A doubly stochastic block Gauss–Seidel algorithm for solving linear equations

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Abstract

We propose a simple doubly stochastic block Gauss–Seidel algorithm for solving linear systems of equations. By varying the row partition parameter and the column partition parameter of the coefficient matrix, we recover the Landweber algorithm, the randomized Kaczmarz algorithm, the randomized Gauss–Seidel algorithm, and the doubly stochastic Gauss–Seidel algorithm. For general (consistent or inconsistent) linear systems, we show the exponential convergence of the *norms of the expected iterates* via exact formulas. For consistent linear systems, we prove the exponential convergence of the *expected norms of the error and the residual*. Numerical experiments are given to illustrate the efficiency of the proposed algorithm.

Keywords. Randomized Kaczmarz, Randomized Gauss–Seidel, Doubly stochastic Gauss–Seidel, Doubly stochastic block Gauss–Seidel, Exponential convergence

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1 Introduction

Randomized iterative algorithms for solving a linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m\times n}, \quad \mathbf{b} \in \mathbb{R}^m \tag{1}$$

have attracted much attention recently; see, for example, [21, 11, 24, 16, 17, 13, 9, 14, 1, 2, 3, 5, 4, 8, 19, 15, 23, 12, 22, 6, 18, 20]. At each step, to generate the next iterate from the current iterate, the randomized Kaczmarz algorithm [21] uses a randomly picked row, the randomized Gauss–Seidel (i.e., randomized coordinate descent) algorithm [11] uses a randomly picked column, and the doubly stochastic Gauss–Seidel algorithm [19] uses a randomly picked entry of the coefficient matrix $\mathbf{A}$. It is natural to ask whether one can design a randomized algorithm which uses a randomly picked submatrix of $\mathbf{A}$.

In this paper, we propose a doubly stochastic block Gauss–Seidel (DSBGS) algorithm which uses a submatrix of $\mathbf{A}$ at each step (see Algorithm 1 in §2). We can view DSBGS as a stochastic gradient descent for solving the following optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \left\{ f(\mathbf{x}) := \frac{1}{2\|\mathbf{A}\|_{\mathrm{F}}^2} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2 \right\}. \tag{2}$$

The Landweber iterative algorithm [10], the randomized Kaczmarz (RK) algorithm, the randomized Gauss–Seidel (RGS) algorithm, and the doubly stochastic Gauss–Seidel (DSGS) algorithm are special cases of our algorithm. Our algorithm does not need to use projections and Moore-Penrose pseudoinverses of submatrices, so it is different from the block algorithms in [16, 17, 9].

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Numerical experiments for both synthetic data and real-world data are given to illustrate the efficiency of DSBGS.

Main contributions. We propose a simple doubly stochastic block Gauss–Seidel algorithm for solving linear equations and prove its convergence theory. More specifically, we show the exponential convergence of the norms of the expected iterates via exact formulas (see Theorems 2 and 5) for general (consistent or inconsistent) linear systems, and prove the exponential convergence of the expected norms of the error and the residual (see Theorems 8 and 11) for consistent linear systems.

Organization of this paper. In the rest of this section, we give some notation. In Section 2 we describe the doubly stochastic block Gauss–Seidel algorithm and prove its convergence theory. In Section 3 we report the numerical results. Finally, we present brief concluding remarks in Section 4.

Notation. For any random variables $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, we use $\mathbb{E}[\boldsymbol{\xi}]$ and $\mathbb{E}[\boldsymbol{\xi}\,|\boldsymbol{\zeta}]$ to denote the expectation of $\boldsymbol{\xi}$ and the conditional expectation of $\boldsymbol{\xi}$ given $\boldsymbol{\zeta}$, respectively. For an integer $m \geq 1$, let $[m] := \{1, 2, 3, \ldots, m\}$. Lowercase (upper-case) boldface letters are reserved for column vectors (matrices). For any vector $\mathbf{u} \in \mathbb{R}^m$, we use $\mathbf{u}_i$, $\mathbf{u}^{\mathrm{T}}$ and $\|\mathbf{u}\|_2$ to denote, the ith entry, the transpose and the Euclidean norm of $\mathbf{u}$, respectively. We use $\mathbf{I}$ to denote the identity matrix whose order is clear from the context. For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we use $\mathbf{A}_{i,j}$, $\mathbf{A}_{i,:}$, $\mathbf{A}_{:,j}$ $\mathbf{A}^{\mathrm{T}}$, $\mathbf{A}^{\dagger}$, $\|\mathbf{A}\|_{\mathrm{F}}$, range($\mathbf{A}$), rank($\mathbf{A}$), $\sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{A}) \geq \cdots \geq \sigma_r(\mathbf{A}) > 0$ to denote the (i, j) entry, the ith row, the jth column, the transpose, the Moore-Penrose pseudoinverse, the Frobenius norm, the column space, the rank, and all the nonzero singular values of $\mathbf{A}$, respectively. Obviously, rank($\mathbf{A}$) $= r$. We call a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ full column rank if rank($\mathbf{A}$) $= n$ and rank deficient if rank($\mathbf{A}$) $< n$. For index sets $\mathcal{I} \subseteq [m]$ and $\mathcal{J} \subseteq [n]$, let $\mathbf{A}_{\mathcal{I},:}$, $\mathbf{A}_{:,\mathcal{J}}$, and $\mathbf{A}_{\mathcal{I},\mathcal{J}}$ denote the row submatrix indexed by $\mathcal{I}$, the column submatrix indexed by $\mathcal{J}$, and the submatrix that lies in the rows indexed by $\mathcal{I}$ and the columns indexed by $\mathcal{J}$, respectively. The linear system (1) is called consistent if $\mathbf{b} \in$ range($\mathbf{A}$), i.e., a solution exists; otherwise, it is called inconsistent.

2 A doubly stochastic block Gauss–Seidel algorithm

Let $\{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_s\}$ denote a partition of $[m]$ such that, for $i, j = 1, 2, \ldots, s$ and $i \neq j$,

$$\mathcal{I}_i \neq \emptyset, \quad \mathcal{I}_i \cap \mathcal{I}_j = \emptyset, \quad \bigcup_{i=1}^{s} \mathcal{I}_i = [m].$$

Let $\{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_t\}$ denote a partition of $[n]$ such that, for $i, j = 1, 2, \ldots, t$ and $i \neq j$,

$$\mathcal{J}_i \neq \emptyset, \quad \mathcal{J}_i \cap \mathcal{J}_j = \emptyset, \quad \bigcup_{i=1}^{t} \mathcal{J}_i = [n].$$

Let

$$\mathcal{P} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_s\} \times \{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_t\}.$$

We propose the following doubly stochastic block Gauss–Seidel algorithm (Algorithm 1) for solving the linear system $\mathbf{Ax} = \mathbf{b}$.

Algorithm 1: A doubly stochastic block Gauss–Seidel algorithm

Let $\alpha > 0$. Initialize $\mathbf{x}^0 \in \mathbb{R}^n$

for $k = 1, 2, \ldots,$ **do**

Pick $(\mathcal{I}, \mathcal{J}) \in \mathcal{P}$ with probability $\dfrac{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}{\|\mathbf{A}\|_{\mathrm{F}}^2}$

Set $\mathbf{x}^k = \mathbf{x}^{k-1} - \alpha \dfrac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}(\mathbf{Ax}^{k-1} - \mathbf{b})$

Here we consider constant step size for simplicity. By varying the row partition parameter s and the column partition parameter t, we recover the following well-known algorithms as special cases:

- Landweber [10] ($s = 1$ and $t = 1$),

$$\mathbf{x}^k = \mathbf{x}^{k-1} - \alpha \frac{\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}).$$

- Randomized Kaczmarz [21] ($s = m$ and $t = 1$),

$$\mathbf{x}^k = \mathbf{x}^{k-1} - \alpha \frac{\mathbf{A}_{i,:}\mathbf{x}^{k-1} - \mathbf{b}_i}{\|\mathbf{A}_{i,:}\|_2^2}(\mathbf{A}_{i,:})^{\mathrm{T}}.$$

- Randomized Gauss–Seidel [11] ($s = 1$ and $t = n$),

$$\mathbf{x}^k = \mathbf{x}^{k-1} - \alpha \frac{(\mathbf{A}_{:,j})^{\mathrm{T}}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b})}{\|\mathbf{A}_{:,j}\|_2^2}\mathbf{I}_{:,j}.$$

- Doubly Stochastic Gauss–Seidel [19] ($s = m$ and $t = n$),

$$\mathbf{x}^k = \mathbf{x}^{k-1} - \alpha \frac{\mathbf{A}_{i,j}(\mathbf{A}_{i,:}\mathbf{x}^{k-1} - \mathbf{b}_i)}{|\mathbf{A}_{i,j}|^2}\mathbf{I}_{:,j}.$$

The conditional expectation of $\mathbf{x}^k$ given $\mathbf{x}^{k-1}$ is

$$\begin{aligned}
\mathbb{E}[\mathbf{x}^k \,|\mathbf{x}^{k-1}] &= \mathbf{x}^{k-1} - \alpha \mathbb{E}\left[\frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right](\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}) \\
&= \mathbf{x}^{k-1} - \alpha \left(\sum_{(\mathcal{I},\mathcal{J})\in\mathcal{P}} \frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\frac{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}) \\
&= \mathbf{x}^{k-1} - \alpha \frac{\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}).
\end{aligned}$$

Note that the gradient of the objective function of the optimization problem (2) is

$$\nabla f(\mathbf{x}) = \frac{\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x} - \mathbf{b}).$$

It follows

$$\mathbb{E}[\mathbf{x}^k \,|\mathbf{x}^{k-1}] = \mathbf{x}^{k-1} - \alpha \nabla f(\mathbf{x}^{k-1}).$$

Therefore, DSBGS can be viewed as a stochastic gradient descent method for solving the optimization problem (2).

2.1 The exponential convergence of the norms of the expected iterates

In this subsection we show the exponential convergence of the norms of the expected iterates for general (consistent or inconsistent) linear systems. The following lemma will be used to prove Theorems 2 and 5. Its proof (via singular value decomposition) is straightforward and we omit the details.

Lemma 1. *Let $\alpha > 0$ and $\mathbf{A}$ be any nonzero real matrix. For every $\mathbf{u} \in \operatorname{range}(\mathbf{A})$, it holds*

$$\left\| \left(\mathbf{I} - \frac{\alpha \mathbf{A}\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k \mathbf{u} \right\|_2 \leq \left(\max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2} \right| \right)^k \|\mathbf{u}\|_2.$$

In Theorem 2, we show the exponential convergence of the norm of the expected error for *consistent* linear systems.

Theorem 2. *Let $\mathbf{x}^k$ denote the kth iterate of* DSBGS *applied to the consistent linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with arbitrary $\mathbf{x}^0 \in \mathbb{R}^n$. It holds*

$$\|\mathbb{E}[\mathbf{x}^k - \mathbf{x}_\star^0]\|_2 \leq \left(\max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2} \right| \right)^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2,$$

where

$$\mathbf{x}_\star^0 := (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}^0 + \mathbf{A}^\dagger \mathbf{b}$$

is the orthogonal projection of $\mathbf{x}^0$ onto the solution set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$.

Proof. The conditional expectation of $\mathbf{x}^k - \mathbf{x}_\star^0$ given $\mathbf{x}^{k-1}$ is

$$\begin{aligned}
\mathbb{E}[\mathbf{x}^k - \mathbf{x}_\star^0 \,|\mathbf{x}^{k-1}] &= \mathbb{E}[\mathbf{x}^k \,|\mathbf{x}^{k-1}] - \mathbf{x}_\star^0 \\
&= \mathbf{x}^{k-1} - \alpha \frac{\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}) - \mathbf{x}_\star^0 \\
&= \mathbf{x}^{k-1} - \alpha \frac{\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{A}\mathbf{x}_\star^0) - \mathbf{x}_\star^0 \\
&= \left(\mathbf{I} - \frac{\alpha \mathbf{A}^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{x}_\star^0).
\end{aligned}$$

Taking expectation gives

$$\begin{aligned}
\mathbb{E}[\mathbf{x}^k - \mathbf{x}_\star^0] &= \mathbb{E}[\mathbb{E}[\mathbf{x}^k - \mathbf{x}_\star^0 \,|\mathbf{x}^{k-1}]] \\
&= \left(\mathbf{I} - \frac{\alpha \mathbf{A}^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)\mathbb{E}[\mathbf{x}^{k-1} - \mathbf{x}_\star^0] \\
&= \left(\mathbf{I} - \frac{\alpha \mathbf{A}^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k (\mathbf{x}^0 - \mathbf{x}_\star^0).
\end{aligned}$$

Applying the norms to both sides we obtain

$$\|\mathbb{E}[\mathbf{x}^k - \mathbf{x}_\star^0]\|_2 \leq \left(\max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2} \right| \right)^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2.$$

Here the inequality follows from the fact that

$$\mathbf{x}^0 - \mathbf{x}_\star^0 = \mathbf{A}^\dagger \mathbf{A}\mathbf{x}^0 - \mathbf{A}^\dagger \mathbf{b} \in \operatorname{range}(\mathbf{A}^{\mathrm{T}})$$

and Lemma 1. □

Remark 3. *If $\mathbf{x}^0 \in \operatorname{range}(\mathbf{A}^{\mathrm{T}})$, then $\mathbf{x}_\star^0 = \mathbf{A}^\dagger \mathbf{b}$.*

Remark 4. *To ensure convergence of the expected iterate, it suffices to have*

$$\max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2} \right| < 1 \quad \textit{i.e.,} \quad 0 < \alpha < \frac{2\|\mathbf{A}\|_{\mathrm{F}}^2}{\sigma_1^2(\mathbf{A})}.$$

In Theorem 5, we show the exponential convergence of $\|\mathbb{E}[\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{x}_\star]\|_2$ for the *consistent or inconsistent* linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{x}_\star$ is any solution of the normal equations

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b}.$$

Theorem 5. *Let $\mathbf{x}^k$ denote the kth iterate of* DSBGS *applied to the consistent or inconsistent linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with arbitrary $\mathbf{x}^0 \in \mathbb{R}^n$. It holds*

$$\|\mathbb{E}[\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{x}_\star]\|_2 \leq \left(\max_{1\leq i\leq r}\left|1 - \frac{\alpha\sigma_i^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right|\right)^k \|\mathbf{A}\mathbf{x}^0 - \mathbf{A}\mathbf{x}_\star\|_2,$$

where $\mathbf{x}_\star$ is any solution of $\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$.

Proof. The conditional expectation of $\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{x}_\star$ given $\mathbf{x}^{k-1}$ is

$$\begin{aligned}
\mathbb{E}[\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{x}_\star \mid \mathbf{x}^{k-1}] &= \mathbf{A}(\mathbb{E}[\mathbf{x}^k \mid \mathbf{x}^{k-1}] - \mathbf{x}_\star) \\
&= \mathbf{A}\left(\mathbf{x}^{k-1} - \alpha\frac{\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}) - \mathbf{x}_\star\right) \\
&= \mathbf{A}\left(\mathbf{x}^{k-1} - \frac{\alpha\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{A}\mathbf{x}_\star) - \mathbf{x}_\star\right) \qquad (\text{by } \mathbf{A}^{\mathrm{T}}\mathbf{b} = \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x}_\star) \\
&= \mathbf{A}\mathbf{x}^{k-1} - \mathbf{A}\mathbf{x}_\star - \frac{\alpha\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{A}\mathbf{x}_\star) \\
&= \left(\mathbf{I} - \frac{\alpha\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{A}\mathbf{x}_\star).
\end{aligned}$$

Taking expectation gives

$$\begin{aligned}
\mathbb{E}[\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{x}_\star] &= \mathbb{E}[\mathbb{E}[\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{x}_\star \mid \mathbf{x}^{k-1}]] \\
&= \left(\mathbf{I} - \frac{\alpha\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)\mathbb{E}[\mathbf{A}\mathbf{x}^{k-1} - \mathbf{A}\mathbf{x}_\star] \\
&= \left(\mathbf{I} - \frac{\alpha\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k(\mathbf{A}\mathbf{x}^0 - \mathbf{A}\mathbf{x}_\star).
\end{aligned}$$

Applying the norms to both sides we obtain

$$\|\mathbb{E}[\mathbf{A}\mathbf{x}^k - \mathbf{A}\mathbf{x}_\star]\|_2 \leq \left(\max_{1\leq i\leq r}\left|1 - \frac{\alpha\sigma_i^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right|\right)^k \|\mathbf{A}\mathbf{x}^0 - \mathbf{A}\mathbf{x}_\star\|_2.$$

Here the inequality follows from the fact that $\mathbf{A}\mathbf{x}^0 - \mathbf{A}\mathbf{x}_\star \in \mathrm{range}(\mathbf{A})$ and Lemma 1. □

2.2 The exponential convergence of the expected norms of the error and the residual

In this subsection we prove the exponential convergence of the expected norms of the error or the residual for consistent linear systems. The convergence depends on the positive number ρ defined as

$$\beta = \max_{(\mathcal{I},\mathcal{J})\in\mathcal{P}} \frac{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_2^2}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}.$$

The following two lemmas will be used. Their proofs are straightforward and we omit the details.

Lemma 6. *For any vector* $\mathbf{u} \in \mathbb{R}^m$ *and any matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$*, it holds*

$$\mathbf{u}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{u} \le \|\mathbf{A}\|_2^2\mathbf{u}^{\mathrm{T}}\mathbf{u}.$$

Lemma 7. *For any matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *with rank* r *and any vector* $\mathbf{u} \in \mathrm{range}(\mathbf{A})$*, it holds*

$$\mathbf{u}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{u} \ge \sigma_r^2(\mathbf{A})\|\mathbf{u}\|_2^2.$$

For *full column rank consistent* linear systems, we prove the exponential convergence of the expected norm of the error in the following theorem. We recall that in this case $\mathbf{A}^\dagger\mathbf{b}$ is the unique solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Theorem 8. *Let* $\mathbf{x}^k$ *denote the kth iterate of* DSBGS *applied to the full column rank consistent linear system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *with arbitrary* $\mathbf{x}^0 \in \mathbb{R}^n$*. Assume* $0 < \alpha < 2/(t\beta)$*. It holds*

$$\mathbb{E}[\|\mathbf{x}^k - \mathbf{A}^\dagger\mathbf{b}\|_2^2] \le \left(1 - \frac{(2\alpha - t\beta\alpha^2)\sigma_n^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k \|\mathbf{x}^0 - \mathbf{A}^\dagger\mathbf{b}\|_2^2.$$

Proof. Note that

$$\begin{aligned}
\|\mathbf{x}^k - \mathbf{A}^\dagger\mathbf{b}\|_2^2 &= \left\|\mathbf{x}^{k-1} - \alpha\left(\frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}) - \mathbf{A}^\dagger\mathbf{b}\right\|_2^2 \\
&= \left\|\mathbf{x}^{k-1} - \alpha\left(\frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)\mathbf{A}(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}) - \mathbf{A}^\dagger\mathbf{b}\right\|_2^2 \\
&= \left\|\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b} - \alpha\left(\frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})\right\|_2^2 \\
&= \|\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}\|_2^2 - 2\alpha(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}) \\
&\quad +\alpha^2(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}^{\mathrm{T}}\mathbf{I}_{:,\mathcal{I}}\mathbf{A}_{\mathcal{I},\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^4}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}) \\
&\le \|\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}\|_2^2 - 2\alpha(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}) \\
&\quad +\alpha^2(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})^{\mathrm{T}}\left(\frac{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_2^2}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\cdot\frac{\mathbf{A}^{\mathrm{T}}\mathbf{I}_{:,\mathcal{I}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}) \quad \text{(by Lemma 6)} \\
&\le \|\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}\|_2^2 - 2\alpha(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}) \\
&\quad +\beta\alpha^2(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}^{\mathrm{T}}\mathbf{I}_{:,\mathcal{I}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}).
\end{aligned}$$

Taking expectation gives

$$\begin{aligned}
\mathbb{E}[\|\mathbf{x}^k - \mathbf{A}^\dagger\mathbf{b}\|_2^2\ |\mathbf{x}^{k-1}] &\le \|\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}\|_2^2 - (2\alpha - t\beta\alpha^2)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)(\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}) \\
&\le \left(1 - \frac{(2\alpha - t\beta\alpha^2)\sigma_n^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)\|\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}\|_2^2. \quad \text{(by Lemma 7)}
\end{aligned}$$

Taking expectation again gives

$$\begin{aligned}
\mathbb{E}[\|\mathbf{x}^k - \mathbf{A}^\dagger\mathbf{b}\|_2^2] &= \mathbb{E}[\mathbb{E}[\|\mathbf{x}^k - \mathbf{A}^\dagger\mathbf{b}\|_2^2\ |\mathbf{x}^{k-1}]] \\
&\le \left(1 - \frac{(2\alpha - t\beta\alpha^2)\sigma_n^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)\mathbb{E}[\|\mathbf{x}^{k-1} - \mathbf{A}^\dagger\mathbf{b}\|_2^2] \\
&\le \left(1 - \frac{(2\alpha - t\beta\alpha^2)\sigma_n^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k \|\mathbf{x}^0 - \mathbf{A}^\dagger\mathbf{b}\|_2^2. \qquad \square
\end{aligned}$$

Remark 9. *For the special case $s = m$, $t = 1$, $\alpha = 1$ (i.e., the randomized Kaczmarz algorithm, we have $\beta = 1$ in this case) and the special case $s = m$, $t = n$, $\alpha = 1/n$ (i.e., the doubly stochastic Gauss–Seidel algorithm, we have $\beta = 1$ in this case), the results of Theorem 8 are given in [21, Theorem 2] and [19, Theorem 1], respectively.*

Remark 10. *For rank deficient consistent linear systems, if $t = 1$ and $\mathbf{x}^0 \in \mathbb{R}^n$, we can show $\mathbf{x}^k - \mathbf{x}_\star^0 \in \mathrm{range}(\mathbf{A}^{\mathrm{T}})$ by induction, where $\mathbf{x}_\star^0 = (\mathbf{I} - \mathbf{A}^\dagger\mathbf{A})\mathbf{x}^0 + \mathbf{A}^\dagger\mathbf{b}$ is the orthogonal projection of $\mathbf{x}^0$ onto the solution set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$. Then by the same approach as used in the proof of Theorem 8, for any $s \in [m]$ and $t = 1$, we can prove the convergence bound*

$$\mathbb{E}[\|\mathbf{x}^k - \mathbf{x}_\star^0\|_2^2] \leq \left(1 - \frac{(2\alpha - \beta\alpha^2)\sigma_r^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k \|\mathbf{x}^0 - \mathbf{x}_\star^0\|_2^2.$$

This result for the special case $s = m$ and $t = 1$ (i.e., the randomized Kaczmarz algorithm, we have $\beta = 1$ in this case) with $\mathbf{x}^0 \in \mathrm{range}(\mathbf{A}^{\mathrm{T}})$ is given in [24, Theorem 3.4].

Next, we prove the exponential convergence of the expected norm of the residual for *full column rank or rank-deficient consistent* linear systems.

Theorem 11. *Let $\mathbf{x}^k$ denote the kth iterate of* DSBGS *applied to the consistent linear system (full column rank or rank deficient) $\mathbf{A}\mathbf{x} = \mathbf{b}$ with arbitrary $\mathbf{x}^0 \in \mathbb{R}^n$. If $t = n$ and $0 < \alpha < 2\sigma_r^2(\mathbf{A})/(\beta\|\mathbf{A}\|_{\mathrm{F}}^2)$, then*

$$\mathbb{E}[\|\mathbf{A}\mathbf{x}^k - \mathbf{b}\|_2^2] \leq \left(1 + \beta\alpha^2 - \frac{2\alpha\sigma_r^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k \|\mathbf{A}\mathbf{x}^0 - \mathbf{b}\|_2^2.$$

If $t < n$ and $0 < \alpha < 2\sigma_r^2(\mathbf{A})/(t\rho\beta)$, then

$$\mathbb{E}[\|\mathbf{A}\mathbf{x}^k - \mathbf{b}\|_2^2] \leq \left(1 - \frac{2\alpha\sigma_r^2(\mathbf{A}) - t\rho\beta\alpha^2}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k \|\mathbf{A}\mathbf{x}^0 - \mathbf{b}\|_2^2,$$

where

$$\rho = \max_{1 \leq j \leq t} \sigma_1^2(\mathbf{A}_{:,\mathcal{J}_j}).$$

Proof. Note that

$$\begin{aligned}
\|\mathbf{A}\mathbf{x}^k - \mathbf{b}\|_2^2 &= \left\|\mathbf{A}\mathbf{x}^{k-1} - \alpha\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}) - \mathbf{b}\right\|_2^2 \\
&= \|\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}\|_2^2 - 2\alpha(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}) \\
&\quad + \alpha^2(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{I}_{:,\mathcal{I}}\mathbf{A}_{\mathcal{I},\mathcal{J}}(\mathbf{I}_{:,\mathcal{J}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^4}\right)(\mathbf{A}\mathbf{x}^{k-1} - \mathbf{b}).
\end{aligned}$$

If $t = n$, then it follows from $(\mathbf{I}_{:,\mathcal{J}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{I}_{:,\mathcal{J}} = \|\mathbf{A}_{:,\mathcal{J}}\|_{\mathrm{F}}^2$ (since $\mathbf{A}\mathbf{I}_{:,\mathcal{J}} = \mathbf{A}_{:,\mathcal{J}}$ is a column vector) that

$$\begin{aligned}
\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2 &= \|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\quad+\alpha^2(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\|\mathbf{A}_{:,\mathcal{J}}\|_{\mathrm{F}}^2\mathbf{I}_{:,\mathcal{I}}\mathbf{A}_{\mathcal{I},\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^4}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\leq \|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\quad+\alpha^2(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_2^2}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\frac{\|\mathbf{A}_{:,\mathcal{J}}\|_{\mathrm{F}}^2\mathbf{I}_{:,\mathcal{I}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\quad\text{(by Lemma 6)}\\
&\leq \|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\quad+\beta\alpha^2(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\|\mathbf{A}_{:,\mathcal{J}}\|_{\mathrm{F}}^2\mathbf{I}_{:,\mathcal{I}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})
\end{aligned}$$

Taking expectation gives

$$\begin{aligned}
\mathbb{E}[\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2\ |\mathbf{x}^{k-1}] &\leq (1+\beta\alpha^2)\|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\leq \left(1+\beta\alpha^2-\frac{2\alpha\sigma_r^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)\|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2.
\end{aligned}$$

The last inequality follows from $\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\in\mathrm{range}(\mathbf{A})$ and Lemma 7. Taking expectation again gives

$$\begin{aligned}
\mathbb{E}[\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2] &= \mathbb{E}[\mathbb{E}[\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2\ |\mathbf{x}^{k-1}]]\\
&\leq \left(1+\beta\alpha^2-\frac{2\alpha\sigma_r^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)\mathbb{E}[\|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2]\\
&\leq \left(1+\beta\alpha^2-\frac{2\alpha\sigma_r^2(\mathbf{A})}{\|\mathbf{A}\|_{\mathrm{F}}^2}\right)^k\|\mathbf{A}\mathbf{x}^0-\mathbf{b}\|_2^2.
\end{aligned}$$

If $t<n$, then it follows from $(\mathbf{I}_{:,\mathcal{J}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{I}_{:,\mathcal{J}}=\mathbf{A}_{:,\mathcal{J}}^{\mathrm{T}}\mathbf{A}_{:,\mathcal{J}}\preceq\rho\mathbf{I}$ (since $\rho=\max_{1\leq j\leq t}\sigma_1^2(\mathbf{A}_{:,\mathcal{J}_j})$) that

$$\begin{aligned}
\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2 &\leq \|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\quad+\alpha^2(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\rho\mathbf{I}_{:,\mathcal{I}}\mathbf{A}_{\mathcal{I},\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^4}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\leq \|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\quad+\alpha^2(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_2^2}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\frac{\rho\mathbf{I}_{:,\mathcal{I}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\quad\text{(by Lemma 6)}\\
&\leq \|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\mathbf{A}\mathbf{I}_{:,\mathcal{J}}(\mathbf{A}_{\mathcal{I},\mathcal{J}})^{\mathrm{T}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\
&\quad+\alpha^2(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^{\mathrm{T}}\left(\frac{\rho\beta\mathbf{I}_{:,\mathcal{I}}(\mathbf{I}_{:,\mathcal{I}})^{\mathrm{T}}}{\|\mathbf{A}_{\mathcal{I},\mathcal{J}}\|_{\mathrm{F}}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}).
\end{aligned}$$

Taking expectation gives

$$\begin{aligned}\mathbb{E}[\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2\ |\mathbf{x}^{k-1}] &\leq \left(1+\frac{t\rho\beta\alpha^2}{\|\mathbf{A}\|_\mathrm{F}^2}\right)\|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2-2\alpha(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})^\mathrm{T}\left(\frac{\mathbf{A}\mathbf{A}^\mathrm{T}}{\|\mathbf{A}\|_\mathrm{F}^2}\right)(\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b})\\ &\leq \left(1-\frac{2\alpha\sigma_r^2(\mathbf{A})-t\rho\beta\alpha^2}{\|\mathbf{A}\|_\mathrm{F}^2}\right)\|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2.\end{aligned}$$

The last inequality follows from $\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\in\mathrm{range}(\mathbf{A})$ and Lemma 7. Taking expectation again gives

$$\begin{aligned}\mathbb{E}[\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2] &= \mathbb{E}[\mathbb{E}[\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|_2^2\ |\mathbf{x}^{k-1}]]\\ &\leq \left(1-\frac{2\alpha\sigma_r^2(\mathbf{A})-t\rho\beta\alpha^2}{\|\mathbf{A}\|_\mathrm{F}^2}\right)\mathbb{E}[\|\mathbf{A}\mathbf{x}^{k-1}-\mathbf{b}\|_2^2]\\ &\leq \left(1-\frac{2\alpha\sigma_r^2(\mathbf{A})-t\rho\beta\alpha^2}{\|\mathbf{A}\|_\mathrm{F}^2}\right)^k\|\mathbf{A}\mathbf{x}^0-\mathbf{b}\|_2^2. \qquad \square\end{aligned}$$

Remark 12. *For the special case $s=1$, $t=n$, $\alpha=\sigma_r^2(\mathbf{A})/\|\mathbf{A}\|_\mathrm{F}^2$ (i.e., the randomized Gauss–Seidel algorithm, we have $\beta=1$ in this case) and the special case $s=m$, $t=n$, $\alpha=\sigma_r^2(\mathbf{A})/\|\mathbf{A}\|_\mathrm{F}^2$ (i.e., the doubly stochastic Gauss–Seidel algorithm, we have $\beta=1$ in this case), the results of Theorem 11 are given in [11, Theorem 3.2] and [19, Theorem 2], respectively.*

Remark 13. *Let $\mathbf{x}_\star$ be any solution of $\mathbf{A}^\mathrm{T}\mathbf{A}\mathbf{x}=\mathbf{A}^\mathrm{T}\mathbf{b}$. We have $(\mathbf{A}_{:,\mathcal{J}})^\mathrm{T}\mathbf{b}=(\mathbf{A}_{:,\mathcal{J}})^\mathrm{T}\mathbf{A}\mathbf{x}_\star$. Then for $s=1$ and $t\in[n]$, by the same approach as used in the proof of Theorem 11, we can prove that the convergence bounds (replacing $\mathbf{b}$ by $\mathbf{A}\mathbf{x}_\star$) in Theorem 11 still hold for inconsistent linear systems. The result for the special case $s=1$ and $t=n$ (i.e., the randomized Gauss–Seidel algorithm) was already given in the literature, for example, [11, Theorem 3.2], [13, Lemma 4.2] and [8, Theorem 3].*

3 Numerical results

In this section, we compare the performance of the doubly stochastic block Gauss–Seidel (DSBGS) algorithm proposed in this paper against the randomized Kaczmarz (RK) algorithm for solving consistent linear systems. We only run on small or medium-scale problems. The purpose is to demonstrate that even in these simple examples, DSBGS offers significant advantages over RK. All experiments are performed using MATLAB (R2019a) on a laptop with 2.7-GHz Intel Core i7 processor, 16 GB memory, and macOS Sierra (version 10.12.6).

We use DSBGS(α,ℓ,τ) to denote the doubly stochastic block Gauss–Seidel algorithm employing the step size α, the row partition $\{\mathcal{I}_i\}_{i=1}^s$ with $s=\lceil\frac{m}{\ell}\rceil$:

$$\begin{aligned}\mathcal{I}_i &= \{(i-1)\ell+1,(i-1)\ell+2,\ldots,i\ell\},\quad i=1,2,\ldots,s-1,\\ \mathcal{I}_s &= \{(s-1)\ell+1,(s-1)\ell+2,\ldots,m\},\end{aligned}$$

and the column partition $\{\mathcal{J}_j\}_{j=1}^t$ with $t=\lceil\frac{n}{\tau}\rceil$:

$$\begin{aligned}\mathcal{J}_j &= \{(j-1)\tau+1,(j-1)\tau+2,\ldots,j\tau\},\quad j=1,2,\ldots,t-1,\\ \mathcal{J}_t &= \{(t-1)\tau+1,(t-1)\tau+2,\ldots,n\}.\end{aligned}$$

The randomized Kaczmarz algorithm with step size $\alpha=1$ is the special case DSBGS$(1,1,n)$.

To construct a consistent linear system, for a given coefficient matrix $\mathbf{A}$, we set $\mathbf{b}=\mathbf{A}\mathbf{x}$ where $\mathbf{x}$ is a vector with entries generated from a standard normal distribution. All algorithms are started from the initial guess $\mathbf{x}^0=\mathbf{0}$, terminated if $\|\mathbf{x}^k-\mathbf{A}^\dagger\mathbf{b}\|_2\leq 10^{-5}$. We report the average number of iterations (denoted as ITER) of RK and DSBGS. We also report the average computing time in seconds (denoted as CPU) and the speed-up of DSBGS against RK, which is defined as

$$\text{speed-up} = \frac{\text{CPU of RK}}{\text{CPU of DSBGS}}.$$

In each experiment, ITER and CPU are averaged over 20 trials.

3.1 Synthetic data

Two types of coefficient matrices are generated as follows.

- Type I: For given m, n, $r = rank(\mathbf{A})$, and $\kappa > 1$, we construct a matrix $\mathbf{A}$ by

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}},$$

  where $\mathbf{U} \in \mathbb{R}^{m\times r}$ and $\mathbf{V} \in \mathbb{R}^{n\times r}$. Entries of $\mathbf{U}$ and $\mathbf{V}$ are generated from a standard normal distribution, and then, columns are orthonormalized:

$$[\mathtt{U}, \sim] = \mathtt{qr}(\mathtt{randn}(\mathtt{m}, \mathtt{r}), \mathtt{0}); \qquad [\mathtt{V}, \sim] = \mathtt{qr}(\mathtt{randn}(\mathtt{n}, \mathtt{r}), \mathtt{0});$$

  The matrix $\mathbf{D}$ is an $r \times r$ diagonal matrix whose diagonal entries are uniformly distributed numbers in $(1, \kappa)$:

$$\mathtt{D} = \mathtt{diag}(\mathtt{1} + (\kappa - \mathtt{1}).*\mathtt{rand}(\mathtt{r}, \mathtt{1}));$$

  So the condition number of $\mathbf{A}$, which is defined as $\sigma_1(\mathbf{A})/\sigma_r(\mathbf{A})$, is upper bounded by κ.

- Type II: For given m, n, entries of $\mathbf{A}$ are generated from a standard normal distribution:

$$\mathtt{A} = \mathtt{randn}(\mathtt{m}, \mathtt{n});$$

  So $\mathbf{A}$ is a full (column or row) rank matrix with probability one.

In Figures 1 and 2 we plot the error $\|\mathbf{x}^k - \mathbf{A}^\dagger\mathbf{b}\|_2$ of DSBGS with different step size α and different block size for full column rank consistent linear systems. From these figures, we observe that appropriate step size and block size improve the convergence remarkably. In Tables 1 and 2, we report the average numbers of iterations and the average computing times for RK and DSBGS. From these results we see DSBGS vastly outperforms RK in terms of computing times with significant speed-ups for general (overdetermined or underdetermined, full column rank or rank deficient) consistent linear systems. It should be noted that the step size $\alpha \in (0, 2/(t\beta))$ given in Theorem 8 is a sufficient condition for DSBGS's convergence. Numerical experiments show that DSBGS with some large step size (for example $\alpha = 15$ in the experiment for Figure 2) converges much faster than with step size $\alpha \in (0, 2/(t\beta))$.

3.2 Real-world data

We test RK and DSBGS on eight real-world problems from the University of Florida sparse matrix collection [7]: `abtaha1`, `WorldCities`, `cari`, `df2177`, `flower_5_1`, `football`, `relat6`, `Sandi_authors`. The first two matrices are of full column rank and the last six matrices are rank-deficient. In Table 3, we report the average numbers of iterations and the average computing times of RK and DSBGS. We observe that DSBGS based on good choices of step size and block size significantly outperforms RK. Moreover, good step size and block size are problem dependent.

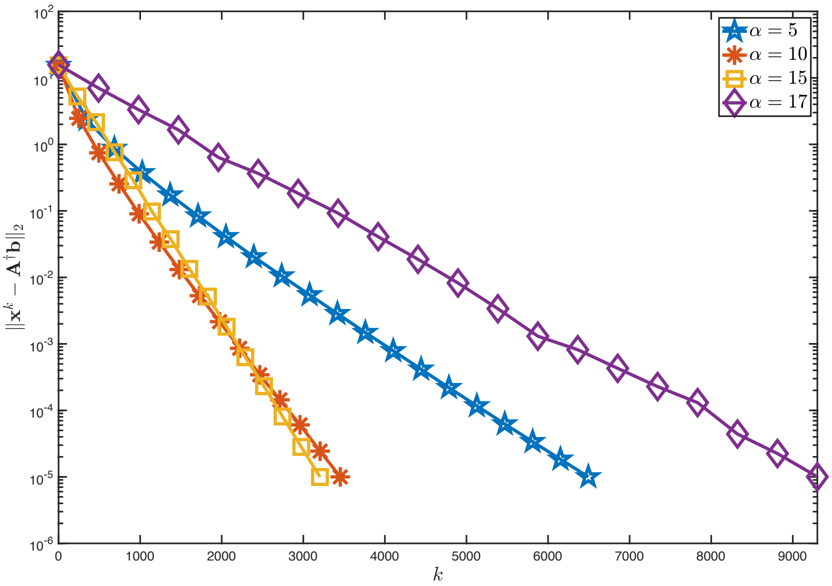

Figure 1: The average (20 trials of each case) convergence history of DSBGS with different step size ($\alpha = 5, 10, 15, 17$) and fixed block size ($\ell = 50, \tau = 50$) for a full column rank consistent linear system with random coefficient matrix $\mathbf{A}$ of Type II (`A=randn(500,250)`).

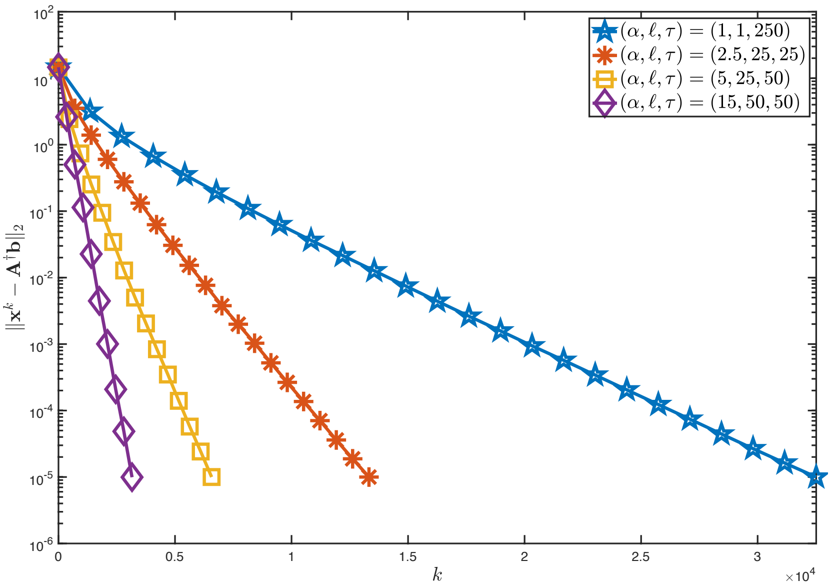

Figure 2: The average (20 trials of each case) convergence history of DSBGS with different step size and different block size for a full column rank consistent linear system with random coefficient matrix $\mathbf{A}$ of Type II (`A=randn(500,250)`).

Table 1: The average (20 trials of each experiment) ITER and CPU of RK and DSBGS(α,ℓ,τ) for consistent linear systems with random coefficient matrices $\mathbf{A}$ of Type I: $\mathbf{A}=\mathbf{UDV}^{\mathrm{T}}$.

$m\times n$	rank	κ	RK: ITER, CPU		DSBGS: ITER, CPU, (α,ℓ,τ)			speed-up
125×250	100	2	3162.55	0.0805	628.85	0.0258	$(5,5,n)$	3.12
125×250	100	10	26413.55	0.6813	5255.70	0.1751	$(5,5,n)$	3.89
125×250	100	20	158506.00	3.8662	31751.35	1.0482	$(5,5,n)$	3.69
250×500	200	2	6791.10	0.2270	638.40	0.0482	$(10,10,n)$	4.71
250×500	200	10	69568.35	2.3030	6912.85	0.5141	$(10,10,n)$	4.48
250×500	200	20	252768.05	8.3663	25328.15	1.8639	$(10,10,n)$	4.49
250×125	125	2	4215.20	0.1138	975.25	0.0291	$(5,25,25)$	3.90
250×125	125	10	38675.35	1.0213	8241.10	0.2396	$(5,25,25)$	4.26
250×125	125	20	101769.10	2.6832	23687.75	0.6682	$(5,25,25)$	4.02
500×250	250	2	8637.00	0.2758	993.80	0.0602	$(10,50,50)$	4.58
500×250	250	10	85328.80	2.7336	9732.80	0.5871	$(10,50,50)$	4.66
500×250	250	20	448211.30	14.1407	45301.45	2.7288	$(10,50,50)$	5.18

Table 2: The average (20 trials of each experiment) ITER and CPU of RK and DSBGS(α,ℓ,τ) for consistent linear systems with random coefficient matrices $\mathbf{A}$ of Type II: `A=randn(m,n)`. Here $\kappa(\mathbf{A})=\sigma_1(\mathbf{A})/\sigma_r(\mathbf{A})$.

$m\times n$	$\kappa(\mathbf{A})$	RK: ITER, CPU		DSBGS: ITER, CPU, (α,ℓ,τ)			speed-up
125×250	5.66	16118.20	0.4160	3210.75	0.1063	$(5,5,n)$	3.91
125×500	2.87	5876.40	0.1719	1134.60	0.0756	$(5,5,n)$	2.28
125×1000	2.09	3867.00	0.1571	727.65	0.0691	$(5,5,n)$	2.27
250×500	5.56	30557.05	1.0139	3091.35	0.2273	$(10,10,n)$	4.46
250×1000	2.96	12015.30	0.5085	1174.55	0.1310	$(10,10,n)$	3.88
250×2000	2.06	7927.15	0.4963	751.75	0.1959	$(10,10,n)$	2.53
500×750	9.66	173700.40	7.2715	17381.35	1.7435	$(10,10,n)$	4.17
500×1500	3.66	33019.55	2.1238	3325.55	0.5445	$(10,10,n)$	3.90
500×3000	2.34	18520.60	2.4642	1813.70	0.7393	$(10,10,n)$	3.33
250×125	5.33	13981.95	0.3709	3053.15	0.0864	$(5,25,25)$	4.29
500×125	2.96	6095.05	0.1811	1338.45	0.0393	$(5,25,25)$	4.60
1000×125	2.07	4112.30	0.1488	1003.05	0.0344	$(5,25,25)$	4.32
500×250	5.77	32563.80	1.0330	6958.60	0.4097	$(5,50,25)$	2.52
1000×250	2.87	11965.45	0.4537	2724.85	0.1763	$(5,50,25)$	2.57
2000×250	2.12	8489.30	0.4305	2086.25	0.1430	$(5,50,25)$	3.01
750×500	9.50	148619.65	5.9663	32345.90	2.8518	$(5,50,50)$	2.09
1500×500	3.68	33655.05	1.6979	7212.25	0.6977	$(5,50,50)$	2.43
3000×500	2.36	18990.70	1.3557	4378.90	0.5279	$(5,50,50)$	2.57

Table 3: The average (20 trials of each experiment) ITER and CPU of RK and DSBGS(α, ℓ, τ) for consistent linear systems with coefficient matrices from the University of Florida sparse matrix collection. Here $\kappa(\mathbf{A}) = \sigma_1(\mathbf{A})/\sigma_r(\mathbf{A})$. The first two matrices are of full column rank and the last six matrices are rank deficient.

Matrix	$m \times n$	$\kappa(\mathbf{A})$	RK: ITER, CPU		DSBGS: ITER, CPU, (α, ℓ, τ)			speed-up
abtaha1	14596×209	12.23	1.83e05	42.2491	3.91e04	2.8988	$(5, 10, n)$	14.57
WorldCities	315×100	66.00	7.38e04	1.9794	3.09e04	0.8624	$(2.5, 10, n)$	2.30
cari	400×1200	3.13	9.79e03	0.5035	2.76e03	0.3186	$(2.5, 5, n)$	1.58
df2177	630×10358	2.01	1.63e04	8.5768	2.66e03	5.0829	$(5, 10, n)$	1.69
flower_5_1	211×201	13.70	9.55e04	2.6053	3.83e04	1.2001	$(2.5, 5, n)$	2.17
football	35×35	166.47	7.88e05	15.1897	3.94e05	8.4485	$(2, 4, n)$	1.80
relat6	2340×157	7.74	2.66e04	1.4331	1.03e04	0.3969	$(2.5, 10, n)$	3.61
Sandi_authors	86×86	189.58	2.16e06	45.6432	8.66e05	21.5159	$(2.5, 5, n)$	2.12

4 Concluding remarks

We have proposed a doubly stochastic block Gauss–Seidel algorithm for solving linear systems and prove its convergence theory. The randomized Kaczmarz algorithm, the randomized Gauss–Seidel algorithm, and the doubly stochastic Gauss–Seidel algorithm are special cases of the doubly stochastic block Gauss–Seidel algorithm. Numerical experiments show that appropriate step size and block size significantly improve the performance. Finding appropriate variable step size, proposing more effective sampling strategies for submatrices, and designing other block variants via the ideas in [15] should be valuable topics in the future study.

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