

The alternating direction method of multipliers for finding the distance between ellipsoids

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Abstract

We study several versions of the alternating direction method of multipliers (ADMM) for solving the convex problem of finding the distance between two ellipsoids and the nonconvex problem of finding the distance between the boundaries of two ellipsoids. In the convex case we present the ADMM with and without automatic penalty updates and demonstrate via numerical experiments on problems of various dimensions that our methods significantly outperform all other existing methods for finding the distance between ellipsoids. In the nonconvex case we propose a heuristic rule for updating the penalty parameter and a heuristic restarting procedure (a heuristic choice of a new starting point for the second run of the algorithm). The restarting procedure was verified numerically with the use of a global method based on KKT optimality conditions. The results of numerical experiments on various test problems showed that this procedure always allows one to find a globally optimal solution in the nonconvex case. Furthermore, the numerical experiments also demonstrated that our version of the ADMM significantly outperforms existing methods for finding the distance between the boundaries of ellipsoids on problems of moderate and high dimensions.

1 Introduction

The alternating direction method of multipliers (ADMM) is an efficient method for solving structured convex optimisation problems [3, 6, 8, 11, 12, 30] that has found a wide variety of applications [5, 7, 15, 17], including applications to some nonconvex and nonsmooth optimisation problems [10, 29]. Although the ADMM was originally developed for structured convex problems, extensions of this method to various nonconvex settings have recently become an active area of research [4, 9, 13, 20, 21, 26, 28, 31].

Recently, a version of the ADMM for finding the Euclidean projection of a point onto an ellipsoid has been developed [15]. Numerical experiments presented in [15] showed that the ADMM is significantly faster than other existing methods for projecting a point onto an ellipsoid.

Being inspired by paper [15], we propose to apply the ADMM to the problem of finding the distance between two ellipsoids or the boundaries of two ellipsoids, which have attracted a considerable attention of researchers. A detailed algebraic analysis of these and related

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problems was presented in [22–24]. An exact penalty method for finding the distance between the boundaries of two ellipsoids was developed in [25]. A geometric method for finding the distance between two ellipsoids was studied in [16], while the so-called charged balls method for solving this problem was proposed in [1]. A global optimisation method for solving a closely related (but more difficult) nonconvex problem of finding the so-called *signed* distance between two ellipsoids was developed in [14].

The main goal of this paper is to develop the ADMM for solving both the convex problem of finding the distance between two ellipsoids and the nonconvex problem of finding the distance between the boundaries of two ellipsoids. In the convex case we present the standard version of the ADMM with fixed penalty parameter and, following the ideas of [12], a version of the ADMM with automatic adjustments of the penalty parameter. To verify the efficiency of the proposed method, we present some results of numerical experiments on problems of various dimensions. These results demonstrate that the ADMM with penalty adjustments is faster than the ADMM with fixed penalty parameter, and both versions of the ADMM significantly outperform all other existing methods for finding the distance between ellipsoids.

In the nonconvex case, we present a version of the ADMM with a heuristic rule for updating the penalty parameter. We provide a theoretical analysis of this method with a different rule for updating the penalty parameter. This rule performed poorly in numerical experiments; nevertheless, its analysis sheds some light on the overall performance of the ADMM and helps one to gain an insight into the choice of parameters of the method and their role in the optimisation process.

Since the problem of finding the distance between the boundaries of two ellipsoids is essentially nonconvex (i.e. there is always a locally optimal solution of this problem, which is not globally optimal), we also propose a heuristic restarting procedure for our method. Namely, we propose a heuristic choice of a new starting point for the second run of the algorithm that is defined by a point computed during the first run. To verify whether this strategy allows one to find a globally optimal solution, we describe a slight modification of the method from [14] that provably computes a globally optimal solution of the problem of finding the distance between the boundaries of two ellipsoids. Then we present and discuss some results of numerical experiments on problems of various dimensions. These results (as well as results of numerous other experiments on various test problems not presented in this paper) demonstrate that the proposed heuristic restarting procedure always allows one to find a globally optimal solution. Furthermore, we applied our restarting procedure to the exact penalty method from [25], and the results of numerical experiments showed that this method with the restarting procedure always finds a globally optimal solution as well, but without this procedure the method often converges to a locally optimal solution. Finally, the numerical experiments show that the ADMM significantly outperforms other methods for finding the distance between the boundaries of ellipsoids on problems of moderate and high dimension, while on small dimensional problems (namely, when the dimension is smaller than 8), the modification of the global method from [14] is the fastest method.

The paper is organised as follows. The convex problem of finding the distance between two ellipsoids is studied in Section 2. In Subsection 2.1 we present two versions of the ADMM for solving this problem, Subsection 2.2 contains a theoretical result on the convergence of these methods, while the results of numerical experiments are given in Subsection 2.3. Section 3 is devoted to the nonconvex problem of finding the distance between the boundaries of two ellipsoids. The ADMM for solving this problem is described in Subsection 3.1. Its theoretical analysis is presented in Subsection 3.2, while a heuristic restarting procedure for the method and a global method for solving the problem are considered in Subsection 3.3.

Finally, some results of numerical experiments in the nonconvex case are given in Subsection 3.4.

2 The distance between ellipsoids: the convex case

In the first part of the paper we study the problem of finding the distance between two ellipsoids \mathcal{E}_1 and \mathcal{E}_2 in \mathbb{R}^d , defined as

$$\mathcal{E}_i = \left\{ x \in \mathbb{R}^d \mid \langle x - z_i, Q_i(x - z_i) \rangle \leq 1 \right\}, \quad i \in \{1, 2\}. \quad (1)$$

Here $\langle \cdot, \cdot \rangle$ is the inner product in \mathbb{R}^d , $z_1, z_2 \in \mathbb{R}^d$ are the centres of the ellipsoids, and $Q_1, Q_2 \in \mathbb{R}^{d \times d}$ are positive definite symmetric matrices. The problem of finding the distance between these ellipsoids is a convex programming problem that can be formalised in the following way:

$$\begin{aligned} \min \quad & \frac{1}{2} \|x_1 - x_2\|^2 \\ \text{subject to} \quad & \langle x_1 - z_1, Q_1(x_1 - z_1) \rangle \leq 1, \quad \langle x_2 - z_2, Q_2(x_2 - z_2) \rangle \leq 1. \end{aligned} \quad (2)$$

Here $\|\cdot\|$ is the Euclidean norm and $x_1, x_2 \in \mathbb{R}^d$. Problem (2) is a convex quadratically constrained quadratic programming problem, which means that almost all general convex programming methods, such as interior point methods, can be used to find its solution. However, it is more efficient to utilise a method exploiting the structure of problem (2). Our main goal in this section is to describe and analyse one such method.

Remark 1. Let us note that the main results of the paper can be easily extended to the case when the ellipsoids are defined as

$$\mathcal{E}_i = \left\{ x \in \mathbb{R}^d \mid \langle x, A_i x \rangle + \langle b_i, x \rangle + \alpha_i \leq 0 \right\}, \quad i \in \{1, 2\},$$

where the matrices A_1 and A_2 are positive definite and $b_1, b_2 \in \mathbb{R}^d$. Namely, if one puts

$$z_i = -\frac{1}{2} A_i^{-1} b_i, \quad Q_i = \frac{1}{-0.5 \langle b_i, z_i \rangle - \alpha_i} A_i, \quad i \in \{1, 2\},$$

then equalities (1) hold true and one can apply the method presented in this article to the problem under consideration. Note that $-0.5 \langle b_i, z_i \rangle - \alpha_i > 0$, provided the ellipsoid \mathcal{E}_i is nondegenerate (i.e. its interior is nonempty), since z_i is the point of global minimum of the quadratic function from the definition of \mathcal{E}_i , while its optimal value is equal to $0.5 \langle b_i, z_i \rangle + \alpha_i < 0$.

2.1 The alternating direction method of multipliers

Being inspired by ideas of Jia, Cai, and Han [15] on algorithms for projecting a point onto an ellipsoid, we propose to solve the problem of finding the distance between two ellipsoids with the use of the alternating direction method of multipliers (ADMM). Recall that this method was originally developed to solve convex optimisation problems of the form:

$$\min_{(x,y)} f(x) + g(y) \quad \text{subject to} \quad Ax + By = c, \quad x \in X, \quad y \in Y. \quad (3)$$

Here $f: \mathbb{R}^d \rightarrow \mathbb{R}$ and $g: \mathbb{R}^m \rightarrow \mathbb{R}$ are convex functions, $A \in \mathbb{R}^{l \times d}$, $B \in \mathbb{R}^{l \times m}$, and $c \in \mathbb{R}^l$, while $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^m$ are closed convex sets. A theoretical scheme of the ADMM for solving problem (3) is as follows:

$$\begin{aligned} x^{n+1} &= \arg \min_{x \in X} \left\{ f(x) - \langle \lambda^n, Ax \rangle + \frac{\tau}{2} \|Ax + By^n - c\|^2 \right\}, \\ y^{n+1} &= \arg \min_{y \in Y} \left\{ g(y) - \langle \lambda^n, By \rangle + \frac{\tau}{2} \|Ax^{n+1} + By - c\|^2 \right\}, \\ \lambda^{n+1} &= \lambda^n - \tau(Ax^{n+1} + By^{n+1} - c). \end{aligned}$$

Here λ^n is an approximation of Lagrange multiplier for problem (3) and $\tau > 0$ is the penalty parameter. It is well-known that, if well-defined, the ADMM converges to a globally optimal solution of problem (3) for any value of the penalty parameter τ under very mild assumptions (see [5, 6, 11, 30] for more details). However, its numerical performance depends on the value of τ . A poor choice of this parameter might significantly slow down the convergence of the method. Therefore, below we will use a version of the ADMM with automatic adjustments of the penalty parameter proposed in [12], which will be described later.

In order to apply the ADMM to problem (2) let us rewrite this problem as an optimisation problem of the form (3). Let S_i be the square root of the matrix Q_i , which exists since the matrices Q_1 and Q_2 are positive definite. Recall that S_i is a positive definite matrix and $S_i S_i = Q_i$. Consequently, one has

$$\mathcal{E}_i = \left\{ x_i \in \mathbb{R}^d \mid \langle x_i - z_i, Q_i(x_i - z_i) \rangle \leq 1 \right\} = \left\{ x_i \in \mathbb{R}^d \mid \|S_i(x_i - z_i)\|^2 \leq 1 \right\}.$$

Denote $c_i = S_i z_i$ and $y_i = S_i x_i - c_i$, $i \in \{1, 2\}$. Then problem (2) can be rewritten as follows:

$$\min_{(x, y)} \frac{1}{2} \|x_1 - x_2\|^2 \quad \text{s.t.} \quad S_i x_i - y_i = c_i, \quad \|y_i\| \leq 1, \quad i \in \{1, 2\}. \quad (4)$$

Here $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2d}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^{2d}$. Note that problem (4) is a particular case of problem (3) with $f(x) = 0.5\|x_1 - x_2\|^2$, $g(y) \equiv 0$, and

$$\begin{aligned} A &= \begin{pmatrix} S_1 & \mathbb{O}_d \\ \mathbb{O}_d & S_2 \end{pmatrix}, \quad B = \begin{pmatrix} -I_d & \mathbb{O}_d \\ \mathbb{O}_d & -I_d \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \\ X &= \mathbb{R}^{2d}, \quad Y = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^{2d} \mid \|y_1\| \leq 1, \|y_2\| \leq 1 \right\}, \end{aligned}$$

where \mathbb{O}_d is the zero matrix of order d and I_d is the identity matrix of order d . Therefore we can apply the ADDM to find its solution. In the case of problem (4) this method takes the following form:

$$\begin{aligned} x^{n+1} &= \arg \min_{x_1, x_2 \in \mathbb{R}^d} \left\{ \frac{1}{2} \|x_1 - x_2\|^2 + \sum_{i=1}^2 \left(-\langle \lambda_i^n, S_i x_i \rangle + \frac{\tau}{2} \|S_i x_i - y_i^n - c_i\|^2 \right) \right\}, \\ y_i^{n+1} &= \arg \min_{\|y_i\| \leq 1} \left(\langle \lambda_i^n, y_i \rangle + \frac{\tau}{2} \|S_i x_i^{n+1} - y_i - c_i\|^2 \right), \quad i \in \{1, 2\} \\ \lambda_i^{n+1} &= \lambda_i^n - \tau(S_i x_i^{n+1} - y_i^{n+1} - c_i), \quad i \in \{1, 2\}. \end{aligned} \quad (5)$$

Observe that in order to find x^{n+1} one simply has to minimise the quadratic function

$$f_n(x) = \frac{1}{2} \|x_1 - x_2\|^2 + \sum_{i=1}^2 \left(-\langle \lambda_i^n, S_i x_i \rangle + \frac{\tau}{2} \|S_i x_i - y_i^n - c_i\|^2 \right). \quad (6)$$

As is easily seen, the Hessian matrix of this function does not depend on n and has the form

$$H(\tau) = \begin{pmatrix} I_d + \tau Q_1 & -I_d \\ -I_d & I_d + \tau Q_2 \end{pmatrix}. \quad (7)$$

This matrix is positive definite, since

$$\langle x, H(\tau)x \rangle = \|x_1 - x_2\|^2 + \tau \langle x_1, Q_1 x_1 \rangle + \tau \langle x_2, Q_2 x_2 \rangle \quad \forall x \in \mathbb{R}^{2d}$$

and the matrices Q_i are positive definite. Therefore the point x^{n+1} is correctly defined. It can be found, e.g. by the direct minimisation of the function f_n via the conjugate gradient method or by solving the system of linear equations $\nabla f_n(x^{n+1}) = 0$ with a positive definite matrix, which has the form:

$$\begin{pmatrix} I_d + \tau Q_1 & -I_d \\ -I_d & I_d + \tau Q_2 \end{pmatrix} x^{n+1} = \begin{pmatrix} S_1 & \mathbb{O}_d \\ \mathbb{O}_d & S_2 \end{pmatrix} (\lambda^n + \tau(y^n + c)). \quad (8)$$

In our numerical experiments we computed x^{n+1} by directly solving this system with the use of the Cholesky decomposition of the Hessian matrix $H(\tau)$. Below we describe an algorithm for finding the distance using this approach, although it should be noted that other methods for computing x^{n+1} can be applied.

Remark 2. One can reduce the dimension of system (8) by resolving it with respect to, say, x_2^{n+1} . Namely, from (8) it follows that

$$x_2^{n+1} = (I_d + \tau Q_2)x_1^{n+1} - S_2(\lambda_2^n + \tau(y_2^n + c_2)).$$

Hence one gets the following system of linear equations for x_1 :

$$\begin{aligned} \left(\tau(Q_1 + Q_2) + \tau^2 Q_2 Q_1 \right) x_1^{n+1} &= S_2(\lambda_2^n + \tau(y_2^n + c_2)) \\ &+ (I_d + \tau Q_2) \left[S_1(\lambda_1^n + \tau(y_1^n + c_1)) \right]. \end{aligned} \quad (9)$$

Note that the dimension of this systems is twice smaller than the dimension of system (8). We performed numerical experiments on various test problems to verify whether it is more efficient to solve the reduced system (9) instead of system (8). The results of these experiments demonstrated that the implementation of the ADMM directly solving system (8) outperforms the implementation based on the reduced system (9) on all test problems with both sparse and dense matrices Q_1 and Q_2 . However, this effect might be due to some peculiarities of our implementation of the ADMM in MATLAB, and other implementations may lead to different results.

Observe that problem (5) for computing the next estimate of y is equivalent to the following optimisation problem:

$$\min_{\|y_i\| \leq 1} \frac{\tau}{2} \left\| y_i - S_i x_i^{n+1} + c_i + \frac{1}{\tau} \lambda_i^n \right\|^2, \quad i \in \{1, 2\}$$

Thus, y_i^{n+1} is nothing but the projection of the point $S_i x_i^{n+1} - c_i - (1/\tau) \lambda_i^n$ onto the unit ball centered at the origin, which can be easily computed analytically.

To define a stopping criterion for the method, let us consider optimality conditions for problem (4) and more general problem (3). The KKT optimality conditions for problem (3)

have the form:

$$\begin{aligned} x^* &\in X, \quad y^* \in Y, \quad Ax^* + By^* = c, \\ \langle \nabla f(x^*) - A^T \lambda^*, x - x^* \rangle &\geq 0 \quad \forall x \in X, \\ \langle \nabla g(y^*) - B^T \lambda^*, y - y^* \rangle &\geq 0 \quad \forall y \in Y. \end{aligned}$$

In the case of problem (4) these conditions take the form:

$$\|y_i^*\| \leq 1, \quad S_i x_i^* - y_i^* = c_i, \quad i \in \{1, 2\}, \quad (10)$$

$$x_1^* - x_2^* - S_1 \lambda_1^* = 0, \quad x_2^* - x_1^* - S_2 \lambda_2^* = 0, \quad (11)$$

$$\langle \lambda_i^*, y_i - y_i^* \rangle \geq 0 \quad \forall y_i: \|y_i\| \leq 1, \quad i \in \{1, 2\}. \quad (12)$$

Note that conditions (12) are satisfied iff $y_i^* = \text{Pr}_{B^d}(y_i^* - \lambda_i^*)$, $i \in \{1, 2\}$, where $\text{Pr}_{B^d}(\cdot)$ is the Euclidean projection onto the unit ball B^d in \mathbb{R}^d centered at the origin. Indeed, by definition $y_i^* = \text{Pr}_{B^d}(y_i^* - \lambda_i^*)$ iff y_i^* is an optimal solution of the convex problem

$$\min \frac{1}{2} \|y_i - (y_i^* - \lambda_i^*)\|^2 \quad \text{subject to} \quad \|y_i\| \leq 1.$$

Bearing in mind the fact that the necessary and sufficient optimality conditions for this problem coincide with (12) we arrive at the required result. Let us note that the equivalence between the optimality condition of the form (12) and the corresponding equality for the projection was pointed out, e.g. in [15].

The violation of optimality conditions (10)–(12) can be measured with the use of the following functions:

$$\begin{aligned} R_x(x, y, \lambda) &= \begin{pmatrix} x_1 - x_2 - S_1 \lambda_1 \\ x_2 - x_1 - S_2 \lambda_2 \end{pmatrix}, \quad R_y(x, y, \lambda) = \begin{pmatrix} y_1 - \text{Pr}_{B^d}(y_1 - \lambda_1) \\ y_2 - \text{Pr}_{B^d}(y_2 - \lambda_2) \end{pmatrix}, \\ R_c(x, y, \lambda) &= \begin{pmatrix} S_1 x_1 - y_1 - c_1 \\ S_2 x_2 - y_2 - c_2 \end{pmatrix}. \end{aligned}$$

The function $R_c(x, y, \lambda)$ measures the violation of the equality constraints of problem (4). The value $\|R_x(x, y, \lambda)\| + \|R_y(x, y, \lambda)\| + \|R_c(x, y, \lambda)\|$ indicates how far a point (x, y) lies from an optimal solution of problem (4), which implies that the condition

$$\|R_x(x^{n+1}, y^{n+1}, \lambda^{n+1})\| + \|R_y(x^{n+1}, y^{n+1}, \lambda^{n+1})\| + \|R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})\| < \varepsilon \quad (13)$$

can be used as a stopping criterion. Let us note that different stopping criteria, such as the standard one based on primal-dual residuals [5, Section 3.3], can be used as well. We opted for criterion (13), since similar stopping criteria for the ADMM performed well in a number of numerical experiments presented in [15].

Thus, we arrive at the following scheme of the ADMM (given in Algorithm 1) for solving problem (4), which is equivalent to the original problem of finding the distance between ellipsoids.

Observe that in Algorithm 1 the Cholesky decomposition $H(\tau) = LL^T$ is computed only once before the main iterations of the method start. The use of this decomposition allows one to significantly reduce the cost of computing the next iterate x^{n+1} in comparison with direct solution of the corresponding system of linear equation. Furthermore, we chose the Cholesky decomposition for solving the corresponding linear systems because of its well-known efficiency and numerical stability. However, the downside of this approach is the

Algorithm 1: The ADMM for finding the distance between ellipsoids.

Step 1. Choose $y^0, \lambda^0 \in \mathbb{R}^{2d}$, $\tau > 0$, and $\varepsilon > 0$, and set $n := 0$.

Step 2. Compute the square roots S_i of the matrices Q_i , vectors $c_i = S_i z_i$, $i \in \{1, 2\}$, and compute the Cholesky decomposition $H(\tau) = LL^T$ of the matrix $H(\tau)$ given in (7).

Step 3. Compute $u^n = \begin{pmatrix} S_1(\lambda_1^n + \tau(y_1^n + c_1)) \\ S_2(\lambda_2^n + \tau(y_2^n + c_2)) \end{pmatrix}$ and solve the linear systems $Lw^n = u^n$ and $L^T x^{n+1} = w^n$.

Step 4. Compute $v_i^n = S_i x_i^{n+1} - c_i - (1/\tau)\lambda_i^n$, $i \in \{1, 2\}$. If $\|v_i^n\| \leq 1$, then $y_i^{n+1} = v_i^n$; otherwise, $y_i^{n+1} = \frac{1}{\|v_i^n\|} v_i^n$, $i \in \{1, 2\}$.

Step 5. Put $\lambda_i^{n+1} = \lambda_i^n - \tau(S_i x_i^{n+1} - y_i^{n+1} - c_i)$, $i \in \{1, 2\}$.

Step 6. Compute $R_x(x^{n+1}, y^{n+1}, \lambda^{n+1})$, $R_y(x^{n+1}, y^{n+1}, \lambda^{n+1})$, and $R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})$. If condition (13) holds true, then **Stop**. Otherwise, set $n := n + 1$ and go to **Step 3**.

fact that one has to store the lower triangular matrix L of order $2d$ throughout iterations. Therefore, for large-scale problems a direct minimization of the corresponding quadratic function might be more efficient due to reduced memory consumption.

Let us also describe a version of the ADMM for finding the distance between ellipsoids with automatic adjustments of the penalty parameter τ . The rule for modifying τ that we use was proposed in [12]. Numerical experiments on various problems [12, 15] clearly demonstrated that the ADMM using this rule significantly outperforms the standard version of the ADMM with fixed τ . Let us also note that the rule for updating the penalty parameter from [12] is defined via an arbitrary sequence $\{\alpha_n\} \subset [0, +\infty)$, and the convergence of the ADMM is established in [12] under the assumption that $\sum_{n=0}^{\infty} \alpha_n < +\infty$.

A version of the ADMM with automatic adjustments of the penalty parameter is given in Algorithm 2.

The only significant difference between Algorithms 1 and 2 is the additional step (Step 8) describing adjustments of the penalty parameter τ_n . However, note that in Algorithm 2, in contrast to Algorithm 1, one needs to recompute the Cholesky decomposition of the Hessian matrix (7) every time the penalty parameter is adjusted. Results of numerical experiments given in Section 2.3 demonstrate that (i) the adjustments of the penalty parameter allow one to substantially reduce the number of iterations of the ADMM needed to find an optimal solution with a prescribed tolerance, and (ii) the benefits of adjusting the penalty parameter largely outweigh the extra time needed to recompute the Cholesky decomposition of the Hessian matrix multiple times.

We used the boolean variable “flag” to indicate whether an update of the Cholesky decomposition is needed. It helps one to avoid unnecessary computations by making the algorithm recompute the decomposition only when the penalty parameter has been changed.

Remark 3. In our numerical experiments with ill-conditioned problems, stopping criterion (13) was sometimes too optimistic in the sense that it led to the termination of the algorithm before the desired accuracy of solution was achieved. To overcome this issue, in our implementations of Algorithms 1 and 2 we additionally checked the conditions

$$|\langle x_i^{n+1} - z_i, Q_i(x_i^{n+1} - z_i) \rangle - 1| < \varepsilon, \quad i \in \{1, 2\}, \quad (14)$$

if condition (13) was satisfied and $\|x_1^{n+1} - x_2^{n+1}\| > \delta$ for some small $\delta > 0$. Let us note that inequalities (14) simply mean that the points x_i^{n+1} lie close to the boundaries of the

Algorithm 2: The ADMM with penalty adjustments.

- Step 1.** Choose $y^0, \lambda^0 \in \mathbb{R}^{2d}$, $\tau_0 > 0$, $\eta \in (0, 1)$, a sequence $\{\alpha_n\} \subset [0, +\infty)$, and $\varepsilon > 0$. Set $n := 0$ and **flag** = **true**.
- Step 2.** Compute the square roots S_i of the matrices Q_i and vectors $c_i = S_i z_i$, $i \in \{1, 2\}$.
- Step 3.** If **flag** = **true**, compute the Cholesky decomposition $H(\tau_n) = LL^T$ of the matrix $H(\tau_n)$ given in (7) and put **flag** = **false**.
- Step 4.** Compute $u^n = \begin{pmatrix} S_1(\lambda_1^n + \tau_n(y_1^n + c_1)) \\ S_2(\lambda_2^n + \tau_n(y_2^n + c_2)) \end{pmatrix}$ and solve the linear systems $Lw^n = u^n$ and $L^T x^{n+1} = w^n$.
- Step 5.** Compute $v_i^n = S_i x_i^{n+1} - c_i - (1/\tau_n)\lambda_i^n$, $i \in \{1, 2\}$. If $\|v_i^n\| \leq 1$, then $y_i^{n+1} = v_i^n$; otherwise, $y_i^{n+1} = \frac{1}{\|v_i^n\|} v_i^n$, $i \in \{1, 2\}$.
- Step 6.** Put $\lambda_i^{n+1} = \lambda_i^n - \tau_n(S_i x_i^{n+1} - y_i^{n+1} - c_i)$, $i \in \{1, 2\}$.
- Step 7.** Compute $R_x(x^{n+1}, y^{n+1}, \lambda^{n+1})$, $R_y(x^{n+1}, y^{n+1}, \lambda^{n+1})$, and $R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})$. If condition (13) holds true, then **Stop**.
- Step 8.** Define

$$\tau_{n+1} = \begin{cases} \tau_n(1 + \alpha_n), & \text{if } \|R_x(x^{n+1}, y^{n+1}, \lambda^{n+1})\| < \eta \|R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})\|, \\ \tau_n/(1 + \alpha_n), & \text{if } \eta \|R_x(x^{n+1}, y^{n+1}, \lambda^{n+1})\| > \|R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})\|, \\ \tau_n, & \text{otherwise.} \end{cases}$$

If $\tau_{n+1} \neq \tau_n$, set **flag** = **true**. Put $n := n + 1$ and go to **Step 3**.

corresponding ellipsoids. Condition (14) is based on the fact that if the ellipsoids \mathcal{E}_1 and \mathcal{E}_2 do not intersect, then a unique solution of problem (2) necessarily lies on the boundaries of these ellipsoids.

The additional stopping criterion (14) led to noticeably improved results in most cases when the stopping criterion (13) failed to recognise the non-optimality of the current iterate. Alternatively, decreasing ε by a factor of 100 we obtained the same or even better improvements in the accuracy of the found solution without any significant increase in the run time.

2.2 Analysis of the methods

Let us briefly discuss convergence of the two methods for finding the distance between ellipsoids described in the previous section. Note that it is sufficient to consider only Algorithm 2, since this algorithm is reduced to Algorithm 1 when $\alpha_n \equiv 0$.

Theorem 1. *For any choice of parameters $y^0, \lambda^0 \in \mathbb{R}^{2d}$, $\tau_0 > 0$, $\eta \in (0, 1)$, $\{\alpha_n\} \subset [0, +\infty)$, $\sum_{n=0}^{\infty} \alpha_n < +\infty$, and $\varepsilon > 0$ Algorithm 2 is well-defined and terminates after a finite number of iterations. Moreover, if $\varepsilon = 0$, then Algorithm 2 generates a sequence $\{(x^n, y^n, \lambda^n)\}$ that converges to a point (x^*, y^*, λ^*) satisfying optimality conditions (10)–(12) and such that x^* is a globally optimal solution of problem (2).*

Proof. The fact that Algorithm 2 is well-defined follows directly from its detailed description given in the previous subsection. Indeed, Step 2 of Algorithm 2 is well-defined, since by our assumption the matrices Q_i , $i \in \{1, 2\}$, are positive definite and any positive definite matrix has a unique square root. Steps 3 and 4 are well-defined, since, as was shown above, the

matrix $H(\tau)$ is positive definite for any $\tau > 0$ and $\tau_n > 0$ for all $n \in \mathbb{N}$ according to Step 8. The correctness of all other steps of the algorithm is obvious.

Let us prove that in the case $\varepsilon = 0$ Algorithm 2 generates a sequence $\{(x^n, y^n, \lambda^n)\}$ that converges to a point (x^*, y^*, λ^*) satisfying optimality conditions (10)–(12). Then the finite termination of this algorithm in the case $\varepsilon > 0$ follows directly from the definition of stopping criterion (13). Moreover, from the fact that problem (4) is convex it follows that (x^*, y^*) is a globally optimal solution of this problem. Consequently, x^* is a globally optimal solution of problem (2), since by construction (x^*, y^*) is a globally optimal solution of problem (4) if and only if x^* is a globally optimal solution of problem (2) and $S_i(x_i^* - z_i) = y_i^*$.

To prove the convergence of the sequence $\{(x^n, y^n, \lambda^n)\}$ to a KKT point of problem (4), note that this problem is obviously equivalent to the problem of finding a solution (x^*, y^*) of the following variational inequality:

$$\begin{aligned} \langle x - x^*, F(x^*) \rangle + \langle y - y^*, G(y^*) \rangle &\geq 0 \\ \forall (x, y) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}: S_i x_i - y_i &= c_i, \quad \|y_i\| \leq 1, \quad i \in \{1, 2\}, \\ S_i x_i^* - y_i^* &= c_i, \quad \|y_i^*\| \leq 1, \quad i \in \{1, 2\}. \end{aligned} \quad (15)$$

Here $F(x) = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_1 \end{pmatrix}$ and $G(y) \equiv 0$. Note that the mappings F and G are monotone as the gradients of convex functions.

As one can easily verify, Algorithm 2 (more precisely, algorithm described in (5)) is a particular case of the ADMM from [12] applied to the variational inequality (15). By [12, Theorem 4.1 and Remark 4.2] the sequence generated by the ADMM for solving problem (15) converges to a solution of this problem for all values of parameters, provided $\sum_{n=0}^{\infty} \alpha_n < +\infty$, the algorithm is well-defined, and there exists a solution of the variational inequality (15).

The fact that the algorithm is well-defined was verified above. Note also that problem (2) and the equivalent problem (4) have globally optimal solutions, since the feasible region of problem (2) is obviously compact. In turn, any globally optimal solution of problem (4) is a solution of the variational inequality (15) due to the convexity of problem (4). Therefore one can conclude that a solution of the variational inequality (15) exists.

Thus, by [12, Theorem 4.1 and Remark 4.2] the sequence $\{(x^n, y^n, \lambda^n)\}$ generated by Algorithm 2 converges to a solution of the variational inequality (15) for all values of the parameters, provided $\sum_{n=0}^{\infty} \alpha_n < +\infty$. Any solution of (15) is a globally optimal solution of problem (4). Therefore the limit point (x^*, y^*, λ^*) must satisfy optimality conditions (10)–(12), since Slater's condition obviously holds true for problem (4) (namely, put $x_i = z_i$ and $y_i = 0$). \square

2.3 Numerical experiments

Without trying to present a thorough comparative analysis of Algorithms 1 and 2 and other existing methods for finding the distance between two ellipsoids on various problem instances (e.g. ellipsoids lying very far apart or very close to each other, ‘flat’ and ‘elongated’ ellipsoids, i.e. one of the eigenvalues of the matrix Q_i is much smaller/greater than others, etc.), let us give some results of preliminary numerical experiments demonstrating the higher efficiency of the ADMM in comparison with other existing methods for finding the distance between ellipsoids.

To generate the problem data for numerical experiments, first we randomly generated matrices A_i , $i \in \{1, 2\}$, of dimension $d \in \mathbb{N}$, whose elements were uniformly distributed in the interval $[-10, 10]$. If the matrix A_i has full rank, we define $Q_i = A_i^T A_i$. Otherwise, the

matrix A_i was randomly generated again till it had full rank, to ensure that the matrix Q_i is positive definite. The centres z_i of the ellipsoids were also randomly generated in such a way that their coordinates are uniformly distributed in the same interval $[-10, 10]$.

The parameters of Algorithms 1 and 2 were chosen as follows: $y^0 = \lambda^0 = 0$, $\varepsilon = 10^{-6}$, $\tau = \tau_0 = 1$. As in paper [12], where the penalty adapting strategy was proposed, we put $\eta = 0.1$ and $\alpha_n = 1$, if $n < 100$, while $\alpha_n = 0$, if $n \geq 100$ in Algorithm 2. Below, Algorithm 1 is denoted as ADMM, while Algorithm 2 is denoted as sa-ADMM (self-adaptive ADMM; see [12]).

We compared Algorithms 1 and 2 with Lin and Han's method [16], the exact penalty method from [25], and the so-called charged balls method [1]. The starting points in Lin and Han's (LH) algorithm were chosen as the centres z_1 and z_2 of the ellipsoids. The parameters γ_1 and γ_2 were chosen as $\gamma_i = 1/\|Q_i\|_1$, where $\|\cdot\|_1$ is the 1-norm (i.e. the maximum absolute column sum of the matrix), and the inequalities $\theta_1 < 10^{-6}$ and $\theta_2 < 10^{-6}$ were used as a stopping criterion.

The penalty parameter λ from the exact penalty method [25] was defined as $\lambda = 100$. We used the inequality $\|G^*(z_k)\| < \varepsilon = 10^{-5}$ as a stopping criterion, since in some cases the algorithm failed to terminate when the value $\varepsilon = 10^{-6}$ was used. Since no rules for choosing the starting points were given in [25] and the exact penalty method cannot start from the centres of the ellipsoids, we tried using two different initial guesses. The first rule for choosing starting points consisted in setting $x_1^0 = z_1$ and defining x_2^0 as the vector z_2 perturbed by some other vector with small coordinates, i.e. $x_2^0 = z_2 + e$. We chose $e = (0.1, \dots, 0.1)^T \in \mathbb{R}^d$. We denote the exact penalty method using this rule as EP_1 . The second rule consisted in defining x_1^0 and x_2^0 as the points at which the segment $\text{co}\{z_1, z_2\}$ intersect the boundaries of the ellipsoids. We denote the exact penalty method with these starting points as EP_2 .

Finally, for the charged balls method [1] (denoted CB) we used the same parameters as given in [1, Section 3]. Namely, we set $p_1 = 10$, $p_2 = 1$, $\delta = 0.1$, and $\varepsilon = 10^{-8}$. The starting points were chosen in the same way as in EP_2 method, since in our experiments the charged balls method produced incorrect results, when the starting points were lying in the interiors of the ellipsoids.

All algorithms were implemented in MATLAB. We terminated the algorithms if the number of iterations exceeded 10^7 for the charged balls method and 10^6 for all other methods. Since the behaviour of the algorithms appeared to be very dependent on a particular problem instance (i.e. a particular algorithm might be very slow on one randomly generated problem and very fast on another), we generated 10 problems for a given dimension $d \in \{10, 20, 30, 50, 100, 200, 300, 500, 1000, 2000\}$ and run each algorithm on these 10 problems. The total run time of each method rounded to the nearest tenth is presented in Figure 1 and Table 1.

The results of our numerical experiments clearly demonstrate that Algorithms 1 and 2 considerably outperform all other existing methods for finding the distance between two ellipsoids and are more suitable for large dimensional problems than other methods.

Let us give some comments about the performance of other methods. Firstly, numerical experiments showed that on average there is no significant difference between the exact penalty methods EP_1 and EP_2 . Thus, it is not clear how to initialise the exact penalty method [25] to improve its performance.

Secondly, one should point out that the charged balls method, in accordance with the example given in [1], produced the most inaccurate results among all methods. In our numerical experiments the distance computed by the charged balls method was at least 10^{-3} (in some examples even 10^{-2}) greater than the distance computed by other methods.

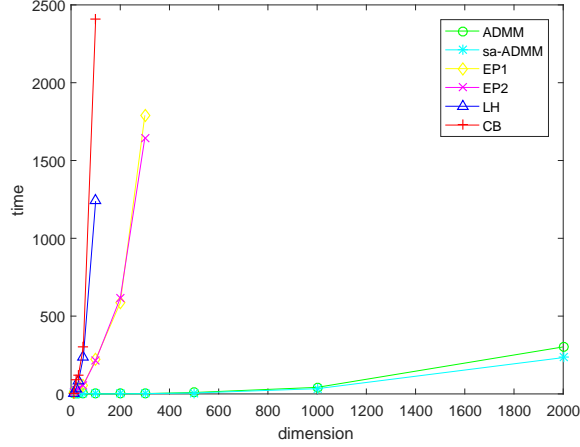


Figure 1: The run time of each method for $d \in \{10, 20, 30, 50, 100, 200, 300, 500, 1000, 2000\}$.

d	ADMM	sa-ADMM	LH	EP_1	EP_2	CB
10	0.1	0.1	3.9	7.4	10.4	6.6
20	0.1	0.1	33.2	26	27	92.1
30	0.1	0.1	81	37.4	32.6	118.8
50	0.1	0.1	236.6	57.5	57.5	303.6
100	0.2	0.2	1239.3	222.1	213.8	2408.6
200	0.9	0.7	—	585	615.4	—
300	2.1	1.8	—	1790.5	1646.6	—
500	9.2	5.9	—	—	—	—
1000	42.1	33.6	—	—	—	—
2000	302.1	233.4	—	—	—	—

Table 1: The run time of each method in seconds. ‘—’ indicates that either the time exceeded one hour or the algorithm failed to find a solution due to reaching the prespecified maximal number of iterations.

We tried decreasing ε , but it did not improve the results. By defining the stepsize δ in the charged balls method as $\delta = 10^{-3}$ (or a smaller value) we managed to obtain the same accuracy as with the use of other methods. However, such choice of the stepsize increased the run time by more than 10 times. That is why we do not report the results of numerical experiments with different stepsizes δ here. Moreover, let us note that, as was pointed out in [1], by changing the parameters p_1 and p_2 in the charged balls method one could significantly improve its performance. We tried to find optimal values of these parameters; however, it turned out that a nearly optimal choice of the parameters for one problem often became a very poor choice for another problem. That is why we used the same values of the parameters p_1 and p_2 as in [1].

Finally, the results of numerical experiments showed that Algorithm 2 with adjustable penalty parameter is faster than the ADMM with the fixed penalty parameter, despite the fact that every change of the penalty parameter requires a recomputation of the Cholesky decomposition of the matrix $H(\tau_n)$ of dimension $2d$. The reason behind this lies in the fact

d	10	20	30	50	100	200	300	500	1000	2000
ADMM	45.3	153.3	113.2	154.5	152.3	244.2	328.4	433.4	425.8	761.5
sa-ADMM	46.6	128.4	113.2	120.1	108.2	213.7	273.9	263.4	321.1	557.6

Table 2: The average number of iterations till termination for Algorithms 1 and 2.

that penalty adjustments allow one to substantially reduce the number of iterations of the algorithm. To illustrate this point, in Table 2 we present the average number of iterations till termination for Algorithms 1 and 2.

3 The distance between ellipsoids: the nonconvex case

In the second part of the paper we study the problem of finding the distance between the boundaries of the two ellipsoids \mathcal{E}_1 and \mathcal{E}_2 in \mathbb{R}^d , defined in (1). One can readily check that the boundary of the ellipsoid \mathcal{E}_i is defined by the corresponding equality constraint:

$$\text{bd } \mathcal{E}_i = \left\{ x \in \mathbb{R}^d \mid \langle x - z_i, Q_i(x - z_i) \rangle = 1 \right\}, \quad i \in \{1, 2\}.$$

Therefore, the problem of finding the distance between the boundaries of the ellipsoids is a *nonconvex* programming problem that can formalised as follows:

$$\begin{aligned} \min \quad & \frac{1}{2} \|x_1 - x_2\|^2 \\ \text{subject to} \quad & \langle x_1 - z_1, Q_1(x_1 - z_1) \rangle = 1, \quad \langle x_2 - z_2, Q_2(x_2 - z_2) \rangle = 1. \end{aligned} \tag{16}$$

Since this problem has exactly the same structure as problem (2), it is natural to extend the alternating direction method of multipliers for solving problem (2) (Algorithms 1 and 2) to the case of problem (16).

3.1 The alternating direction method of multipliers

In order to apply the ADMM to problem (16) let us rewrite this problem as an optimisation problem of the form (3). To this end, as in Section 2.1, let S_i be the square root of the matrix Q_i . Then

$$\text{bd } \mathcal{E}_i = \left\{ x_i \in \mathbb{R}^d \mid \langle x_i - z_i, Q_i(x_i - z_i) \rangle = 1 \right\} = \left\{ x_i \in \mathbb{R}^d \mid \|S_i(x_i - z_i)\|^2 = 1 \right\}.$$

Denote $c_i = S_i z_i$ and $y_i = S_i x_i - c_i$, $i \in \{1, 2\}$. Then problem (16) can be rewritten as follows:

$$\min_{(x,y)} \quad \frac{1}{2} \|x_1 - x_2\|^2 \quad \text{s.t.} \quad S_i x_i - y_i = c_i, \quad \|y_i\| = 1, \quad i \in \{1, 2\}. \tag{17}$$

This problem is very similar to problem (4) with the only difference being the fact that the inequality constraints $\|y_i\| \leq 1$ were replaced by the corresponding equality constraints $\|y_i\| = 1$, $i \in \{1, 2\}$. Therefore the only change one has to make in the ADMM for problem (4) in order to apply it to problem (17) consists in the way one computes the next iterate y_i^{n+1} . Namely, one has to define y_i^{n+1} as a projection of the point $S_i x_i^{n+1} - c_i - (1/\tau)\lambda_i^n$ onto the unit sphere, not the unit ball. Note, however, that in the case when $S_i x_i^{n+1} - c_i - (1/\tau)\lambda_i^n = 0$ this projection is not unique and one can define y_i^{n+1} as any point from the unit sphere.

It should be noted that since problem (17) is nonconvex, one cannot expect the ADMM for solving this problem to converge for all values of the penalty parameter τ . Furthermore, the penalty adjustments strategy from [12] becomes inviable. To define a rule for updating the penalty parameter that ensures convergence, observe that the ADMM can be viewed as a modification of the classical augmented Lagrangian methods based on the Hestenes-Powell-Rockafellar augmented Lagrangian [2] to the case of problems of the form (3). Therefore, it is natural to adopt the rule for updating τ similar to the one used in general augmented Lagrangian methods [2, 18, 19, 27].

Since the ADMM performs two separate steps (x -step and y -step) on every iteration, from a purely theoretical point of view it seems natural to use the following inequality as a criterion for updating the penalty parameter:

$$\max_{i \in \{1, 2\}} \left\{ \|S_i x_i^{n+1} - y_i^n - c_i\| - \eta \|S_i x_i^n - y_i^{n-1} - c_i\|, \right. \\ \left. \|S_i x_i^{n+1} - y_i^{n+1} - c_i\| - \eta \|S_i x_i^n - y_i^n - c_i\| \right\} > 0 \quad (18)$$

(here $\eta \in (0, 1)$ is a fixed parameter). Namely, if this inequality is satisfied, one defines $\tau_{n+1} = \beta \tau_n$ for some $\beta > 1$. Otherwise, one sets $\tau_{n+1} = \tau_n$. As we will show in the following section, this rule significantly simplifies convergence analysis and under some additional assumptions guarantees that the sequence $\{(x^n, y^n, \lambda^n)\}$ converges to a KKT point of problem (17). However, this penalty updating rule performed very poorly in our numerical experiments. The experiments showed that the sequence $\|S_i x_i^{n+1} - y_i^n - c_i\|$ is not monotone at initial stages, which in accordance to (18) results in a rapid increase of the penalty parameter, ill-conditioning and, ultimately, the divergence of the method. Therefore, we propose to use a different penalty updating rule, which showed itself best in numerical experiments. It should be noted that this rule is completely heuristic and its theoretical analysis is a challenging open problem.

We used the following two inequalities as a criterion for updating the penalty parameter:

$$\|S_i x_i^n - y_i^n - c_i\| \geq \varkappa, \quad \|S_i x_i^{n+1} - y_i^{n+1} - c_i\| > \eta \|S_i x_i^n - y_i^n - c_i\|, \quad i \in \{1, 2\}, \quad (19)$$

where $\varkappa > 0$ and $\eta \in (0, 1)$ are fixed parameters. If inequalities (19) are satisfied, then $\tau_{n+1} = \beta \tau_n$ for some $\beta > 1$. Otherwise, we put $\tau_{n+1} = \tau_n$. This way the penalty parameter is *not* updated, if the infeasibility measure $\|S_i x_i^n - y_i^n - c_i\|$, $i \in \{1, 2\}$, is sufficiently small or decreases with linear rate.

Remark 4. Let us explain the motivation behind criterion (19) for increasing the penalty parameter. Multiple numerical experiments with fixed penalty parameter τ demonstrated that one needs to increase τ only if the infeasibility measure $\|S_i x_i^n - y_i^n - c_i\|$, $i \in \{1, 2\}$, does not decrease with iterations. Therefore, it is natural to use the second inequality in (19) as a criterion for updating τ . In addition, numerical experiments showed that at the first stage the ADMM converges to a nearly feasible point and only later the sequence starts to converge to a point satisfying KKT optimality conditions. As a result, at later stages, when the value $\|S_i x_i^n - y_i^n - c_i\|$ is very small but nonzero, the second inequality in (19) might be violated, although there is no need to increase τ at this stage. Therefore, the first inequality in (19) is used as a safeguard to avoid an unnecessary increase of the penalty parameter. A safe choice of parameter \varkappa is $\varkappa < \varepsilon$, where $\varepsilon > 0$ is from the stopping criterion discussed below, since this choice makes penalty updating criterion (19) consistent with the stopping criterion and ensures that the penalty parameter is increased until the infeasibility measure is within the limits specified by the stopping criterion. Nevertheless, in

our experiments we defined $\varkappa = 0.1$, since this value ensured convergence to a KKT point for all test problems and allowed one to avoid unnecessary penalty updates, which might slow down the convergence.

It remains to define a stopping criterion. To this end, let us consider optimality conditions as in the convex case. The KKT conditions for problem (17) have the form:

$$x_1^* - x_2^* - S_1 \lambda_1^* = 0, \quad x_2^* - x_1^* - S_2 \lambda_2^* = 0, \quad (20)$$

$$\lambda_i^* + \mu_i^* y_i^* = 0, \quad S_i x_i^* - y_i^* = c_i, \quad \|y_i^*\| = 1 \quad i \in \{1, 2\}, \quad (21)$$

where $\mu_i^* \in \mathbb{R}$ is a Lagrange multiplier corresponding to the equality constraint $\|y_i\| - 1 = 0$. Note that the first condition in (21) is satisfied iff $\lambda_i^* = \pm \|\lambda_i^*\| y_i^*$.

The violation of optimality conditions (20) and (21) can be measured with the use of the following functions:

$$R_x(x, y, \lambda) = \begin{pmatrix} x_1 - x_2 - S_1 \lambda_1 \\ x_2 - x_1 - S_2 \lambda_2 \end{pmatrix}, \quad R_{y_i}^\pm(x, y, \lambda) = \left(\lambda_i \pm \|\lambda_i\| y_i \right), \quad i \in \{1, 2\},$$

$$R_c(x, y, \lambda) = \begin{pmatrix} S_1 x_1 - y_1 - c_1 \\ S_2 x_2 - y_2 - c_2 \end{pmatrix}.$$

It is natural to use the condition

$$\begin{aligned} \|R_x(x^{n+1}, y^{n+1}, \lambda^{n+1})\| + \sum_{i=1}^2 \min \left\{ \|R_{y_i}^-(x^{n+1}, y^{n+1}, \lambda^{n+1})\|, \|R_{y_i}^+(x^{n+1}, y^{n+1}, \lambda^{n+1})\| \right\} \\ + \|R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})\| < \varepsilon \end{aligned} \quad (22)$$

as a stopping criterion for the method.

Thus, one can propose the following scheme of the ADMM (given in Algorithm 3) for solving nonconvex problem (17), which is equivalent to the original problem (16) of finding the distance between the boundaries of two ellipsoids.

As in Algorithm 2, we used the boolean variable “flag” in order to indicate whether the Cholesky decomposition must be updated. This way the decomposition $H(\tau_n) = LL^T$ is recomputed only when the penalty parameter τ_n has been updated (i.e. increased). Let us also note that the situation when $v_i^n = 0$ on Step 5 of Algorithm 3 never occurred in our numerical experiments, provided the initial guess y^0 satisfied the constraints $\|y_1^0\| = \|y_2^0\| = 1$. Nevertheless, to avoid potentially incorrect behaviour of the algorithm one must include the case $v_i^n = 0$ into Step 5.

Finally, observe that on Step 8 of Algorithm 3 one can check the validity of the inequality

$$\|R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})\| \leq \eta \|R_c(x^n, y^n, \lambda^n)\|$$

instead of verifying two similar inequalities involving $\|S_i x_i^n - y_i^n - c_i\|$.

Remark 5. Let us point out that various existing versions of the ADMM for nonconvex problems (e.g. the proximal ADMM from [4]) can be directly applied to problem (17) instead of Algorithm 3. A comparative analysis of different versions of the ADMM for solving problem (17) is an interesting problem for future research.

3.2 Analysis of the method

Although convergence analysis of the ADMM for nonconvex problems has recently become an active area of research (see [4, 9, 10, 13, 20, 21, 26, 28, 29, 31] and the references therein),

Algorithm 3: The ADMM for finding the distance between the boundaries of ellipsoids.

- Step 1.** Choose $y^0, \lambda^0 \in \mathbb{R}^{2d}$, $\tau_0 > 0$, $\eta \in (0, 1)$, $\beta > 1$, $\varkappa > 0$, and $\varepsilon > 0$. Set $n := 0$ and **flag** = **true**.
- Step 2.** Compute the square roots S_i of the matrices Q_i and vectors $c_i = S_i z_i$, $i \in \{1, 2\}$.
- Step 3.** If **flag** = **true**, compute the Cholesky decomposition $H(\tau_n) = LL^T$ of the matrix $H(\tau_n)$ given in (7) and put **flag** = **false**.
- Step 4.** Compute $u^n = \begin{pmatrix} S_1(\lambda_1^n + \tau_n(y_1^n + c_1)) \\ S_2(\lambda_2^n + \tau_n(y_1^n + c_1)) \end{pmatrix}$ and solve the linear systems $Lw^n = u^n$ and $L^T x^{n+1} = w^n$.
- Step 5.** Compute $v_i^n = S_i x_i^{n+1} - c_i - (1/\tau_n)\lambda_i^n$, $i \in \{1, 2\}$. If $v_i^n \neq 0$, put $y_i^{n+1} = \frac{1}{\|v_i^n\|} v_i^n$; otherwise, choose $z_i \in \mathbb{R}^d$ with $\|z_i\| = 1$ and define $y_i^{n+1} = z_i$, $i \in \{1, 2\}$.
- Step 6.** Put $\lambda_i^{n+1} = \lambda_i^n - \tau_n(S_i x_i^{n+1} - y_i^{n+1} - c_i)$, $i \in \{1, 2\}$.
- Step 7.** Compute $R_x(x^{n+1}, y^{n+1}, \lambda^{n+1})$, $R_{y_i}^\pm(x^{n+1}, y^{n+1}, \lambda^{n+1})$, and $R_c(x^{n+1}, y^{n+1}, \lambda^{n+1})$. If condition (22) holds true, then **Stop**.
- Step 8.** If $n \geq 1$, define

$$\tau_{n+1} = \begin{cases} \beta \tau_n, & \text{if inequalities (19) hold true,} \\ \tau_n, & \text{otherwise} \end{cases}$$

If $\tau_{n+1} \neq \tau_n$, set **flag** = **true**. Put $n := n + 1$ and go to **Step 3**.

to the best of the author's knowledge no existing results on convergence of various modifications of this method are applicable to Algorithm 3. One of the main differences between our algorithm and other existing versions of the ADMM for nonconvex problems is the fact that the penalty parameter in the ADMM is usually assumed to be constant throughout iterations (cf. [4, 9, 10, 20, 26, 28, 29, 31]) or to increase unboundedly [20], while in Algorithm 3 we update the penalty parameter τ_n adaptively. It should be noted that although adaptive penalty updates can improve overall performance of the ADMM, they significantly complicate convergence analysis even in the convex case, as is pointed out in [6].

Since inequalities (19) were chosen heuristically without any theoretical foundation, we will analyse the method under the assumption that inequality (18) is used as a criterion for penalty updates. Although, this criterion performed poorly in our numerical experiments, its analysis provides an insight into the performance of the ADDM in the nonconvex case and the choice of parameters of this method.

We will analyse convergence in two different cases: when the sequence of penalty parameters $\{\tau_n\}$ is bounded (i.e. when Algorithm 3 updates the penalty parameter τ_n only a finite number of times) and when this sequence is unbounded. In the first case the method converges to a point satisfying KKT optimality conditions for problem (17) with linear rate. In the second case the analysis of convergence is much more complicated and we provide only a partial result on the convergence of the method.

Theorem 2. *Let $\{x^n\}$, $\{y^n\}$, and $\{\lambda^n\}$ be the sequences generated by Algorithm 3 with inequalities (19) on Step 8 replaced by inequality (18). Suppose also that the sequence of penalty parameters $\{\tau_n\}$ is bounded. Then the sequence $\{(x^n, y^n, \lambda^n)\}$ converges to a point (x^*, y^*, λ^*) satisfying KKT optimality conditions for problem (17) and there exists $M > 0$*

such that

$$\|x^* - x^n\| \leq M\eta^n, \quad \|y^* - y^n\| \leq M\eta^n, \quad \|\lambda^* - \lambda^n\| \leq M\eta^n \quad \forall n \in \mathbb{N}. \quad (23)$$

Proof. From the fact that the sequence $\{\tau_n\}$ is bounded it follows that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ inequality (18) is not satisfied, i.e. for all $n \geq n_0$ one has

$$\begin{aligned} \|S_i x_i^{n+1} - y_i^n - c_i\| &\leq \eta \|S_i x_i^n - y_i^{n-1} - c_i\|, \\ \|S_i x_i^{n+1} - y_i^{n+1} - c_i\| &\leq \eta \|S_i x_i^n - y_i^n - c_i\| \end{aligned} \quad (24)$$

(see Step 8 of Algorithm 3). Therefore, according to Step 6 for any $i \in \{1, 2\}$ and $n \geq n_0$ one has

$$\|\lambda_i^{n+1} - \lambda_i^n\| = \tau_n \|S_i x_i^{n+1} - y_i^{n+1} - c_i\| \leq \left(\sup_{n \in \mathbb{N}} \tau_n \right) \|S_i x_i^{n_0} - y_i^{n_0} - c_i\| \eta^{n-n_0}.$$

Consequently, for any $m > n \geq n_0$ one has

$$\|\lambda_i^m - \lambda_i^n\| \leq \sum_{k=n}^{m-1} \|\lambda_i^{k+1} - \lambda_i^k\| \leq \left(\sup_{n \in \mathbb{N}} \tau_n \right) \|S_i x_i^{n_0} - y_i^{n_0} - c_i\| \eta^{n-n_0} \frac{1 - \eta^{m-n}}{1 - \eta}.$$

Hence bearing in mind the fact that $\eta \in (0, 1)$ one obtains that $\{\lambda^n\}$ is a Cauchy sequence, which implies that it converges to some $\lambda^* \in \mathbb{R}^{2d}$. Moreover, for any $n \geq n_0$ one has

$$\|\lambda_i^* - \lambda_i^n\| = \left\| \sum_{k=n}^{\infty} (\lambda_i^{k+1} - \lambda_i^k) \right\| \leq \sum_{k=n}^{\infty} \left(\sup_{n \in \mathbb{N}} \tau_n \right) \|S_i x_i^{n_0} - y_i^{n_0} - c_i\| \eta^{k-n_0} = M_\lambda \eta^n,$$

where

$$M_\lambda = \left(\sup_{n \in \mathbb{N}} \tau_n \right) \|S_i x_i^{n_0} - y_i^{n_0} - c_i\| \frac{1}{(1 - \eta) \eta^{n_0}}.$$

Thus, there exists $M > 0$ such that the third inequality in (23) holds true for all $n \in \mathbb{N}$.

Fix $i \in \{1, 2\}$. Recall that for all $n \geq n_0$ inequality (24) holds true. Therefore

$$S_i x_i^{n+1} - y_i^n - c_i = \rho_i^n, \quad \|\rho_i^n\| \leq \|S_i x_i^{n_0} - y_i^{n_0-1} - c_i\| \eta^{n-n_0} \quad \forall n \geq n_0. \quad (25)$$

Taking into account the definition of λ_i^{n+1} (see Step 6 of Algorithm 3) one obtains that

$$y_i^{n+1} = \frac{1}{\tau_n} (\lambda_i^{n+1} - \lambda_i^n) + S_i x_i^{n+1} - c_i = \frac{1}{\tau_n} (\lambda_i^{n+1} - \lambda_i^n) + y_i^n + \rho_i^n.$$

Hence with the use of the third inequality in (23) one gets that for any $n \geq n_0$ the following inequalities hold true:

$$\|y_i^{n+1} - y_i^n\| \leq \frac{1}{\tau_n} \|\lambda_i^{n+1} - \lambda_i^n\| + \|\rho_i^n\| \leq \frac{M}{\tau_0} \eta^n + \|S_i x_i^{n_0} - y_i^{n_0-1} - c_i\| \eta^{n-n_0}.$$

Now, arguing in the same way as in the case of the sequence $\{\lambda^n\}$ one can check that $\{y_i^n\}$ is a Cauchy sequence, which implies that it converges to some y_i^* , and the second inequality in (23) is satisfied for all $n \in \mathbb{N}$.

As was noted above, under the assumptions of the theorem inequality (24) is satisfied for all $n \geq n_0$. Therefore

$$S_i x_i^n - y_i^n - c_i = \xi_i^n, \quad \|\xi_i^n\| \leq \|S_i x_i^{n_0} - y_i^{n_0} - c_i\| \eta^{n-n_0} \quad \forall n \geq n_0, i \in \{1, 2\}.$$

Hence with the use of the facts that $y^n \rightarrow y^*$ as $n \rightarrow \infty$ and $\eta \in (0, 1)$ one obtains that the sequence $\{x_i^n\}$ converges to the point $x_i^* = S_i^{-1}(y_i^* + c_i)$, that is, $S_i x_i^* - y_i^* = c_i$. Furthermore, observe that $\|y_i^*\| = 1$, since $\|y_i^n\| = 1$ for all $n \in \mathbb{N}$ (see Step 5 of Algorithm 3). Thus, (x^*, y^*) is a feasible point of problem (17). In addition, one has

$$\|x_i^* - x_i^n\| = \|S_i^{-1}\| \|y_i^* - y_i^n - \xi_i^n\| \leq \|S_i^{-1}\| \|y_i^* - y_i^n\| + \|S_i^{-1}\| \|S_i x_i^{n_0} - y_i^{n_0} - c_i\| \eta^{n-n_0}$$

for all $n \geq n_0$. Therefore, the first inequality in (23) follows directly from the second one.

Let us finally check that the triplet (x^*, y^*, λ^*) satisfies KKT optimality conditions for problem (17) (see (20), (21)). Indeed, as was noted above, one has $S_i x_i^* - y_i^* = c_i$ and $\|y_i^*\| = 1$, $i \in \{1, 2\}$, that is, (x^*, y^*) is a feasible point of problem (17).

By definition the point x^{n+1} is the solution of the following system of equations

$$\begin{pmatrix} I_d + \tau_n Q_1 & -I_d \\ -I_d & I_d + \tau_n Q_2 \end{pmatrix} x^{n+1} = \begin{pmatrix} S_1 & \mathbb{O}_d \\ \mathbb{O}_d & S_2 \end{pmatrix} (\lambda^n + \tau_n (y^n + c)).$$

(see Steps 3 and 4 of Algorithm 3 and equality (8)). Therefore, for any $n \in \mathbb{N}$ one has

$$\begin{aligned} x_1^{n+1} - x_2^{n+1} - S_1 \lambda_1^n &= \tau_n S_1 \left(-S_1 x_1^{n+1} + y_1^n + c_1 \right), \\ x_2^{n+1} - x_1^{n+1} - S_2 \lambda_2^n &= \tau_n S_2 \left(-S_2 x_2^{n+1} + y_2^n + c_2 \right) \end{aligned}$$

(here we used the equality $Q_i = S_i S_i$). Hence adding and subtracting y_i^{n+1} and taking into account the definition of λ_i^{n+1} (see Step 6) one gets that

$$\begin{aligned} x_1^{n+1} - x_2^{n+1} - S_1 \lambda_1^n &= S_1 (\lambda_1^{n+1} - \lambda_1^n) + \tau_n S_1 (y_1^n - y_1^{n+1}), \\ x_2^{n+1} - x_1^{n+1} - S_2 \lambda_2^n &= S_2 (\lambda_2^{n+1} - \lambda_2^n) + \tau_n S_2 (y_2^n - y_2^{n+1}). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$ with the use of the fact that the sequence τ_n is bounded one obtains

$$x_1^* - x_2^* - S_1 \lambda_1^* = 0, \quad x_2^* - x_1^* - S_2 \lambda_2^* = 0, \quad (26)$$

i.e. optimality condition (20) holds true.

Fix $i \in \{1, 2\}$. If $\lambda_i^* = 0$, then $\lambda_i^* + 0y_i^* = 0$. Therefore, suppose that $\lambda_i^* \neq 0$. Let us consider two cases. Suppose at first that there exists $n^* \in \mathbb{N}$ such that $v_i^n \neq 0$ for all $n \geq n^*$ (see Step 5 of Algorithm 3). Then with the use of the definitions of y_i^{n+1} and λ_i^{n+1} (Steps 5 and 6 of Algorithm 3) one obtains that

$$y_i^* = \lim_{n \rightarrow \infty} y_i^{n+1} = \lim_{n \rightarrow \infty} \frac{y_i^{n+1} - \frac{1}{\tau_n} \lambda_i^{n+1}}{\|y_i^{n+1} - \frac{1}{\tau_n} \lambda_i^{n+1}\|} = \frac{y_i^* - \frac{1}{\tau_{n_0}} \lambda_i^*}{\|y_i^* - \frac{1}{\tau_{n_0}} \lambda_i^*\|}.$$

Consequently, $\lambda_i^* + \mu_i^* y_i^* = 0$, where $\mu_i^* = \tau_{n_0} - \|\tau_{n_0} y_i^* - \lambda_i^*\|$. Thus, the triplet (x^*, y^*, λ^*) satisfies KKT optimality conditions (20), (21) for problem (17).

Suppose now that there exists a subsequence $\{v_i^{n_k}\}$ such that $v_i^{n_k} = 0$ for all $k \in \mathbb{N}$. Then by definition (see Step 5 of Algorithm 3) one has

$$S_i x_i^{n_k+1} - c_i = \frac{1}{\tau_{n_k}} \lambda_i^{n_k} = y_i^{n_k} + \rho_i^{n_k} \quad \forall k \in \mathbb{N},$$

where $\rho_i^n = S_i x_i^{n+1} - y_i^n - c_i$. As was noted above, $\rho_i^n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, passing to the limit as $k \rightarrow \infty$ one obtains that $\lambda_i^* = \tau_{n_0} y_i^*$, which completes the proof. \square

Remark 6. Note that the parameter $\eta \in (0, 1)$ from criterion (18) for penalty updates is used in the upper estimates (23) of the rate of convergence of the ADMM. Therefore η must be greater than the actual rate of convergence of the method, i.e.

$$\max \left\{ \limsup_{n \rightarrow \infty} \frac{\|x^* - x^{n+1}\|}{\|x^* - x^n\|}, \limsup_{n \rightarrow \infty} \frac{\|y^* - y^{n+1}\|}{\|y^* - y^n\|}, \limsup_{n \rightarrow \infty} \frac{\|\lambda^* - \lambda^{n+1}\|}{\|\lambda^* - \lambda^n\|} \right\} < \eta,$$

since otherwise the method would start increasing the penalty parameter τ_n , when it is absolutely unnecessary. In particular, it seems advisable to choose η (in both (18) and (19)) to be sufficiently close to 1 in order to avoid an unbounded increase of the penalty parameter τ_n for ill-conditioned problems.

Let us now consider the case when the penalty parameter τ_n increases unboundedly with iterations. In this case a convergence analysis of the method is much more complicated and we present only an incomplete result, which can be viewed as an intermediate lemma in a comprehensive convergence analysis of the ADMM for finding the distance between the boundaries of ellipsoids. Nevertheless, this result highlights some peculiarities of the method in the case when the penalty parameter increases unboundedly, and we hope that it might help the interested reader to develop a more complete convergence theory for the ADMM in the nonconvex case.

Let us first prove an auxiliary lemma, which, in particular, implies that for any sufficiently large value of the penalty parameter $v_i^n \neq 0$ on Step 5 of Algorithm 3, provided multipliers $\{\lambda^n\}$ lie within a bounded set.

Lemma 1. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a bounded set. Then there exists $\tau^* > 0$ such that for any $\tau \geq \tau^*$ and for all $\lambda \in \Lambda$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^{2d}$ with $\|y_1\| = \|y_2\| = 1$ solution x^* of the system of linear equations*

$$\begin{pmatrix} I_d + \tau Q_1 & -I_d \\ -I_d & I_d + \tau Q_2 \end{pmatrix} x^* = \begin{pmatrix} S_1 & \mathbb{O}_d \\ \mathbb{O}_d & S_2 \end{pmatrix} (\lambda + \tau(y + c)).$$

satisfies the conditions $S_i x_i^ - c_i - (1/\tau)\lambda_i \neq 0$, $i \in \{1, 2\}$.*

Proof. Let us verify that the statement of the lemma holds true for any $\tau^* > \max\{1, K\}$, where

$$K = \left(\sum_{i=1}^2 \|S_i^{-1}\| (1 + \|c_i\| + \sup_{\lambda \in \Lambda} \|\lambda_i\|) \right)^2.$$

Indeed, fix any $\tau \geq \tau^* > \max\{1, K\}$, $\lambda \in \Lambda$, and $y \in \mathbb{R}^{2d}$ with $\|y_1\| = \|y_2\| = 1$. Let x^* be a solution of the corresponding system of linear equations.

As is easily seen, x^* is a point of global minimum of the convex function

$$g(x) = \frac{1}{2} \|x_1 - x_2\|^2 + \sum_{i=1}^2 \frac{\tau}{2} \left\| S_i x_i - y_i - c_i - \frac{1}{\tau} \lambda_i \right\|^2,$$

since it satisfies the equality $\nabla g(x^*) = 0$. Consequently, $g(x^*) \leq g(\hat{x})$, where

$$\hat{x}_i = S_i^{-1} \left(y_i + c_i + \frac{1}{\tau} \lambda_i \right), \quad i \in \{1, 2\}.$$

Taking into account the fact that $\tau \geq \tau^* > 1$ one obtains that

$$\|\hat{x}_1 - \hat{x}_2\| \leq \|\hat{x}_1\| + \|\hat{x}_2\| \leq \sum_{i=1}^2 \|S_i^{-1}\| \left(1 + \|c_i\| + \frac{1}{\tau} \|\lambda_i\| \right) \leq \sqrt{K}.$$

Therefore

$$g(x^*) \leq g(\hat{x}) = \frac{1}{2} \|\hat{x}_1 - \hat{x}_2\|^2 \leq \frac{K}{2} < \frac{\tau_*}{2} \leq \frac{\tau}{2}.$$

On the other hand, if $S_i x_i^* - c_i - (1/\tau)\lambda_i = 0$ for some $i \in \{1, 2\}$, then

$$g(x^*) \geq \frac{\tau}{2} \|y_i\|^2 = \frac{\tau}{2} > g(x^*),$$

which is impossible. Thus, $S_i x_i^* - c_i - (1/\tau)\lambda_i \neq 0$, $i \in \{1, 2\}$. \square

Next we present a partial result on convergence of Algorithm 3 in the case when the penalty parameter $\{\tau_n\}$ increases unboundedly as $n \rightarrow \infty$.

Proposition 1. *Let $\{x^n\}$, $\{y^n\}$, and $\{\lambda^n\}$ be the sequences generated by Algorithm 3 with inequalities (19) on Step 8 replaced by inequality (18). Suppose that the sequence of penalty parameters $\{\tau_n\}$ is unbounded, but the sequence of multipliers $\{\lambda^n\}$ is bounded. Then the sequence $\{(x^n, y^n)\}$ is bounded as well and all its limit points are feasible for problem (17). Moreover, all limit points of the sequence $\{(x^n, y^n, \lambda^n)\}$ satisfy KKT optimality conditions for problem (17) if and only if $\tau_n \|y^{n+1} - y^n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. For any $n \in \mathbb{N}$ denote $\xi_i^n = S_i x_i^{n+1} - y_i^n - c_i$, $i \in \{1, 2\}$. Let us show that $\xi_i^n \rightarrow 0$ as $n \rightarrow \infty$. Indeed, denote $\hat{x}_i^n = S_i^{-1}(y_i^n + c_i + (1/\tau_n)\lambda_i^n)$ for all $n \in \mathbb{N}$ and $i \in \{1, 2\}$. The sequence $\{\hat{x}^n\}$ is bounded due to the boundedness of the sequence $\{\lambda^n\}$ and the fact that $\|y_i^n\| = 1$ for all $n \in \mathbb{N}$ (see Step 5 of Algorithm 3).

Recall that by definition x^{n+1} is a point of global minimum of the quadratic function $f_n(x)$ defined in (6) with $\tau = \tau_n$. Therefore x^{n+1} is also a point of global minimum of the function

$$\begin{aligned} g_n(x) &= f_n(x) + \sum_{i=1}^2 \left(\langle \lambda_i^n, y_i^n + c_i \rangle + \frac{1}{2\tau_n} \|\lambda_i^n\|^2 \right) \\ &= \frac{1}{2} \|x_1 - x_2\|^2 + \sum_{i=1}^2 \frac{\tau_n}{2} \left\| S_i x_i - y_i^n - c_i - \frac{1}{\tau_n} \lambda_i^n \right\|^2, \end{aligned}$$

which implies that $g_n(x^{n+1}) \leq g_n(\hat{x}^n)$ for all $n \in \mathbb{N}$. Hence with the use of the equality $g_n(\hat{x}^n) = 0.5 \|\hat{x}_1^n - \hat{x}_2^n\|^2$ and the boundedness of the sequence $\{\hat{x}^n\}$ one obtains that the sequence $\{g_n(x^{n+1})\}$ is bounded above.

Arguing by reductio ad absurdum, suppose that the sequence $\{\xi_1^n\}$ does not converge to zero (the convergence of the sequence $\{\xi_2^n\}$ to zero is proved in the same way). Then there exist $\delta > 0$ and a subsequence $\{\xi_1^{n_k}\}$ such that $\|\xi_1^{n_k}\| \geq \delta$ for all $k \in \mathbb{N}$. Observe that for any $k \in \mathbb{N}$ one has

$$\begin{aligned} g_{n_k}(x^{n_k+1}) &\geq \frac{1}{2} \|x_1^{n_k+1} - x_2^{n_k+1}\|^2 + \sum_{i=1}^2 \left(\frac{\tau_{n_k}}{2} \|\xi_i^{n_k}\|^2 - \|\lambda_i^{n_k}\| \|\xi_i^{n_k}\| + \frac{1}{2\tau_{n_k}} \|\lambda_i^{n_k}\|^2 \right) \\ &\geq \frac{1}{2} \left(\sqrt{\tau_{n_k}} \|\xi_1^{n_k}\| - \frac{1}{\sqrt{\tau_{n_k}}} \|\lambda_1^{n_k}\| \right)^2. \end{aligned}$$

Hence bearing in mind the facts that by our assumptions the sequence $\{\lambda^n\}$ is bounded, while the sequence $\{\tau_n\}$ increases unboundedly one can readily verify that $g_{n_k}(x^{n_k+1}) \rightarrow +\infty$ as $k \rightarrow \infty$, which contradicts the fact that the sequence $\{g_n(x^{n+1})\}$ is bounded above.

Thus, $\xi_i^n \rightarrow 0$ as $n \rightarrow \infty$, $i \in \{1, 2\}$. Recall that $\|y_i^n\| = 1$ for all $n \in \mathbb{N}$ (see Step 5 of Algorithm 3). Therefore

$$\|x_i^{n+1}\| = \|S_i^{-1}(y_i^n + c_i + \xi_i^n)\| \leq \|S_i^{-1}\|(1 + \|c_i\| + \|\xi_i^n\|),$$

which implies that the sequence $\{(x^n, y^n)\}$ is bounded. Let (x^*, y^*) be a limit point of this sequence, i.e. there exists a subsequence $\{(x^{n_k}, y^{n_k})\}$ converging to (x^*, y^*) . Let us check that this point is feasible for problem (17).

Indeed, from the fact that $\|y_i^n\| = 1$ for all $n \in \mathbb{N}$ it follows that $\|y_i^*\| = 1$. Taking into account the definition of λ_i^{n+1} (see Step 6 of Algorithm 3) one obtains that

$$\|S_i x_i^{n+1} - y_i^{n+1} - c_i\| \leq \frac{2}{\tau_n} \sup_{n \in \mathbb{N} \cup \{0\}} \|\lambda_i^n\|.$$

Therefore, $\zeta_i^{n+1} = S_i x_i^{n+1} - y_i^{n+1} - c_i \rightarrow 0$ as $n \rightarrow \infty$ due the boundedness of the sequence $\{\lambda^n\}$ and the fact that $\tau_n \rightarrow +\infty$ as $n \rightarrow \infty$. Hence passing to the limit as $k \rightarrow \infty$ in the equality

$$x_i^{n_k} = S_i^{-1}(y_i^{n_k} + c_i + \zeta_i^{n_k})$$

one obtains that $S_i x_i^* - y_i^* = c_i$, that is, the point (x^*, y^*) is feasible for problem (17).

Suppose now that all limit points of the sequence $\{(x^n, y^n, \lambda^n)\}$ satisfy KKT optimality conditions for problem (17). Arguing by reductio ad absurdum, assume that the sequence $\tau_n \|y^{n+1} - y^n\|$ does not converge to zero. Then there exist a subsequence of this sequence and $\varepsilon > 0$ such that $\tau_{n_k-1} \|y^{n_k} - y^{n_k-1}\| \geq \varepsilon$. The sequence $\{(x^{n_k}, y^{n_k}, \lambda^{n_k})\}$ is bounded. Replacing this sequence, if necessary, with its subsequence one can suppose that $\{(x^{n_k}, y^{n_k}, \lambda^{n_k})\}$ converges to some (x^*, y^*, λ^*) . By definition the point x^{n+1} is the solution of the following system of equations

$$\begin{pmatrix} I_d + \tau_n Q_1 & -I_d \\ -I_d & I_d + \tau_n Q_2 \end{pmatrix} x^{n+1} = \begin{pmatrix} S_1 & \mathbb{O}_d \\ \mathbb{O}_d & S_2 \end{pmatrix} (\lambda^n + \tau_n(y^n + c)).$$

(see Steps 3 and 4 of Algorithm 3 and equality (7)). Therefore, for any $k \in \mathbb{N}$ one has

$$\begin{aligned} x_1^{n_k} - x_2^{n_k} - S_1 \lambda_1^{n_k-1} &= \tau_{n_k-1} S_1 (-S_1 x_1^{n_k} + y_1^{n_k-1} + c_1), \\ x_2^{n_k} - x_1^{n_k} - S_2 \lambda_2^{n_k-1} &= \tau_{n_k-1} S_2 (-S_2 x_2^{n_k} + y_2^{n_k-1} + c_2) \end{aligned}$$

(here we used the equality $Q_i = S_i S_i$). Adding and subtracting $y_i^{n_k}$ and taking into account the definition of λ_i^{n+1} (see Step 6) one obtains that

$$\begin{aligned} x_1^{n_k} - x_2^{n_k} - S_1 \lambda_1^{n_k} &= \tau_{n_k-1} S_1 (y_1^{n_k-1} - y_1^{n_k}), \\ x_2^{n_k} - x_1^{n_k} - S_2 \lambda_2^{n_k} &= \tau_{n_k-1} S_2 (y_2^{n_k-1} - y_2^{n_k}) \end{aligned} \quad \forall k \in \mathbb{N}. \quad (27)$$

Therefore for all $k \in \mathbb{N}$ one has

$$\begin{aligned} \|x_1^{n_k} - x_2^{n_k} - S_1 \lambda_1^{n_k}\| &\geq \frac{1}{\|S_1^{-1}\|} \tau_{n_k-1} \|y_1^{n_k-1} - y_1^{n_k}\| \geq \frac{\varepsilon}{\|S_1^{-1}\|}, \\ \|x_2^{n_k} - x_1^{n_k} - S_2 \lambda_2^{n_k}\| &\geq \frac{1}{\|S_2^{-1}\|} \tau_{n_k-1} \|y_2^{n_k-1} - y_2^{n_k}\| \geq \frac{\varepsilon}{\|S_2^{-1}\|}, \end{aligned}$$

which contradicts the fact that by our assumption all limit points of the sequence $\{(x^n, y^n, \lambda^n)\}$ (including (x^*, y^*, λ^*)) satisfy KKT optimality conditions for problem (17) (see (20)).

Suppose finally that $\tau_n \|y^{n+1} - y^n\| \rightarrow 0$ as $n \rightarrow \infty$. Let (x^*, y^*, λ^*) be a limit point of the sequence $\{(x^n, y^n, \lambda^n)\}$, i.e. there exists a subsequence $\{(x^{n_k}, y^{n_k}, \lambda^{n_k})\}$ converging to (x^*, y^*, λ^*) . Passing to the limit as $k \rightarrow \infty$ in (27) one obtains that

$$x_1^* - x_2^* - S_1 \lambda_1^* = 0, \quad x_2^* - x_1^* - S_2 \lambda_2^* = 0.$$

Recall that by our assumption the sequence $\{\lambda_n\}$ is bounded and $\tau_n \rightarrow +\infty$ as $n \rightarrow \infty$. Therefore by Lemma 1 there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has $v_i^n \neq 0$, $i \in \{1, 2\}$ (see Step 5 of Algorithm 3). Hence by the definitions of y^{n+1} and λ^{n+1} (Steps 5 and 6 of Algorithm 3) one has

$$y_i^{n+1} = \frac{y_i^{n+1} - \tau_n^{-1} \lambda_i^{n+1}}{\|y_i^{n+1} - \tau_n^{-1} \lambda_i^{n+1}\|}, \quad i \in \{1, 2\} \quad \forall n \geq n_0.$$

or, equivalently,

$$\lambda_i^{n+1} = \mu_i^{n+1} y_i^{n+1}, \quad \mu_i^{n+1} = \tau_n - \|\tau_n y_i^{n+1} - \lambda_i^{n+1}\| \quad \forall n \geq n_0. \quad (28)$$

Recall that $\|y_i^n\| = 1$ for all $n \in \mathbb{N}$, $i \in \{1, 2\}$. Therefore $|\mu_i^{n+1}| = \|\lambda_i^{n+1}\|$ for all $n \geq n_0$ and the sequence $\{\mu_i^n\}$ is bounded by our assumption on the boundedness of multipliers λ^n . Consequently, there exists a subsequence of the sequence $\{\mu_i^{n_k}\}$, which we denote again by $\{\mu_i^{n_k}\}$, converging to some μ_i^* . With the use of (28) one obtains that $\lambda_i^* = \mu_i^* y_i^*$, which implies the required result (see KKT optimality conditions (20), (21) for problem (17)). \square

Remark 7. The assumption on the boundedness of multipliers $\{\lambda^n\}$ is standard in the theory of augmented Lagrangian methods (cf. [2, 18]). There are several well-known techniques guaranteeing the boundedness of multipliers and convergence of augmented Lagrangian methods: safeguarding, conditional multiplier updating, normalization of multipliers, etc. All these techniques were discussed in detail, e.g. in [18, 19, 27], and can be applied to Algorithm 3. However, as is pointed out in the monograph [2], in applications multipliers usually remain bounded without the use of any special techniques ensuring their boundedness. Moreover, in all our numerical experiments on various types of test problems multipliers $\{\lambda^n\}$ always remained bounded. That is why we did not include any of the aforementioned techniques into the description of Algorithm 3.

3.3 A heuristic restarting procedure and a global method

The problem of finding the distance between the boundaries of ellipsoids is nonconvex. In particular, not all KKT points of this problem are its globally optimal solutions. Consequently, a sequence generated by Algorithm 3 might converge to a locally optimal solution or even just a stationary point of problem (17). To overcome this difficulty, we propose to use a simple heuristic restarting procedure (a new starting point), which can be applied to any local method for solving problem (16) (or problem (17)).

Namely, if Algorithm 3 terminates by finding a point (x^*, y^*, λ^*) , then we propose to restart the algorithm at a point $x^0 = (x_1^0, x_2^0)$ such that the points x_i^0 are, in a sense, diametrically opposed to x_i^* with respect to the centre of the corresponding ellipsoid. Namely, we define

$$x_i^0 = (-x_i^* + z_i) + z_i = -x_i^* + 2z_i, \quad i \in \{1, 2\}.$$

Algorithm 4: The ADMM for finding the distance between the boundaries of ellipsoids with a heuristic restart.

Step 1. Choose $\varepsilon_0 > 0$ and apply Algorithm 3 with an arbitrary starting point.

The algorithm terminates at a point x^* .

Step 2. If $\|x_1^* - x_2^*\| < \varepsilon_0$, then **return** x^* . Otherwise, choose $\lambda_i^0 \in \mathbb{R}^d$, $i \in \{1, 2\}$, and apply Algorithm 3 with $y_i^0 = -S_i x_i^* + c_i$, and the rest of the parameters defined as during the first run. The algorithm terminates at a point x^{**} .

Step 3. If $\|x_1^* - x_2^*\| \leq \|x_1^{**} - x_2^{**}\|$, **return** x^* . Otherwise, **return** x^{**} .

Bearing in mind the constraints of problem (17) we put $y_i^0 = S_i x_i^0 - c_i = -S_i x_i^* + c_i$, $i \in \{1, 2\}$. The initial value of multipliers can be defined arbitrarily.

Roughly speaking, we restart the algorithm at the opposite ‘sides’ of the ellipsoids. In the case $z_1 = z_2 = 0$ one simply sets $x^0 = -x^*$. To obtain the formula for x^0 in the general case one has to shift the origin to the centre z_i of the corresponding ellipsoid, take the vector opposite to $x_i^* - z_i$, and then shift the origin back to its place.

Note that if the algorithm finds a point at which the ellipsoids intersect, there is no need to restart it at a new initial point. Therefore, we arrive at the following restarting scheme for Algorithm 3 given in Algorithm 4.

To verify the proposed heuristic restarting procedure we used a global optimisation method, which is a slight modification of the global method for finding the so-called *signed* distance between ellipsoids developed in [14]. This method is based on the use of KKT optimality conditions for problem (16) and a direct computation of all Lagrange multipliers for this problem via an auxiliary generalised eigenvalue problem.

Recall that the problem of finding the distance between the boundaries of ellipsoids has the form:

$$\begin{aligned} \min \quad & \|x_1 - x_2\|^2 \\ \text{subject to} \quad & \langle x_1 - z_1, Q_1(x_1 - z_1) \rangle = 1, \quad \langle x_2 - z_2, Q_2(x_2 - z_2) \rangle = 1. \end{aligned}$$

The KKT optimality conditions for this problem can be written as follows:

$$x_1 - x_2 = \mu Q_1(x_1 - z_1), \quad x_2 - x_1 = \gamma Q_2(x_2 - z_2). \quad (29)$$

Here $\mu, \gamma \in \mathbb{R}$ are Lagrange multipliers. In [14, Section 4.1] it was shown that these Lagrange multipliers are solutions of the two following linear generalised eigenvalue problems:

$$\det L_1(\mu) := \det \left[\mu(F_{11} \otimes G_{10} - F_{10} \otimes G_{11}) + (F_{01} \otimes G_{10} - F_{10} \otimes G_{01}) \right] = 0, \quad (30)$$

$$\det L_2(\gamma) := \det \left[\gamma(F_{11} \otimes G_{01} - F_{01} \otimes G_{11}) + (F_{10} \otimes G_{01} - F_{01} \otimes G_{10}) \right] = 0, \quad (31)$$

where \otimes is the Kronecker product,

$$\begin{aligned} F_{10} &= \begin{pmatrix} \mathbb{O}_d & -Q_2^{-1} \\ -Q_2^{-1} & \mathbb{O}_d \end{pmatrix}, \quad F_{01} = \begin{pmatrix} Q_1^{-1} & -Q_1^{-1} \\ -Q_1^{-1} & (z_1 - z_2)(z_1 - z_2)^T \end{pmatrix}, \\ G_{01} &= \begin{pmatrix} Q_2^{-1} & -Q_2^{-1} \\ -Q_2^{-1} & (z_1 - z_2)(z_1 - z_2)^T \end{pmatrix}, \quad G_{10} = \begin{pmatrix} \mathbb{O}_d & -Q_1^{-1} \\ -Q_1^{-1} & \mathbb{O}_d \end{pmatrix}, \\ F_{11} &= G_{11} = \begin{pmatrix} \mathbb{O}_d & I_d \\ I_d & \mathbb{O}_d \end{pmatrix}, \end{aligned}$$

and, as above, \mathbb{O}_d is the zero matrix of order d , while I_d is the identity matrix of order d .

Thus, one can globally solve the problem of finding the distance between the boundaries of ellipsoids by solving generalised eigenvalue problems (30) and (31), computing corresponding x_1 and x_2 for real values of μ and γ from the KKT optimality conditions (29), and then comparing the distances $\|x_1 - x_2\|$ for those x_1 and x_2 that satisfy the constraints. The pair with the least distance is a globally optimal solution.

Remark 8. Let us note that by [14, Theorem 4.4] for almost all positive definite matrices Q_1 and Q_2 generalised eigenvalue problems (30) and (31) have only a finite number of solutions. However, these problems might be degenerate, if both Q_1 and Q_2 have eigenvectors orthogonal to $z_1 - z_2$. In [14, Section 5] regularity tests for problems (30) and (31) and some techniques for handling degenerate cases are discussed. However, we did not employ them in our numerical experiments, since we used the method from [14] only as a tool for verifying Algorithm 4. In the case when the generalised eigenvalue problems were degenerate, we simply restarted computations with different matrices Q_1 and Q_2 .

3.4 Numerical experiments

Let us present some results of preliminary numerical experiments demonstrating the higher efficiency of the ADMM in comparison with other methods for finding the distance between the boundaries of ellipsoids in the case of nonconvex high-dimensional problems.

Our main goal was to test Algorithm 4 in the *nonconvex* case. Therefore the problem data was generated in such a way that in most cases one ellipsoid lies within the other, which makes the problem multiextremal (i.e. there are locally optimal solutions of this problem that are not globally optimal). To this end, we randomly generated a matrix A of dimension $d \in \mathbb{N}$, whose elements were uniformly distributed in the interval $[-100, 100]$. If the matrix A has full rank, we define $Q_1 = A^T A$. Otherwise, the matrix A was randomly generated again till it had full rank, to ensure that the matrix Q_1 is positive definite. We chose as Q_2 a random diagonal matrix whose diagonal elements are uniformly distributed in the interval $[0.1, 0.6]$. The centres z_i of the ellipsoids were also randomly generated in such a way that their coordinates are uniformly distributed in the interval $[-0.05, 0.05]$. Our numerical experiments demonstrated that this choice of matrices Q_i and vectors z_i ensured that in most cases ellipsoid \mathcal{E}_1 lies within \mathcal{E}_2 . Moreover, the numerical experiments also showed that when the dimension d is small, the boundaries of the ellipsoids do not intersect in almost all cases. However, as the dimension increases, the average distance between the boundaries of the ellipsoids tends to zero and cases when the boundaries intersect appear more frequently.

The parameters of Algorithms 3 and 4 were chosen in the following way. We set $\eta = 0.99$, $\beta = 2$, $\tau_0 = 10$, $\varepsilon = \varepsilon_0 = 10^{-6}$, and $\varkappa = 0.1$. We also defined $\lambda^0 = 0$ and $y_1^0 = y_2^0 = (1, 0, 0, \dots, 0)^T$ to make sure that in the case $z_1 = z_2 = 0$ the first iteration of the algorithm is not spent on correcting the initial data. Note that in the case when $\lambda^0 = y^0 = 0$ and $z_1 = z_2 = 0$ one has $x^1 = 0$, y^1 is defined as an arbitrary vector from the unit sphere, and $\lambda^1 = \tau_0 y^1$, which can be viewed as a random reinitialization of the algorithm. Let us point out that in our numerical experiments any choice of y_i^0 from the unit sphere guaranteed that $v_i^n \neq 0$ for all $n \in \mathbb{N}$ on Step 5 of Algorithm 3.

We compared Algorithm 4 with the exact penalty method from [25], since to the best of the author's knowledge this is the only available numerical method for finding the distance between the boundaries of ellipsoids. The penalty parameter λ for this method was defined as $\lambda = 100$. To make a fair comparison we applied the same restarting procedure to the exact penalty method [25], as described in Algorithm 4. We used the inequality $\|G^*(z_k)\| <$

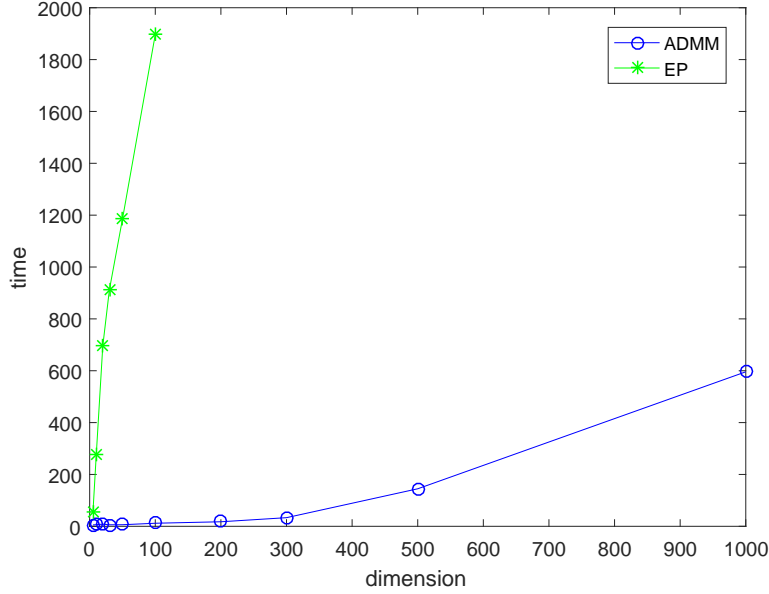


Figure 2: The run time of the ADMM and the exact penalty method for $d \in \{5, 10, 20, 30, 50, 100, 200, 300, 500, 1000\}$.

d	5	10	20	30	50	100	200	300	500	1000
ADMM	5.3	6.2	8.2	5.8	6.1	11.8	17.5	33.3	144.7	595.8
EP	55	274.1	695.7	911.2	1185.1	1896.8	—	—	—	—
Global	0.7	12.8	709.7	—	—	—	—	—	—	—

Table 3: The run time of each method in seconds. ‘—’ indicates that either the time exceeded one hour or the algorithm failed to find a solution due to reaching the prespecified maximal number of iterations.

$\varepsilon = 10^{-4}$ as a stopping criterion, since in some cases the algorithm failed to terminate before reaching the prespecified maximal number of iterations when the value $\varepsilon = 10^{-5}$ was used. We chose an initial point for the method in the same way as in the first version of the exact penalty method in the convex case (see Section 2.3). Namely, we defined $x_1^0 = z_1$ and $x_2^0 = z_2 + (0.1, \dots, 0.1)^T$. Below we denote the exact penalty method as *EP*.

Both algorithms were implemented in MATLAB. We terminated the algorithms if the number of iterations exceeded 10^6 . Similar to the convex case, we generated 10 problems for a given dimension $d \in \{5, 10, 20, 30, 50, 100, 200, 300, 500, 1000\}$ and run each algorithm on these 10 problems. The total run time of each method rounded to the nearest tenth is presented in Figure 2 and Table 3. In addition, we implemented a modification of the global method from [14] described in Section 3.3 to verify whether the proposed heuristic restarting procedure allows one to find a *globally* optimal solution. Since the complexity of the global method is very high (it is equal to $O(d^6)$; see [14]), we applied it only to problems of dimensions $d \in \{5, 10, 20\}$. For the sake of completeness, the total run time of the global method is given in Table 3.

Firstly, let us note that for *all* problems of dimension $d \in \{5, 10, 20\}$ (and numerous other

test problems not reported here) both Algorithm 4, based on the ADMM, and the exact penalty method from [25] with the restarting procedure described in the previous section found *exactly the same* solutions as the global method. Thus, one can conclude that the proposed heuristic choice of a new starting point for the second run of an algorithm indeed allows one to find a *globally* optimal solution for various problem instances. It seems that the reason behind this phenomenon lies in the fact that for any dimension $d \in \mathbb{N}$ there are always only *two* points of local minimum in the problem of finding the distance between the boundaries of ellipsoids, while the rest of the KKT points are local/global maxima. Furthermore, the two points of local minimum roughly lie on the ‘opposite sides’ of the ellipsoids. Therefore, any reasonable minimisation method would converge to one of the local minima, while restarting the method at the ‘opposite sides’ of ellipsoids allows one to compute the second point of local minimum and, as a result, find a global minimum between the two computed points. A rigorous theoretical justification of this observation is a challenging open problem that lies beyond the scope of this article.

Let us also note that for problems of dimension $d \in \{30, 50, 100\}$ Algorithm 4 and the exact penalty method always found the same solution. In addition, we tested Algorithm 4 and the exact penalty method without the restarting procedure. The results of our experiments showed that without the restart these methods often converge to a locally optimal solution, which is not globally optimal. In particular, in the case $d = 5$ Algorithm 4 converged to a locally optimal solution, which is not globally optimal, for 16 out of 100 test problems, while the exact penalty method converged to such solutions for 42 out of 100 test problems. Thus, the proposed heuristic restarting procedure is not redundant, since without it both the ADMM and the exact penalty method often cannot find a globally optimal solution.

Secondly, the results of numerical experiments demonstrate that Algorithm 4 significantly outperforms the exact penalty method from [25] for all dimensions d . On the other hand, for small dimensional problems ($d \leq 7$) the global method was consistently faster than Algorithm 4. Thus, for small dimensional problems ($d \leq 7$) it is reasonable to apply the global method, while for problems of higher dimension Algorithm 4 is the method of choice.

Finally, let us comment on the unexpected decrease of the run time of Algorithm 4 for dimensions d in the range between 20 and 100. It seems that this phenomenon is at least partially connected with some peculiarities of our implementation of Algorithm 4 in MATLAB. In particular, we used standard MATLAB routines to compute the Cholesky decomposition, to solve corresponding systems of linear equations, etc. These routines might work more efficiently for problems of dimension more than 20 due to the effect of parallelisation.

Moreover, it should be noted that for problems with $d \leq 20$ reported here the boundaries of the ellipsoids did not intersect. As our numerical experiments on various test problems showed, both Algorithm 4 and the exact penalty method are extremely efficient, when it comes to finding an intersection point of the boundaries of two ellipsoids. Even in the case $d = 1000$ it usually takes Algorithm 4 less than 5 seconds to find an intersection point, provided the boundaries of the ellipsoids intersect. Therefore, the run time of the algorithm significantly decreases with the increase of the number of problem instances when the boundaries of the ellipsoids intersect. As was mentioned above, for our choice of the problem data the average distance between the boundaries of the ellipsoids tends to zero as d increases and the cases when the boundaries of the ellipsoids intersect appear more frequently. This peculiarity of the problem data might be another factor that contributed to the decrease in the run time.

We tried generating matrices Q_i in multiple different ways to ensure that the boundaries of the ellipsoids either do not intersect for all d or intersect with approximately the

same frequency for all dimensions. However, generating such data turned out to be a very complicated problem, whose detailed discussion lies outside the scope of this paper.

4 Conclusions

We developed several versions of the alternating direction method of multipliers for computing the distance between either two ellipsoids or their boundaries. In the first case we presented the ADMM with both fixed penalty parameter and automatic adjustments of the penalty parameter. In the second case we presented the ADMM with a heuristic rule for updating the penalty parameter and a heuristic restarting procedure for finding a global minimum in the nonconvex case. We numerically verified this procedure with the use of a slight modification of a global method for finding the so-called signed distance between ellipsoids. The results of our numerical experiments showed that the proposed heuristic restarting procedure always allows one to find a globally optimal solution. Furthermore, the results of numerical experiments both in the convex and the nonconvex cases clearly demonstrate that the versions of the ADMM developed in this paper significantly outperform all existing methods for finding the distance between two ellipsoids and all methods for finding the distance between the boundaries of ellipsoids, except for the case of small-dimensional problem.

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