 1 and all arcs with tail in Y with colour 2, then we clearly obtain a properly connected digraph.

Proposition 3 *If D is a strong bipartite digraph, $D \neq \overleftrightarrow{K}_2$, then $\overrightarrow{pc}(D) = 2$.*

If D has a spanning subdigraph H with $\overrightarrow{pc}(H) \leq 2$, then $\overrightarrow{pc}(D) \leq 2$. We thus obtain the following corollary.

Corollary 4 *If a digraph D has a strong bipartite spanning subdigraph, then $\overrightarrow{pc}(D) \leq 2$.*

In the next theorem we give another family of digraphs having proper connection number at most 2. Consequently, every digraph which contains a spanning digraph belonging to this family also has proper connection number at most 2.

Theorem 5 *Let D be a strong digraph. If there is a partition $V(D) = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, such that $D[V_1]$ is strong bipartite and V_2 is an independent set of vertices, then $\overrightarrow{pc}(D) \leq 2$.*

Proof. We first consider the subdigraph $D[V_1]$. Let $V_1 = V'_1 \cup V''_1$ be the partition of V_1 into two independent sets. We then colour the arcs with tail in V'_1 with 1, and the arcs with tail in V''_1 with 2. Observe that every path linking two vertices in $D[V_1]$ is properly coloured and, moreover, the first colour of every path starting from V'_1 (resp. from V''_1) is 1 (resp. 2) and the last colour of every path ending in V'_1 (resp. in V''_1) is 2 (resp. 1).

We now colour the remaining arcs of D as follows. Let xy be an arc with $x \in V_1$ and $y \in V_2$. We then colour xy with colour 1 if $x \in V'_1$ and with colour 2 if $x \in V''_1$. On the other hand, if xy is an arc with $x \in V_2$ and $y \in V_1$, then we colour xy with colour 1 if $y \in V''_1$ and with colour 2 if $y \in V'_1$. Observe that every directed path of length 2 starting from V_2 or ending in V_2 is properly coloured.

We now claim that the so-constructed arc-colouring makes D properly connected. Let (x, y) be any ordered pair of distinct vertices in D . Let x' be any out-neighbour of x in V_1 if $x \in V_2$ (such a vertex necessarily exists since D is strong), and $x' = x$ otherwise. Similarly, let y' be any in-neighbour of y in V_1 if $y \in V_2$ (again, such a vertex necessarily exists since D is strong), and $y' = y$ otherwise. From the above observations, we get that for any path P in $D[V_1]$ from x' to y' (with possibly $x' = y'$, in which case P is the empty path), the path xPy is necessarily properly coloured. Therefore, the so-constructed arc-colouring makes D properly connected, and thus $\overrightarrow{pc}(D) \leq 2$. ■

Corollary 6 *If a digraph D has a spanning subdigraph satisfying the conditions of Theorem 5, then $\overrightarrow{pc}(D) \leq 2$.*

4 Proper connection of digraphs

In this section we consider the proper connection number of Hamiltonian digraphs. Observe that if a Hamiltonian digraph D has even order, then it has a spanning even cycle, which has proper connection number 2, and it follows that $\vec{pc}(D) \leq 2$. We thus focus on Hamiltonian digraphs of odd order. First we consider Hamiltonian digraphs having even chords.

Theorem 7 *Let D be a Hamiltonian digraph of odd order and C be a Hamiltonian cycle of D . If every vertex of D is the tail of an even chord of C , then $\vec{pc}(D) \leq 2$.*

Proof. We will show that D contains a strong bipartite spanning subdigraph. Let $C = x_0 \dots x_{n-1}x_0$. Renaming vertices if necessary, we may assume that x_px_0 , for some p , $1 \leq p \leq n-2$, is an even chord of C with the minimum value of $|C[x_p, x_0]|$.

Let $V_1 \cup V_2$ be the partition of $V(D)$ given by $V_1 = \{x_{2i} : i \in \{0, \dots, (n-1)/2\}\}$ and $V_2 = \{x_{2i+1} : i \in \{0, \dots, (n-3)/2\}\}$. We claim that there is a strong bipartite spanning subdigraph of D with vertex partition (V_1, V_2) . Observe that every arc of $A(C) \setminus \{x_{n-1}x_0\}$ joins two vertices from different sets of the partition. Furthermore, every even chord x_ix_j such that $x_{n-1}x_0 \subseteq C[x_i, x_j]$ also joins two vertices from different sets of the partition. Our initial assumption that $|C[x_p, x_0]|$ is minimum implies that, for every even chord x_rx_s with $x_r \in C[x_{p+1}, x_{n-1}]$, we have $x_s \in C[x_1, x_{r-1}]$, and thus $x_{n-1}x_0 \subseteq C[x_r, x_s]$. Therefore, every such chord joins vertices of two different sets of the partition.

Let A' be the set of even chords with tail in $C[x_{p+1}, x_{n-1}]$. Then the spanning subdigraph D^* , with set of arcs $A(D^*) = A(C) \setminus \{x_{n-1}x_0\} \cup \{x_px_0\} \cup A'$, is bipartite. We finally claim that D^* is a strong subdigraph. Note that $x_0x_1 \dots x_px_0$ is a cycle and $x_px_{p+1} \dots x_{n-1}$ a path in D^* . It thus suffices to prove that there is a path from any vertex in $C[x_{p+1}, x_{n-1}]$ to v_p . Indeed, this follows from the fact that for every vertex x_r in $C[x_{p+1}, x_{n-1}]$, there is an arc x_rx_s in A' with $s < r$. ■

Thanks to Observation 1, and the fact that an even chord of some cycle C in a digraph D is an even chord of the “reversed cycle” C^{-1} in D^{-1} , we get the following corollary.

Corollary 8 *Let D be a Hamiltonian digraph of odd order and C be a Hamiltonian cycle of D . If every vertex of D is the head of an even chord of C , then $\vec{pc}(D) \leq 2$.*

Now, from Theorem 7 and Corollary 8, we get the following corollary.

Corollary 9 *Let D be a Hamiltonian digraph and C be a Hamiltonian cycle of D . If C contains only even chords and $d_D^-(v) \geq 2$ for every $v \in V(D)$ (or $d_D^+(v) \geq 2$ for every $v \in V(D)$), then $\vec{pc}(D) = 2$.*

In the rest of this section we consider circulant digraphs and prove the following theorem.

Theorem 10 *If $n \geq 4$, $S \subseteq \{1, \dots, n-1\}$, $|S| \geq 2$, and $1 \in S$, then $\vec{pc}(C_n(S)) \leq 2$.*

Observe first that Theorem 10 obviously holds for n even since $C_n(S)$ is Hamiltonian. When n is odd, Theorem 7 and Corollary 8 give the following result.

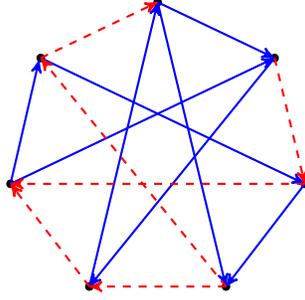


Figure 1: A properly connected 2-arc-colouring of $C_7(\{1, 3\})$.

Corollary 11 *If n is odd, $1 \in S$, and S contains an even integer, then $\vec{pc}(C_n(S)) \leq 2$.*

Thus to complete the proof of Theorem 10 it is enough to show that for every two odd integers n and k , $n \geq 5$, $3 \leq k < n$, we have $\vec{pc}(C_n(\{1, k\})) = 2$. To prove this result we need the following lemma, which has been proved in [6], using a construction that we recall in Figure 1.

Lemma 12 [6] $\vec{pc}(C_7(\{1, 3\})) = 2$.

Observe that $C_7(\{1, 3\})$ and $C_7(\{1, 5\})$ are isomorphic graphs, so that we also have $\vec{pc}(C_7(\{1, 5\})) = 2$. We are now able to deal with the remaining cases of Theorem 10.

Lemma 13 *If n and k are odd, $n \geq 5$ and $3 \leq k \leq n - 1$, then $\vec{pc}(C_n(\{1, k\})) = 2$.*

Proof. For $n = 5$, there is exactly one circulant digraph with odd k , namely $C_5(\{1, 3\})$. Observe that $C_5(\{1, 3\})$ is a strong tournament. Since the proper connection number of every strong tournament with at least four vertices is 2 (see [15]), we get $\vec{pc}(C_5(\{1, 3\})) \leq 2$ and the lemma holds in this case.

For $n = 7$, the result follows from Lemma 12 and the fact that $C_7(\{1, 3\})$ and $C_7(\{1, 5\})$ are isomorphic. Thus we may assume $n \geq 9$. We will show that, in that case, the digraph $C_n(\{1, k\})$ contains a strong bipartite spanning subdigraph, which will imply the desired result. This subdigraph will be constructed step by step, starting from an even cycle, and adding ears in such a way that the so-obtained subdigraph is still bipartite, until we get a spanning subdigraph. Doing so, the constructed subdigraph will clearly be strong.

The bipartition of the subdigraph will be given by means of a vertex 2-colouring c , simply referred to as a 2-colouring in the rest of the proof.

Let $V(C_n(\{1, k\})) = \{x_0, \dots, x_{n-1}\}$ and $n = \alpha k + r$, with $0 \leq r \leq k - 1$ and $\alpha \in \mathbb{N}$. We consider three cases, depending on the value of k .

Case 1. $3k - 2 \leq n$.

We consider three subcases, depending on the value of r .

Subcase 1.1. $r \leq k - 3$.

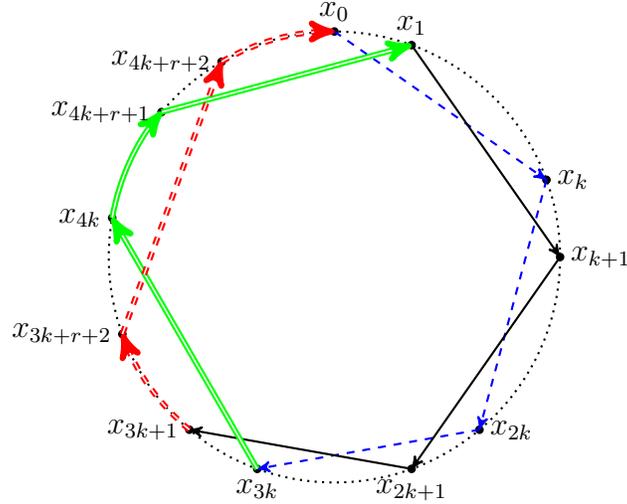


Figure 2: (Subcase 1.1) A sample digraph with $\alpha = 5$, together with the paths P_1 (dashed, blue), P_2 (black), P_3 (double, green) and P_4 (double dashed, red).

In that case, we necessarily have $\alpha \geq 3$. Consider the four following paths (see Figure 2):

$$\begin{aligned}
P_1 &= x_0 \curvearrowright x_k \curvearrowright x_{2k} \curvearrowright \dots \curvearrowright x_{(\alpha-2)k}, \\
P_2 &= x_1 \curvearrowright x_{k+1} \curvearrowright x_{2k+1} \curvearrowright \dots \curvearrowright x_{(\alpha-2)k+1}, \\
P_3 &= x_{(\alpha-2)k} \curvearrowright x_{(\alpha-1)k} \rightarrow \dots \rightarrow x_{(\alpha-1)k+r+1} \curvearrowright x_1, \\
P_4 &= x_{(\alpha-2)k+1} \rightarrow \dots \rightarrow x_{(\alpha-2)k+r+2} \curvearrowright x_{(\alpha-1)k+r+2} \rightarrow \dots \rightarrow x_0.
\end{aligned}$$

Observe that $C = P_1 \cup P_2 \cup P_3 \cup P_4$ is a cycle. Furthermore, since $|P_1| = |P_2| = \alpha - 2$, $|P_3| = r + 3$ and $|P_4| = k + r$, we get $|A(C)| = 2\alpha + 2r + k - 1$, and thus C is an even cycle. Let c be a 2-colouring of C with $c(x_1) = 1$.

The path $P_2 \cup P_4$ goes from x_1 to x_0 in C , thus $d_C(x_1, x_0) = \alpha + k + r - 2$. Since $n = \alpha k + r$ is odd and k is odd, α and r are of different parity. Hence, $d_C(x_1, x_0)$ is even and thus $c(x_0) = c(x_1) = 1$. More generally, we have $c(x_{ik}) = c(x_{ik+1}) = 1$ for $i \in \{0, \dots, \alpha - 2\}$, i even, and $c(x_{ik}) = c(x_{ik+1}) = 2$ for $i \in \{0, \dots, \alpha - 2\}$, i odd.

We are now ready to add ears to the cycle C in order to get a spanning subdigraph of $C_n(\{1, k\})$. We consider the following paths of $C_n(\{1, k\})$:

$$\begin{aligned}
Q_i &= x_{ik+2} \rightarrow x_{ik+3} \rightarrow \dots \rightarrow x_{(i+1)k-1}, \text{ for every } i \in \{0, \dots, \alpha - 3\}, \\
Q &= x_{(\alpha-2)k+r+3} \rightarrow x_{(\alpha-2)k+r+4} \rightarrow \dots \rightarrow x_{(\alpha-1)k-1}.
\end{aligned}$$

Observe that $V(C_n(\{1, k\})) = V(C) \cup \bigcup_{i \in \{0, \dots, \alpha-3\}} V(Q_i) \cup V(Q)$. Moreover, since k is odd, the length of each path Q_i , $0 \leq i \leq \alpha - 3$, is even.

We will add to the cycle C an ear E_0 containing Q_0 and then, sequentially, we will add to $C \cup E_0 \cup \dots \cup E_{i-1}$ an ear E_i containing Q_i , for each i , $1 \leq i \leq \alpha - 3$. (We will add later an ear E containing Q to $C \cup E_0 \cup \dots \cup E_{\alpha-3}$). Since each path Q_i has even length, the ends of each ear E_i must have the same colour. We proceed as follows.

- We first let $E_0 = x_{(\alpha-1)k+r+2} \curvearrowright Q_0 \rightarrow x_k$. As observed above, we have $c(x_k) = 2$. Moreover, since $d_C(x_{(\alpha-1)k+r+2}, x_0) = k-2$ and $c(x_0) = 1$, we also have $c(x_{(\alpha-1)k+r+2}) = 2$. Hence, $C \cup E_0$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ to which we can extend the 2-colouring c . In particular, we have $c(x_2) = 1$.
- We now let $E_1 = x_2 \curvearrowright Q_1 \rightarrow x_{2k}$. Since $c(x_2) = c(x_{2k}) = 1$, $C \cup E_0 \cup E_1$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ to which we can extend the 2-colouring c . Note here that we have $c(x_{k+2}) = 2$.
- Assume finally that we have added ears E_0, \dots, E_{i-1} , for some $i < \alpha - 3$, in such a way that $C \cup E_0 \cup \dots \cup E_{i-1}$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ and the corresponding 2-colouring c is such that $c(x_{jk+2}) = 1$ if j is even and $c(x_{jk+2}) = 2$ if j is odd, for every $j \in \{0, \dots, i-1\}$.

We then let $E_i = x_{(i-1)k+2} \curvearrowright Q_i \rightarrow x_{(i+1)k}$. Since $c(x_{(i-1)k+2}) = c(x_{(i+1)k}) = 1$ if i is odd, and $c(x_{(i-1)k+2}) = c(x_{(i+1)k}) = 2$ if i is even, $C \cup E_0 \cup \dots \cup E_i$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ to which we can extend the 2-colouring c . Note that we then have $c(x_{ik+2}) = 1$ if i is even and $c(x_{ik+2}) = 2$ if i is odd.

We finally let $E = x_{(\alpha-2)k+r+2} \rightarrow Q \curvearrowright x_{\alpha k-1}$. In order to obtain a bipartite subdigraph, we need to have $c(x_{(\alpha-2)k+r+2}) = c(x_{\alpha k-1})$ if $|E|$ is even, and $c(x_{(\alpha-2)k+r+2}) \neq c(x_{\alpha k-1})$ if $|E|$ is odd. To prove that this holds, it is enough to show that $|E|$ and $d_C(x_{(\alpha-2)k+r+2}, x_{\alpha k-1})$ have the same parity. Indeed, we have $|E| = k - r - 2$ and

$$\begin{aligned} d_C(x_{(\alpha-2)k+r+2}, x_{\alpha k-1}) &= d_{P_4}(x_{(\alpha-2)k+r+2}, x_{\alpha k-1}) \\ &= 1 + d_{P_4}(x_{(\alpha-1)k+r+2}, x_{\alpha k-1}) = k - r - 2, \end{aligned}$$

which completes this subcase.

Subcase 1.2. $r = k - 1$.

In that case, we have $n = \alpha k + k - 1$. Since n and k are odd, it follows that α is odd and thus $\alpha \geq 3$. Consider the four following paths (see Figure 3):

$$\begin{aligned} P_1 &= x_0 \curvearrowright x_k \curvearrowright \dots \curvearrowright x_{\alpha k}, \\ P_2 &= x_1 \curvearrowright x_{k+1} \curvearrowright \dots \curvearrowright x_{\alpha k+1}, \\ P_3 &= x_{\alpha k} \curvearrowright x_1, \\ P_4 &= x_{\alpha k+1} \rightarrow x_{\alpha k+2} \rightarrow \dots \rightarrow x_{\alpha k+k-2} \rightarrow x_0. \end{aligned}$$

Observe that $C = P_1 \cup P_2 \cup P_3 \cup P_4$ is a cycle. Furthermore, since $|P_1| = |P_2| = \alpha$, $|P_3| = 1$ and $|P_4| = k - 2$, we get $|A(C)| = 2\alpha + k - 1$, and thus C is an even cycle. Let c be a 2-colouring of C with $c(x_1) = 1$.

The path $P_1 \cup P_3$ goes from x_0 to x_1 in C , thus $d_C(x_0, x_1) = |P_1 \cup P_3| = \alpha + 1$. This implies $c(x_1) = c(x_0)$ and thus $c(x_{ik}) = c(x_{ik+1}) = 1$ for $i \in \{0, \dots, \alpha\}$, i even, and $c(x_{ik}) = c(x_{ik+1}) = 2$ for $i \in \{0, \dots, \alpha\}$, i odd.

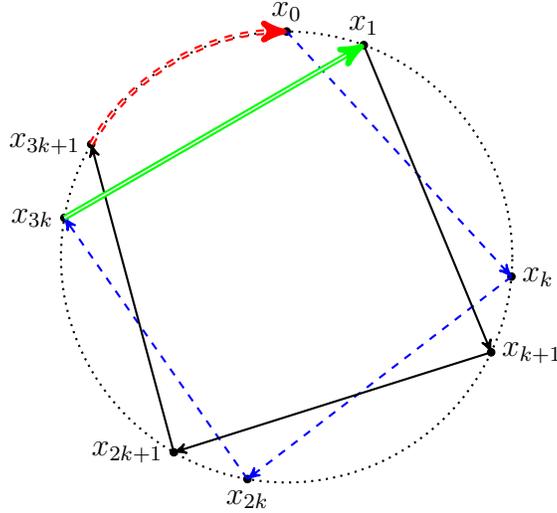


Figure 3: (Subcase 1.2) A sample digraph with $\alpha = 3$, together with the paths P_1 (dashed, blue), P_2 (black), P_3 (double, green) and P_4 (double dashed, red).

We now consider the following paths of $C_n(\{1, k\})$:

$$Q_i = x_{ik+2} \rightarrow x_{ik+3} \rightarrow \dots \rightarrow x_{(i+1)k-1} \text{ for every } i \in \{0, \dots, \alpha - 1\}.$$

Observe that $V(C_n(\{1, k\})) = V(C) \cup \bigcup_{i \in \{0, \dots, \alpha-1\}} V(Q_i)$. Similarly as before, we will add to the cycle C an ear E_0 containing Q_0 and then, sequentially, we will add to $C \cup E_0 \cup \dots \cup E_{i-1}$ an ear E_i containing Q_i , for each i , $1 \leq i \leq \alpha - 1$. Since every path Q_i has even length, the ends of each ear E_i must have the same colour. We proceed as follows.

- We first let $E_0 = x_{\alpha k+1} \curvearrowright Q_0 \rightarrow x_k$. Since $d_C(x_{\alpha k+1}, x_k) = |P_4 \curvearrowright x_k| = k - 1$ is even, we have $c(x_{\alpha k+1}) = c(x_k)$. Hence, $C \cup E_0$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ to which we can extend the 2-colouring c . In particular, since $c(x_{\alpha k+1}) = 2$ and $x_{\alpha k+1}x_2 \in A(C \cup E_0)$, we have $c(x_2) = 1$.
- Assume now that we have added ears E_0, \dots, E_{i-1} , for some $i < \alpha - 1$, in such a way that $C \cup E_0 \cup \dots \cup E_{i-1}$ is a strong bipartite subdigraph of $C_n(\{1, k\})$, and the corresponding 2-colouring c is such that $c(x_{jk+2}) = 1$ if j is even and $c(x_{jk+2}) = 2$ if j is odd, for every $j \in \{0, \dots, i - 1\}$.

We then let $E_i = x_{(i-1)k+2} \curvearrowright E_i \rightarrow x_{(i+1)k}$. Since $c(x_{(i-1)k+2}) = c(x_{(i+1)k}) = 1$ if i is odd and $c(x_{(i-1)k+2}) = c(x_{(i+1)k}) = 2$ if i is even, $C \cup E_0 \cup \dots \cup E_i$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ to which we can extend the 2-colouring c . Note that we then have $c(x_{ik+2}) = 1$ if i is even and $c(x_{ik+2}) = 2$ if i is odd.

The so-obtained subdigraph $C \cup E_0 \cup \dots \cup E_{i_{\alpha-1}}$ is a strong bipartite spanning subdigraph of $C_n(\{1, k\})$, which completes this subcase.

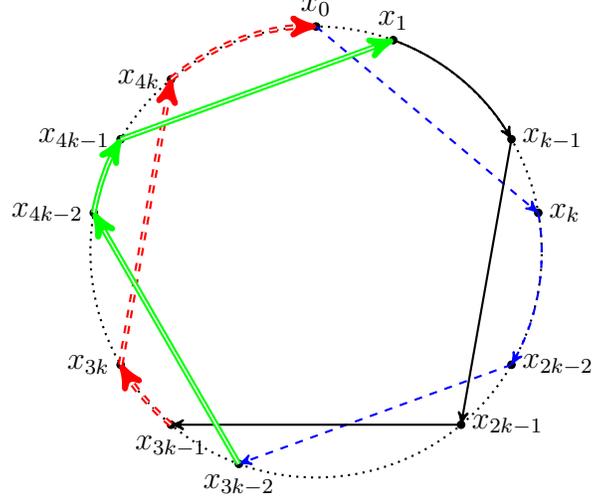


Figure 4: (Subcase 1.3.1) A sample digraph with $\alpha = 4$, together with the paths P_1 (dashed, blue), P_2 (black), P_3 (double, green) and P_4 (double dashed, red).

Subcase 1.3. $r = k - 2$.

In this case, we have $n = \alpha k + k - 2$. Since n and k are odd, it follows that α is even. We consider two subcases, depending on the value of α .

Subcase 1.3.1. $\alpha \geq 4$ (α even).

Consider the four following paths (see Figure 4):

$$\begin{aligned}
 P_1 &= x_0 \curvearrowright x_k \rightarrow \dots \rightarrow x_{2k-2} \curvearrowright x_{3k-2} \curvearrowright \dots \curvearrowright x_{(\alpha-1)k-2}, \\
 P_2 &= x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-1} \curvearrowright x_{2k-1} \curvearrowright x_{3k-1} \curvearrowright \dots \curvearrowright x_{(\alpha-1)k-1}, \\
 P_3 &= x_{(\alpha-1)k-2} \curvearrowright x_{\alpha k-2} \rightarrow x_{\alpha k-1} \curvearrowright x_1, \\
 P_4 &= x_{(\alpha-1)k-1} \rightarrow x_{(\alpha-1)k} \curvearrowright x_{\alpha k} \rightarrow x_{\alpha k+1} \rightarrow \dots \rightarrow x_{\alpha k+k-3} \rightarrow x_0.
 \end{aligned}$$

Observe that $C = P_1 \cup P_2 \cup P_3 \cup P_4$ is a cycle. Furthermore, since $|P_1| = |P_2| = \alpha + k - 4$, $|P_3| = 3$ and $|P_4| = k$, we get $|A(C)| = 2\alpha + 3k - 5$, and thus C is an even cycle. Let c be a 2-colouring of C with $c(x_0) = 1$.

We first claim that $c(x_{ik-2}) = c(x_{ik-1}) = 1$ if i is even and $c(x_{ik-2}) = c(x_{ik-1}) = 2$ if i is odd for every $i \in \{2, \dots, \alpha - 1\}$. Indeed, the path $P_1 \cup P_3$ goes from x_0 to x_1 in C , and thus $d_C(x_0, x_1) = |P_1 \cup P_3| = \alpha + k - 1$. Since α is even, $d_C(x_0, x_1)$ is even and thus $c(x_0) = c(x_1) = 1$. Since $c(x_0) = 1$ and $x_0 x_k \in A(P_1)$, we have $c(x_k) = 2$. Since $c(x_1) = 1$ and $d_{P_2}(x_1, x_{k-1}) = k - 2$ is odd, we have $c(x_{k-1}) = 2$. Since $c(x_{k-1}) = 2$ and $x_{k-1} x_{2k-1} \in A(P_2)$, we have $c(x_{2k-1}) = 1$. Since $c(x_k) = 2$ and $d_{P_1}(x_k, x_{2k-2}) = k - 2$ is odd, we have $c(x_{2k-2}) = 1$. Eventually, we get $c(x_{ik-2}) = c(x_{ik-1}) = 1$ if i is even and $c(x_{ik-2}) = c(x_{ik-1}) = 2$ if i is odd, for every $i \in \{2, \dots, \alpha - 1\}$.

We now consider the following paths of $C_n(\{1, k\})$:

$$\begin{aligned} Q_i &= x_{ik} \rightarrow x_{i(k+1)} \rightarrow \dots \rightarrow x_{(i+1)(k-3)} \text{ for every } i \in \{2, \dots, \alpha - 2\}, \\ Q &= x_{(\alpha-1)(k+1)} \rightarrow x_{(\alpha-1)(k+2)} \rightarrow \dots \rightarrow x_{\alpha k-3}. \end{aligned}$$

Observe that $V(C_n(\{1, k\})) = V(C) \cup \bigcup_{i \in \{2, \dots, \alpha-2\}} V(Q_i) \cup V(Q)$. Again, we will add to the cycle C an ear E_2 containing Q_2 and then, sequentially, we will add to $C \cup E_2 \cup \dots \cup E_{i-1}$ an ear E_i containing Q_i , for each i , $3 \leq i \leq \alpha - 2$. (We will add later an ear E containing Q to $C \cup E_2 \cup \dots \cup E_{\alpha-2}$). Since every path Q_i has even length, the ends of each ear E_i must have the same colour. We proceed as follows.

- We first let $E_2 = x_k \curvearrowright Q_2 \rightarrow x_{3k-2}$. Since $c(x_k) = 2$ and $c(x_{3k-2}) = 2$, $C \cup E_2$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ and we can extend the 2-colouring c . In particular, since $c(x_k) = 2$ and $x_k x_{2k}$ is an arc in E_2 , we have $c(x_{2k}) = 1$.
- Assume now that we have added ears E_2, \dots, E_{i-1} for some $i < \alpha - 2$, in such a way that $C \cup E_2 \cup \dots \cup E_{i-1}$ is a strong bipartite subdigraph of $C_n(\{1, k\})$, and the corresponding 2-colouring c is such that $c(x_{jk}) = 1$ if j is even and $c(x_{jk}) = 2$ if j is odd, for every $j \in \{2, \dots, i-1\}$.

We then let $E_i = x_{(i-1)k} \curvearrowright Q_i \rightarrow x_{(i+1)(k-2)}$. Since $c(x_{(i-1)k}) = c(x_{(i-1)(k-1)}) = c(x_{(i-1)(k-2)})$, we have $c(x_{(i-1)k}) = c(x_{(i+1)(k-2)})$. Therefore, $C \cup E_2 \cup \dots \cup E_{i-1} \cup E_i$ is a strong bipartite subdigraph of $C_n(\{1, k\})$ to which we can extend the 2-colouring c . Note that we have $c(x_{ik}) = 1$ if i is even and $c(x_{ik}) = 2$ if i is odd.

We finally let $E = x_{(\alpha-2)(k+1)} \curvearrowright Q \rightarrow x_{\alpha k-2}$. Since the path Q is of odd length, we need to have $c(x_{(\alpha-2)(k+1)}) \neq c(x_{\alpha k-2})$ for $C \cup E_2 \cup \dots \cup E_{\alpha-2} \cup E$ to be a bipartite subdigraph of $C_n(\{1, k\})$. This is indeed the case since a path in $C \cup E_2 \cup \dots \cup E_{\alpha-2}$ from $x_{(\alpha-2)(k+1)}$ to $x_{\alpha k-2}$ is given by $x_{(\alpha-2)(k+1)} \rightarrow x_{(\alpha-2)(k+2)} \rightarrow \dots \rightarrow x_{(\alpha-1)(k-2)} \curvearrowright x_{\alpha k-2}$, whose length is $k-2$, an odd number. This completes this subcase.

Subcase 1.3.2. $\alpha = 2$.

In that case, we get $n = 3k - 2$ and, since $n \geq 9$, $k \geq 4$.

Consider the following cycle (see Figure 5):

$$C = x_0 \curvearrowright x_k \curvearrowright x_{2k} \rightarrow x_{2k+1} \rightarrow \dots \rightarrow x_{3k-4} \curvearrowright x_{k-2} \curvearrowright x_{2k-2} \curvearrowright x_0.$$

Since $|A(C)| = k + 1$, C is an even cycle. Let c be a 2-colouring of C with $c(x_0) = 1$.

We consider the two following paths of $C_n(\{1, k\})$ (see Figure 5):

$$\begin{aligned} Q_1 &= x_2 \rightarrow \dots \rightarrow x_{k-3} \\ Q_2 &= x_{k+1} \rightarrow \dots \rightarrow x_{2k-4} \end{aligned}$$

We first add to C the two ears $E_1 = x_{2k} \curvearrowright Q_1 \rightarrow x_{k-2}$ and $E_2 = x_k \rightarrow Q_2 \curvearrowright x_{3k-4}$. Since the length of both paths Q_1 and Q_2 is $k-5$, an even number, we need to have $c(x_{2k}) = c(x_{k-2})$ and $c(x_k) = c(x_{3k-4})$ for $C \cup E_1 \cup E_2$ to be a bipartite subdigraph of $C_n(\{1, k\})$. This is

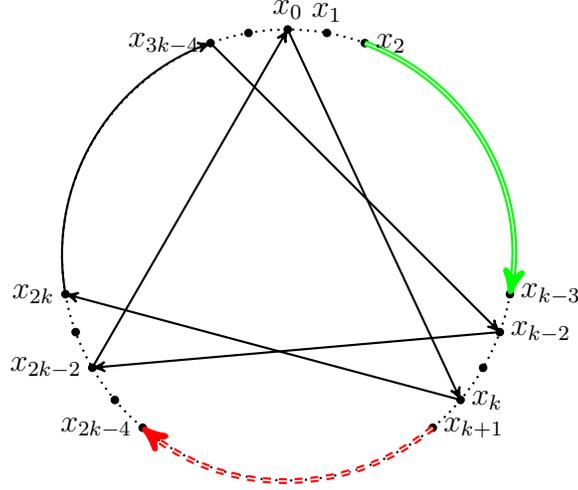


Figure 5: (Subcase 1.3.2) A sample digraph, together with the cycle C (black) and the paths Q_1 (double, green) and Q_2 (double dashed, red).

indeed the case since $x_0 \rightsquigarrow x_k \rightsquigarrow x_{2k}$ and $x_{k-2} \rightsquigarrow x_{2k-2} \rightsquigarrow x_0$ are paths in C . We can thus extend the 2-colouring c to $C \cup E_1 \cup E_2$.

There are now five remaining vertices, namely x_1 , x_{k-1} , x_{2k-3} , x_{2k-1} and x_{3k-3} , not included in $C \cup E_1 \cup E_2$. We then sequentially add the five ears

$$\begin{aligned}
 E_3 &= x_0 \rightarrow x_1 \rightsquigarrow x_{k+1}, \\
 E_4 &= x_{2k-2} \rightarrow x_{2k-1} \rightsquigarrow x_1, \\
 E_5 &= x_{k-2} \rightarrow x_{k-1} \rightsquigarrow x_{2k-1}, \\
 E_6 &= x_{3k-4} \rightarrow x_{3k-3} \rightsquigarrow x_{k-1}, \text{ and} \\
 E_7 &= x_{2k-4} \rightarrow x_{2k-3} \rightsquigarrow x_{3k-3},
 \end{aligned}$$

getting at each step a bipartite subdigraph of $C_n(\{1, k\})$, so that the 2-colouring c can be sequentially extended. This is indeed the case since $x_0 \rightsquigarrow x_k \rightarrow x_{k+1}$, $x_{2k-2} \rightsquigarrow x_0 \rightarrow x_1$, $x_{k-2} \rightsquigarrow x_{2k-2} \rightarrow x_{2k-1}$, $x_{3k-4} \rightsquigarrow x_{k-2} \rightarrow x_{k-1}$ and $x_{2k-4} \rightsquigarrow x_{3k-4} \rightarrow x_{3k-3}$ are paths in $C \cup E_2$, $C \cup E_3$, $C \cup E_4$, $C \cup E_5$ and $E_2 \cup E_6$, respectively, and thus the endvertices of the five above defined ears have the same colour.

We hence get that $C \cup E_1 \cup \dots \cup E_7$ is a strong bipartite spanning subdigraph of $C_n(\{1, k\})$, which completes this subcase.

Case 2. $2k + 1 \leq n \leq 3k - 4$

The assumptions of this case imply $n = 2k + r$ with $1 \leq r \leq k - 4$. Let us consider the following cycle (see Figure 6):

$$C = x_0 \rightsquigarrow x_k \rightarrow \dots \rightarrow x_{k+r+1} \rightsquigarrow x_1 \rightarrow \dots \rightarrow x_{r+2} \rightsquigarrow x_{k+r+2} \rightarrow \dots \rightarrow x_0.$$

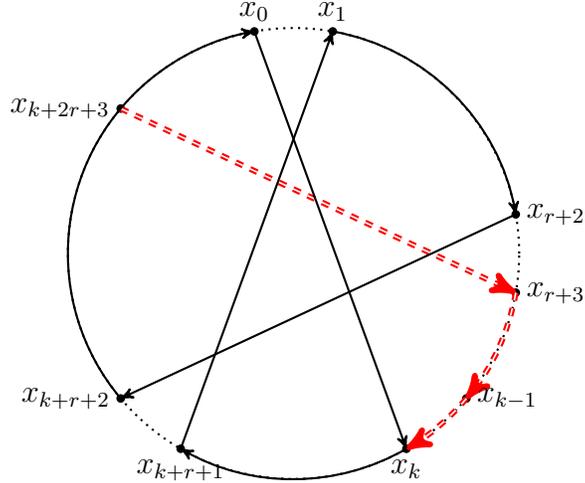


Figure 6: (Case 2) A sample digraph, together with the cycle C (black) and the ear E (double dashed, red).

Observe that the vertices of $C_n(\{1, k\})$ that are not in C form a path $Q = x_{r+3} \rightarrow \dots \rightarrow x_{k-1}$ (since $r \leq k - 4$, Q has at least one vertex) and that C is an even cycle, since $|V(C)| = n - |V(Q)| = n - (k - r - 3) = k + 2r + 3$.

We then add to C the ear $E = x_{k+2r+3} \curvearrowright Q \rightarrow x_k$. Since $d_C(x_{k+2r+3}, x_k) = k - r - 2$ and $|E| = k - r - 2$, $C \cup E$ is a strong bipartite spanning subdigraph of $C_n(\{1, k\})$, which completes this subcase.

Case 3. $k + 2 \leq n \leq 2k - 1$.

In this case we will construct an even cycle C by concatenating several paths of length $n - k + 1$. Our assumptions imply $3 \leq n - k + 1 \leq k$. Let $n = (n - k + 1)t + s$ with $0 \leq s \leq n - k$.

Observe that if $t = 1$, then $s = k - 1$. In that case, since $s \leq n - k$, we have $s = k - 1 \leq n - k$. On the other hand, we have $n - k + 1 \leq k$, that is, $k - 1 \geq n - k$. We finally get $s = n - k$ if $t = 1$.

We will consider five subcases.

Subcase 3.1. $t \geq 1$ and $s = n - k$.

In that case, $n = t(n - k + 1) + n - k$. Since n is odd, it follows that t is also odd. Consider the following paths (see Figure 7):

$$P_i = x_{ik-(i-1)n-i} \curvearrowright x_{(i+1)k-in-i} \rightarrow \dots \rightarrow x_{ik-(i-1)n-(i+1)} \curvearrowright x_{(i+1)k-in-(i+1)},$$

for every $i \in \{0, \dots, t-1\}$,

$$P = x_{n-k} \curvearrowright x_0.$$

Observe that $C = \bigcup_{i=0}^{t-1} P_i \cup P$ is an even cycle, since $|P_i| = n - k + 1$ for every i and $|P| = 1$. Furthermore, the vertices of $C_n(\{1, k\})$ not belonging to C form a path $Q = x_1 \rightarrow \dots \rightarrow x_{n-k-1}$.

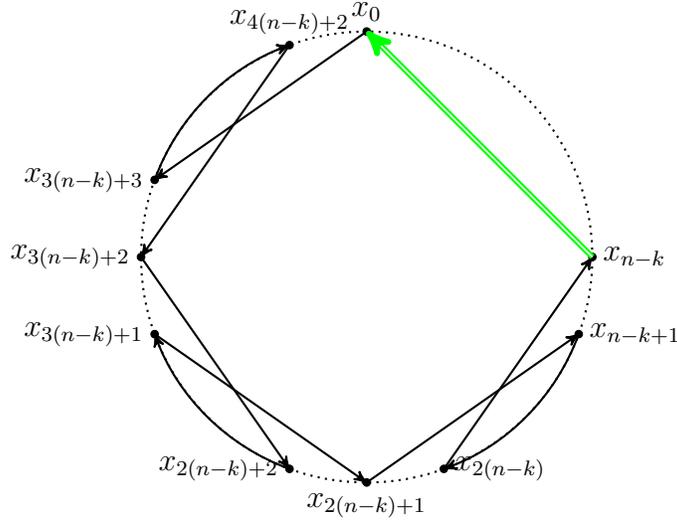


Figure 7: (Subcase 3.1) A sample digraph for $t = 3$, together with the cycle $P_0 \cup P_1 \cup P_2$ (black) and the path P (double, green).

We then add to C the ear $E = x_{n-k+1} \curvearrowright x_1 \rightarrow \dots \rightarrow x_{n-k-1} \rightarrow x_{n-k}$. Since $d_C(x_{n-k+1}, x_{n-k}) = n - k$ and $|E| = n - k + 2$ (these two values have thus the same parity), we get that $C \cup E$ is a strong bipartite spanning subdigraph of $C_n(\{1, k\})$, which completes this subcase.

Subcase 3.2. $t = 2$ and $0 \leq s \leq n - k - 2$.

In that case, we have $n = 2(n - k + 1) + s$, and thus s must be odd, so that $1 \leq s \leq n - k - 3$. Let us consider the following cycle (see Figure 8):

$$C = x_0 \curvearrowright x_{n-k+s+2} \rightarrow \dots \rightarrow x_{2(n-k)} \curvearrowright x_{n-k} \curvearrowright x_0.$$

Note that $|A(C)| = n - k - s + 1$, and thus C is an even cycle.

We first add to C the ear

$$E = x_{n-k+s+2} \curvearrowright x_{s+2} \rightarrow \dots \rightarrow x_{n-k-1} \rightarrow x_{n-k}.$$

Since $d_C(x_{n-k+s+2}, x_{n-k}) = 2$ and $|E| = n - k - s - 1$ is even, $C \cup E$ is a strong bipartite subdigraph of $C_n(\{1, k\})$.

Now, the vertices of $C_n(\{1, k\})$ not belonging to $C \cup E$ can be partitioned into the following three sets,

$$\begin{aligned} & \{x_{2(n-k)+1}, x_{2(n-k)+2}, \dots, x_{2(n-k)+1+s}\} \\ & \{x_1, x_2, \dots, x_{s+1}\}, \text{ and} \\ & \{x_{n-k+1}, x_{n-k+2}, \dots, x_{n-k+s+1}\}, \end{aligned}$$

each containing $s + 1$ vertices.

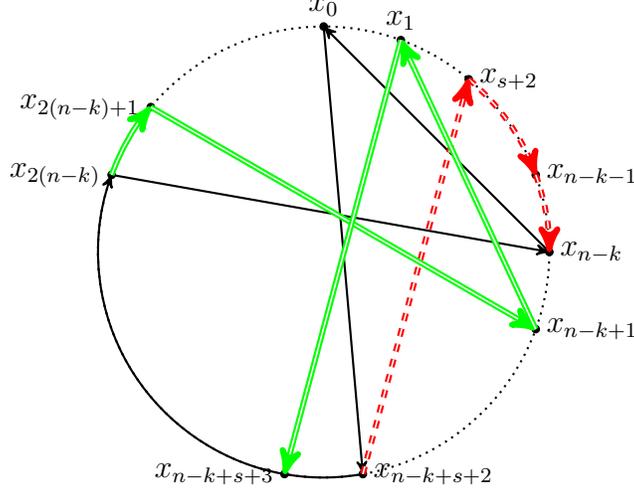


Figure 8: (Subcase 3.2) A sample digraph, together with the cycle C (black) and the ears E (double dashed, red) and E_0 (double, green).

We now add sequentially the following $s + 1$ ears:

$$E_i = x_{2(n-k)+i} \rightarrow x_{2(n-k)+1+i} \curvearrowright x_{n-k+1+i} \curvearrowright x_{1+i} \curvearrowright x_{n-k+s+3+i}, \text{ for every } i \in \{0, \dots, s\}.$$

Note that each of these ears is of length 4 and contains exactly one vertex of each set. Since the distance between $x_{n-k+s+3+i}$ and $x_{2(n-k)+i}$ in $C \cup E \cup E_0 \cup \dots \cup E_{i-1}$, for every $i \in \{0, \dots, s\}$, is $n - k - s - 3$, an even number, we get that $C \cup E \cup E_0 \cup \dots \cup E_s$ is a strong bipartite spanning subdigraph of $C_n(\{1, k\})$, which completes this subcase.

Subcase 3.3. $t = 2$ and $s = n - k - 1$.

In that case, we have $n = 3(n - k) + 1$. We claim that $C_n(\{1, k\})$ is isomorphic to $C_n(\{1, 3\})$, the digraph considered in **Case 1**.

Observe first that the chords of $C_n(\{1, k\})$ form the cycle C_n . Indeed, let D' be the digraph obtained from $C_n(\{1, k\})$ by reversing every arc which is not a chord. Observe that D' is isomorphic to $C_n(\{1, r\})$, with $r = n - k$. Since $n = 3r + 1$, we get that n and r are relatively prime. Therefore, the chords of D' form a cycle that goes through all vertices of D' . Let C' be the cycle of $C_n(\{1, k\})$ induced by the chords. The distance on C' between any two consecutive vertices of C is 3, and thus $C_n(\{1, k\})$ and $C_n(\{1, 3\})$ are isomorphic digraphs, which completes this subcase.

Subcase 3.4. $t \geq 3$ and $0 \leq s \leq n - k - 2$.

Consider the following paths (see Figure 9):

$$P_i = x_{(t-i)(n-k+1)+s} \curvearrowright x_{(t-i-1)(n-k+1)+s+1} \rightarrow \dots \rightarrow x_{(t-i)(n-k+1)+s-1} \curvearrowright x_{(t-i-1)(n-k+1)+s},$$

for every $i \in \{0, \dots, t - 3\}$,

$$P = x_{2n-2k+s+2} \curvearrowright x_{n-k+s+2} \rightarrow \dots \rightarrow x_{2n-2k} \curvearrowright x_{n-k}, \text{ and}$$

$$P' = x_{n-k} \curvearrowright x_0.$$

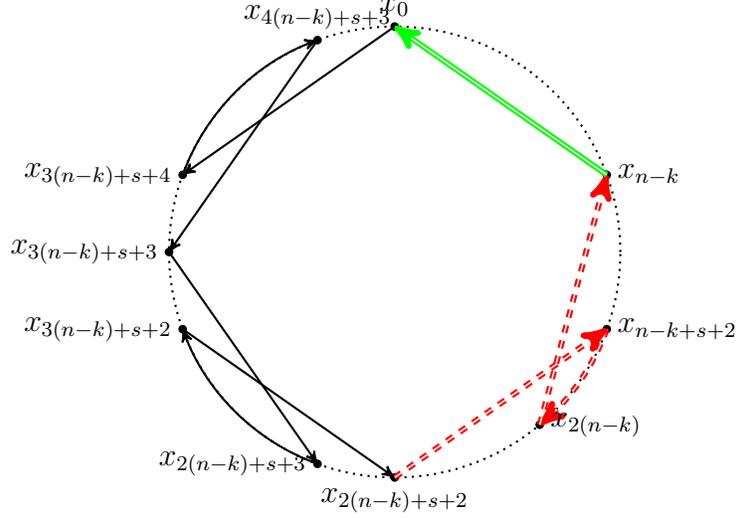


Figure 9: (Subcase 3.4) A sample digraph for $t = 4$, together with the cycle $C = P_0 \cup P_1$ (black) and the paths P (double dashed, red) and P' (double, green).

Note that $|P'| = 1$, $|P| = n - k - s$ and $|P_i| = n - k + 1$. Since n is odd and $n = (n - k + 1)t + s$, it follows that t and s are of different parity. Hence, since $|A(C)| = (t - 2)(n - k + 1) + n - k - s + 1$, we get that $C = \bigcup_{i=0}^{t-3} P_i \cup P \cup P'$ is an even cycle. Let c be a 2-colouring of C with $c(x_0) = 1$.

We are now ready to add ears to the cycle C in order to get a spanning subdigraph of $C_n(\{1, k\})$. We consider the following three paths of $C_n(\{1, k\})$:

$$\begin{aligned} Q_1 &= x_{2n-2k+1} \rightarrow \dots \rightarrow x_{2n-2k+s+1}, \\ Q_2 &= x_{n-k+1} \rightarrow \dots \rightarrow x_{n-k+s+1}, \text{ and} \\ Q_3 &= x_1 \rightarrow \dots \rightarrow x_{n-k-1}. \end{aligned}$$

Observe that $V(C_n(\{1, k\})) = V(C) \cup V(Q_1) \cup V(Q_2) \cup V(Q_3)$. We now sequentially add the following three ears to C :

$$\begin{aligned} E_1 &= x_{3n-3k+1} \curvearrowright Q_1 \rightarrow x_{2n-2k+s+2}, \\ E_2 &= x_{2n-2k+1} \curvearrowright Q_2 \rightarrow x_{n-k+s+2}, \text{ and} \\ E_3 &= x_{n-k+1} \curvearrowright Q_3 \rightarrow x_{n-k}. \end{aligned}$$

Since

$$\begin{aligned} d_C(x_{3n-3k+1}, x_{2n-2k+s+2}) &= d_{P_{t-3}}(x_{3n-3k+1}, x_{2n-2k+s+2}) = s + 2 \text{ and } |E_1| = s + 2, \\ d_{C \cup E_1}(x_{2n-2k+1}, x_{n-k+s+2}) &= s + 2 \text{ and } |E_2| = s + 2, \\ d_C(x_{n-k+1}, x_{n-k}) &= d_{E_2 \cup P}(x_{n-k+1}, x_{n-k}) = n - k \text{ and } |E_3| = n - k, \end{aligned}$$

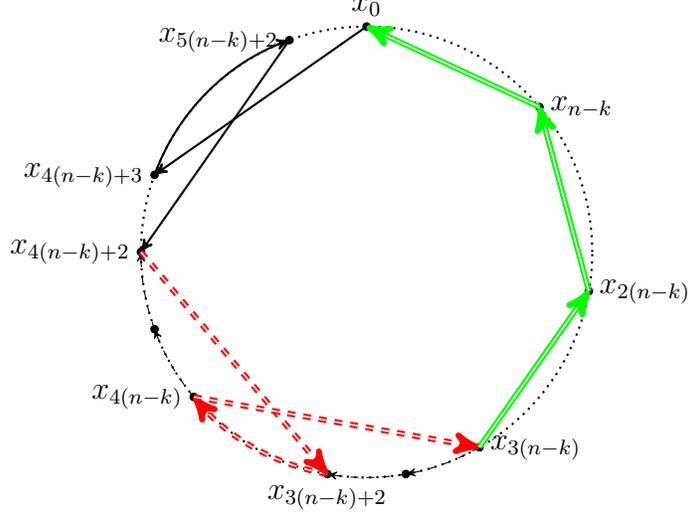


Figure 10: (Subcase 3.5) A sample digraph for $t = 4$, together with the paths $C = P_0$ (black), P (double dashed, red) and P' (double, green).

the so-obtained subdigraph $C \cup E_1 \cup E_2 \cup E_3$ is a strong bipartite spanning subdigraph of $C_n(\{1, k\})$. This completes this subcase.

Subcase 3.5. $t \geq 3$ and $s = n - k - 1$.

In this case, t must be even since n is odd. We thus have $t \geq 4$. Consider the following paths (see Figure 10):

$$P_i = x_{(t-i+1)(n-k)+t-i-1} \curvearrowright x_{(t-i)(n-k)+t-i-1} \rightarrow \dots \rightarrow x_{(t-i+1)(n-k)+t-i-2} \curvearrowright x_{(t-i)(n-k)+t-i-2},$$

for every $i \in \{0, \dots, t-4\}$,

$$P = x_{4n-4k+2} \curvearrowright x_{3n-3k+2} \rightarrow \dots \rightarrow x_{4n-4k} \curvearrowright x_{3n-3k}, \text{ and}$$

$$P' = x_{3n-3k} \curvearrowright x_{2n-2k} \curvearrowright x_{n-k} \curvearrowright x_0.$$

Let $C = \bigcup_{i=0}^{t-4} P_i \cup P \cup P'$. Since $|P_i| = n - k + 1$, $|P| = n - k$ and $|P'| = 3$, we get $|A(C)| = (t-3)(n-k+1) + n - k + 3$ and thus C is an even cycle.

We now sequentially add the five following ears to C :

$$E_1 = x_{5n-5k+1} \curvearrowright x_{4n-4k+1} \rightarrow x_{4n-4k+2},$$

$$E_2 = x_{4n-4k+1} \curvearrowright x_{3n-3k+1} \rightarrow x_{3n-3k+2},$$

$$E_3 = x_{3n-3k+1} \curvearrowright x_{2n-2k+1} \rightarrow \dots \rightarrow x_{3n-3k-1} \rightarrow x_{3n-3k},$$

$$E_4 = x_{2n-2k+1} \curvearrowright x_{n-k+1} \rightarrow \dots \rightarrow x_{2n-2k-1} \rightarrow x_{2n-2k}, \text{ and}$$

$$E_5 = x_{n-k+1} \curvearrowright x_1 \rightarrow \dots \rightarrow x_{n-k-1} \rightarrow x_{n-k}.$$

Note that the ends of the ear E_1 belong to C , while the ends of each ear E_i , $2 \leq i \leq 5$, belong to $C \cup E_1 \cup \dots \cup E_{i-1}$.

Moreover, since

$$\begin{aligned}
d_C(x_{5n-5k+1}, x_{4n-4k+2}) &= d_{P_{t-4}}(x_{5n-5k+1}, x_{4n-4k+2}) = 2 \text{ and } |E_1| = 2, \\
d_{C \cup E_1}(x_{4n-4k+1}, x_{3n-3k+2}) &= 2 \text{ and } |E_2| = 2, \\
d_{C \cup E_1 \cup E_2}(x_{3n-3k+1}, x_{3n-3k}) &= n - k \text{ and } |E_3| = n - k, \\
d_{C \cup E_1 \cup E_2 \cup E_3}(x_{2n-2k+1}, x_{n-k}) &= n - k \text{ and } |E_4| = n - k, \\
d_{C \cup E_1 \cup E_2 \cup E_3 \cup E_4}(x_{n-k+1}, x_{n-k}) &= n - k \text{ and } |E_5| = n - k,
\end{aligned}$$

the so-obtained subdigraph $C \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ is a strong bipartite spanning subdigraph of $C_n(\{1, k\})$. This completes this last subcase.

This concludes the proof of Lemma 13. ■

As observed before, the proof of Theorem 10 then directly follows from Corollary 11 and Lemma 13.

5 Proper-walk connection of digraphs

Each properly coloured path is also a properly coloured walk, so $\vec{wc}(D) \leq \vec{pc}(D)$ for every digraph D . Therefore, similarly as the proper connection number, the proper-walk connection number of every digraph D is upper bounded by 3. On the other hand, the only digraphs with directed proper-walk connection number 1 are the symmetric complete graphs \vec{K}_n .

In this section, we give some sufficient conditions for a digraph D to have proper-walk connection number at most 2. If D has an Eulerian closed walk and an even number of arcs then, going along the Eulerian walk and alternately assigning colours 1 and 2 to the arcs, we get a properly-walk connected arc-colouring of D . We thus have the following observation.

Observation 14 *If D is a strong digraph with $d^+(v) = d^-(v)$ for every vertex $v \in V(D)$ and $|A(D)|$ even, then $\vec{wc}(D) \leq 2$.*

If $|V(D)|$ is even and $d_D^+(v) = d_D^-(v)$ for every vertex $v \in V(D)$, then $|A(D)|$ is even, so we also have the following observation.

Observation 15 *If D is a strong digraph with $d^+(v) = d^-(v)$ for every vertex $v \in V(D)$ and $|V(D)|$ even, then $\vec{wc}(D) \leq 2$.*

In the next theorems, we consider the case of Hamiltonian digraphs.

Theorem 16 *If D is a Hamiltonian digraph with $d_D^+(v) = d_D^-(v)$ for every vertex $v \in V(D)$, then $\vec{wc}(D) \leq 2$.*

Proof. Let C be a Hamiltonian cycle of D . If $|V(D)|$ is even, then C is a strong spanning bipartite subdigraph of D and so $\vec{wc}(D) \leq 2$ by Corollary 4.

Suppose that $|V(D)|$ is odd and let $D' = D - A(C)$. Note that $d_{D'}^+(v) = d_{D'}^-(v)$ for every vertex $v \in V(D') = V(D)$. However, D' may be not strong since, for instance, it

could be a vertex-disjoint sum of strong digraphs. The condition $d_{D'}^+(v) = d_{D'}^-(v)$ for every vertex v implies that there exists a decomposition of $A(D')$ into arc-disjoint cycles, i.e., $D' = C_1 \cup \dots \cup C_k$, with $A(C_i) \cap A(C_j) = \emptyset$ for every i, j , $1 \leq i < j \leq k$.

Since $V(D')$ is odd, at least one cycle is odd, say C_1 . Since C is a Hamiltonian cycle, the subdigraph $C \cup C_1$ is strong. Let v be any vertex of C_1 . We colour the arcs of C and of C_1 alternately with colours 1 and 2, except the arcs incident with v . In C we colour the two arcs incident with v with 1, and in C_1 we colour the two arcs incident with v with 2. It is not difficult to check that the so-obtained colouring is a properly-walk connected arc-colouring of $C \cup C_1$, which gives $\vec{wc}(D) \leq \vec{wc}(C \cup C_1) = 2$. ■

Recall that the length of a chord $x_p x_q$ in a cycle $C = x_0 \dots x_{n-1} x_0$ is $|C[x_p, x_q]|$.

Theorem 17 *Let D be a Hamiltonian digraph with $d_D^+(v) \geq 2$ and $d_D^-(v) \geq 2$ for every vertex $v \in V(D)$, and C be a Hamiltonian cycle of D . If every chord of C has length at most $\lceil |V(D)|/2 \rceil$, then $\vec{wc}(D) = 2$.*

Proof. Since every chord of C has length at most $\lceil |V(D)|/2 \rceil$, $D \neq \overleftrightarrow{K}_n$ and so $\vec{wc}(D) \geq 2$. If $|V(D)|$ is even, then C is a strong spanning bipartite subdigraph of D and so $\vec{wc}(D) \leq 2$ by Corollary 4. We thus assume that $|V(D)|$ is odd. We will prove that D contains a spanning subdigraph D' with $\vec{wc}(D') = 2$. To this aim, we will choose some arcs from $A(D)$ and colour them with two colours in such a way they form a properly connected spanning subdigraph of D .

Let $C = x_0 x_1 \dots x_{n-1} x_0$ and φ be a 2-colouring of the arcs of C such that $\varphi(x_0 x_1) = \varphi(x_1 x_2) = 1$, $\varphi(x_{n-1} x_0) = 2$, $\varphi(x_k x_{k+1}) = 1$ for k odd, and $\varphi(x_k x_{k+1}) = 2$ for k even, for every $k \in \{2, \dots, n-2\}$. Since the arcs $x_0 x_1$ and $x_1 x_2$ have the same colour, (C, φ) is not properly connected yet.

We consider two cases.

Case 1. C has an even chord.

Renaming vertices if necessary, we assume that $x_0 x_p$ is an even chord of C with minimum length (see Figure 11). We colour $x_0 x_p$ with 1, so that $x_0 x_p x_{p+1} \dots x_{n-1}$ is a properly connected subdigraph of D (an even cycle). Moreover, for every vertex x_i in $\{2, \dots, x_{p-1}\}$, there is a properly connected path $x_i x_0$ whose last arc is $x_{n-1} x_0$.

In order to get a properly-walk connected spanning subdigraph of D , it is thus enough to colour some chords of C in such a way that there is a properly coloured $x_0 x_i$ -walk starting with the arc $x_0 x_p$, for every vertex x_i in $\{x_2, \dots, x_{p-1}\}$.

Since $|C[x_0, x_p]|$ is minimum and every chord of C has length at most $\lceil |V(D)|/2 \rceil$, it follows that every even chord whose head is in $\{x_2, \dots, x_{p-1}\}$ has its tail in $\{x_{p+2}, \dots, x_{n-1}\}$. Furthermore, since $d_D^-(v) \geq 2$ for every $v \in V(D)$, every vertex in $\{x_2, \dots, x_{p-1}\}$ is the head of a chord.

Let j be the smallest integer in $\{x_2, \dots, x_p\}$ such that x_j is the head of an even chord (we may have $j = p$). Note that $\varphi(x_{t-1} x_t) = \varphi(x_j x_{j+1})$. If $j < p$, we colour $x_t x_j$ with the colour different from $\varphi(x_{t-1} x_t)$, so that we now have a properly coloured $x_0 x_i$ -walk for every vertex $x_i \in \{x_j, \dots, x_{n-1}\}$.

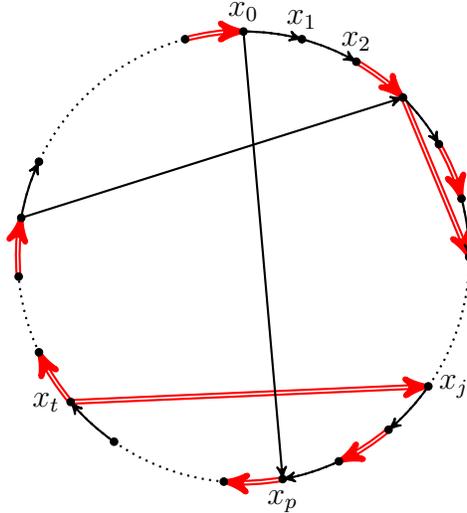


Figure 11: (Theorem 17, Case 1) The Hamiltonian digraph D with its arc-colouring; x_0x_p is a shortest even chord of C , and $x_t x_j$ an even chord which may not exist.

All vertices in $\{x_2, \dots, x_{j-1}\}$ are not the head of an even chord, so they are the head of at least one odd chord. We claim that, one by one, we can colour one such odd chord for every vertex x_i in $\{x_2, \dots, x_{j-1}\}$, in such a way that there will be a properly coloured x_0x_i -walk (starting with the arc x_0x_p), which will allow us to conclude this case.

Let x_sx_2 be an odd chord. Since $|C[x_s, x_2]| \leq (n-1)/2$, we have $p \leq s \leq n-1$ and, since x_sx_2 is odd, $\varphi(x_{s-1}x_s) = 1$. We then colour x_sx_2 with 2, so that the walk $x_0x_p \dots x_sx_2$ is properly coloured.

Assume now that for every $i \in \{2, \dots, \ell-1\}$, $\ell < j-1$, we have found a properly coloured x_0x_i -walk starting with x_0x_p and ending with an odd chord. Let x_qx_ℓ be an odd chord. Since $|C[x_q, x_\ell]| \leq (n-1)/2$, we have either $p+1 \leq q \leq n-1$ or $0 \leq q \leq \ell-3$. If $p+1 \leq q \leq n-1$, then we colour x_qx_ℓ with the colour different from $\varphi(x_{q-1}x_q)$, so that $x_0x_p \dots x_qx_\ell$ is a properly coloured walk. Otherwise, if $q=0$, then we colour x_qx_ℓ with 1, if $q=1$, then we colour x_qx_ℓ with 2, and if $2 \leq q \leq \ell-3$, then we colour x_qx_ℓ with the colour different from the colour of the odd chord with head in x_q . Let P be a properly coloured x_0x_q -walk starting with x_0x_p and ending with a coloured chord. Then Px_qx_ℓ is the walk we are looking for. This completes this case.

Case 2. *Every chord in C is odd.*

Renaming vertices if necessary, we assume that x_0x_p is an odd chord of C with maximum length (see Figure 12). We colour x_0x_p with 1. Next, we choose a path containing only odd chords, $x_p \curvearrowright x_{p_1} \curvearrowright x_{p_2} \curvearrowright \dots \curvearrowright x_{p_{j-1}} \curvearrowright x_{p_j}$, until we “jump over x_0 ” (i.e., $p_{j-1} > p_j$ and $p_k < p_{k+1}$ for every $k < p$). Since x_0x_p has maximum length, we have $p_j < p$ (since n is odd, we cannot have $x_{j-1} = x_0$). We colour the chords of this path alternately, starting with colour 1 on x_0x_p . Observe that since all the chords are odd, the colour of $x_{p_i} \curvearrowright x_{p_{i+1}}$ is the same as the colour of $x_{p_{i+1}} \rightarrow x_{p_{i+1}+1}$ for every $i \in \{1, \dots, j-2\}$. However, the colour

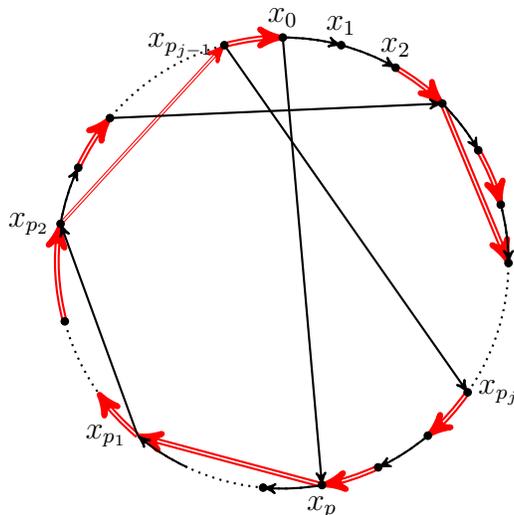


Figure 12: (Theorem 17, Case 2) The Hamiltonian digraph D with its arc-colouring; x_0x_p is a longest odd chord of C .

of $x_{p_{j-1}} \curvearrowright x_{p_j}$ is different from the colour of $x_{p_j} \rightarrow x_{p_{j+1}}$. Thus, the closed walk

$$x_0 \curvearrowright x_p \curvearrowright x_{p_1} \curvearrowright x_{p_2} \curvearrowright \dots \curvearrowright x_{p_j} \rightarrow x_{p_{j+1}} \rightarrow \dots \rightarrow x_{n-1} \rightarrow x_0$$

is properly coloured.

Hence, for every vertex $x_i \in \{x_2, \dots, x_{p-1}\}$, there is a properly coloured $x_i x_0$ -walk ending with $x_{n-1} x_0$. In order to get a properly-walk connected spanning subdigraph of D , it is thus enough to colour some chords of C in such a way that there is a properly coloured $x_0 x_i$ -walk starting with the arc $x_0 x_p$, for every vertex x_i in $\{x_2, \dots, x_{j-1}\}$.

Similarly as in **Case 1**, we claim that, one by one, we can colour, for every vertex x_i in $\{x_2, \dots, x_{p-1}\}$ one chord with head at x_i , in such a way that there will be a properly coloured $x_0 x_i$ -walk (starting with $x_0 x_p$ and ending with the chosen chord), which will allow us to conclude this case.

First observe that, for every vertex $x_i \in \{x_{p+1}, \dots, x_{n-1}\}$, there is a properly coloured $x_0 x_i$ -walk starting with $x_0 x_p$ and ending with an arc in $A(C)$. Let $x_s x_2$ be an odd chord. Since $|C[x_s, x_2]| \leq (n-1)/2$, we have $p+1 \leq s \leq n-1$. Since $x_s x_2$ is odd, $\varphi(x_{s-1} x_s) = 1$. We then colour $x_s x_2$ with 2, so that we obtain the properly coloured walk $P_2 x_s x_2$, where P_2 is the properly coloured $x_0 x_s$ -walk starting with $x_0 x_p$ and ending with an arc in $A(C)$.

Assume now that for every $i \in \{2, \dots, \ell-1\}$, $\ell < j-1$, we have found a properly coloured $x_0 x_i$ -walk starting with $x_0 x_p$ and ending with an odd chord. Let $x_q x_\ell$ be an odd chord. Since $|C[x_q, x_\ell]| \leq (n-1)/2$, we have either $p+1 \leq q \leq n-1$ or $0 \leq q \leq \ell-3$. If $p+1 \leq q \leq n-1$, then we colour $x_q x_\ell$ with the colour different from the colour of $x_{q-1} x_q$. Thus $P_\ell x_q x_\ell$ (where P_ℓ is the properly coloured $x_0 x_q$ -walk starting with $x_0 x_p$ and ending with an arc in $A(C)$) is a properly coloured walk. Otherwise, if $q = 0$, then we colour $x_q x_\ell$ with 1, if $q = 1$, then we colour $x_q x_\ell$ with 2, and if $2 \leq q \leq \ell-3$, then we colour $x_q x_\ell$ with the colour different from

the colour of the odd chord with head x_q . Let P be a properly coloured x_0x_q -walk starting with x_0x_p and ending with a coloured chord. Then Px_qx_ℓ is a path that we are looking for.

This completes the proof of Theorem 17. ■

6 Concluding remarks

In Section 5 we have proved that if C is a Hamiltonian cycle of a digraph D satisfying $d_D^+(v) \geq 2$ and $d_D^-(v) \geq 2$ for every vertex v of D , then $\overrightarrow{wc}(D) \leq 2$, provided that every chord of C has length at most $\lceil |V(D)|/2 \rceil$. We expect that the condition on the length of chords can be removed. We thus propose the following conjecture.

Conjecture 1 *If D is a Hamiltonian digraph such that $d_D^+(v) \geq 2$ and $d_D^-(v) \geq 2$ for every vertex $v \in V(D)$, then $\overrightarrow{wc}(D) \leq 2$.*

In Section 4, we gave some necessary conditions for a Hamiltonian digraph to have proper connection number at most 2. In fact, we proved stronger results. Namely, we proved that such a digraph has a spanning strong bipartite subdigraph. Bang-Jensen *et al.* studied in [2] the existence, in a given digraph D , of a spanning bipartite subdigraph with some special properties. A 2-partition of a digraph D is a partition of its set of vertices in two parts. If (V_1, V_2) is a 2-partition of a digraph D , then the bipartite digraph induced by the set of arcs having one end in each part is denoted by $B_D(V_1, V_2)$. A 2-partition (V_1, V_2) of D is *strong* if $B_D(V_1, V_2)$ is strong. The authors of [2], Bang-Jensen *et al.* studied the complexity of determining whether 2-partitions with some restrictions on the degree of vertices, or strong 2-partitions, exist in digraphs. Among others results, they proved the following theorem.

Theorem 18 *It is NP-complete to decide whether a strong digraph has a strong 2-partition.*

From the proofs of Theorem 7 and Corollary 8, and the observation that a Hamiltonian digraph of even order has a strong 2-partition, the following result follows.

Theorem 19 *Let D be a Hamiltonian digraph and C be a Hamiltonian cycle of D . If for every $v \in V(D)$ there is an even chord of C with tail v , then D has a strong 2-partition.*

Theorem 20 *Let D be a Hamiltonian digraph and C be a Hamiltonian cycle of D . If for every $v \in V(D)$ there is an even chord of C with head v , then D has a strong 2-partition.*

In the proof of Lemma 13, we have shown that, for $n \geq 9$ and k odd, every circulant digraph $C_n(\{1, k\})$ has a strong 2-partition. One can see that $C_5(\{1, 3\})$ has a strong 2-partition. Furthermore, using Theorems 19 and 20, we obtain the following result.

Theorem 21 *If $n \neq 7$, $S \subseteq \{1, \dots, n-1\}$, $|S| \geq 2$, and $1 \in S$, then $C_n(S)$ has a strong 2-partition.*

It would be interesting to consider the case of Hamiltonian digraphs with respect to Theorem 18. We can thus ask the following question.

Question 1 *What is the complexity of deciding whether a Hamiltonian digraph D has a strong 2-partition?*

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