Least squares estimation for path-distribution dependent stochastic differential equations

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Abstract

We study a least squares estimator for an unknown parameter in the drift coefficient of a pathdistribution dependent stochastic differential equation involving a small dispersion parameter $\varepsilon > 0$. The estimator, based on n (where $n \in \mathbb{N}$) discrete time observations of the stochastic differential equation, is shown to be convergent weakly to the true value as $\varepsilon \to 0$ and $n \to \infty$. This indicates that the least squares estimator obtained is consistent with the true value. Moreover, we obtain the rate of convergence and derive the asymptotic distribution of least squares estimator.

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1 Introduction

Nowadays, stochastic differential equations (SDEs) are widely used in modelling time evolution of dynamical systems influenced by random noise, see, e.g., the monographs [5, 10, 25, 27] (and references therein). Usually, there exist unknown parameters in such modelled systems, such as those stochastic models with comparably easier structured stochastic differential equations involving unknown quantities (see, e.g., [1, P.2-4]). Fundamental issues are then to estimate certain parameters (i.e., deterministic quantities) appearing in the stochastic models by certain observations (or by experimental data). Viewing the drift part of the SDEs as the averaging evolution of the systems, estimating the drift parameter of SDEs is hence an important topic. To approach the true value of the unknown parameter, the asymptotic approach to statistical estimation is frequently taken an advantage due to its general applicability and relative simplicity (cf. [1]). As we know, the estimations upon the unknown quantities are based generally on continuous-time or discrete-time observations. Whereas, the parameter estimation relied on continuous-time observations is a mathematical idealisation although there is a vast literature concerned with such topic. On the other hand, no measuring device can follow continuously the sample paths of the diffusion processes involved, which are indeed rather tricky. Whence, in practice the investigation on the parameter

estimations with the help of discrete-time observations has been received much more attention recently. Most importantly, the parameter estimation by the aid of discrete-time observations can be implemented conveniently with a powerful theory of simulation schemes and numerical analysis of diffusion processes.

So far, there are numerous methods to investigate the parameter estimations on the unknown parameters in the drift coefficients; see, e.g., [15, 19, 26, 28] by maximum likelihood estimator (MLE for short), [3, 12, 15, 26] via least squares estimator (LSE for abbreviation), and [23] through trajectory-fitting estimator, to name a few. Diffusion processes with small noises have been applied considerably in mathematical finance, see, e.g., [14, 30, 37] and references within. In the past forty years, the asymptotic theory on parameter estimations for diffusion processes with small noises has also been developed very well, see, for instance, [6, 18, 29, 31, 32] for SDEs driven by Lévy processes with arbitrary moments, and [7, 20, 21] for SDEs driven by α -stable Lévy noises which enjoy heavy tail properties.

Recently, from the stochastic modelling perspective and diverse demanding in practical problems, there has been increasing interest on studying stochastic differential equations with pathdistribution coefficients, see e.g. [8, 9, 35] (and references therein). The distribution-dependent SDEs are also named as McKean-Vlasov SDEs or mean-filed SDEs, which have been studied intensively in the literature, see e.g. [4, 17] and references therein. Such kind of SDEs has been applied successfully in stochastic differential games and stochastic optimal optimisation, see, e.g., [16] and references within. Although McKean-Vlasov SDEs have been applied diffusively in different research areas, so far there is little work on parameter estimations except the existing literature [36], to the best of our knowledge. In the present paper, we are concerned with the LSE problem for the path-distribution stochastic differential equations with small dispersion noise and involving unknown parameter in the drift. Our key start point is the associated discrete-time observations of path-distribution dependent SDEs (see (2.1) below). We then investigate parameter estimation for McKean-Vlasov SDEs which are not only path-dependent but also dependent on the law of the path. Since the state space of the window process is infinite dimensional, some new procedures need to be put forward. We succeeded the task by carefully construct the Euler-Maruyama (EM) discretion scheme of our path-distribution dependent SDEs. It is also interesting to consider other type estimations for such equations and we will study them in another paper.

The rest of the paper is arranged as follows. In Section 2, we introduce some notation, present the framework of our paper, and construct the LSE; Section 3 is devoted to the consistency of LSE; Section 4 focus on the asymptotic distribution of LSE. Throughout this paper, we emphasise that c > 0 is a generic constant which may change from line to line.

2 Preliminaries

We start with some notation and terminology which will be used later. For $d, m \in \mathbb{N}$, the set of all positive integers, let $(\mathbb{R}^d, \langle \cdot, \cdot \rangle, |\cdot|)$ be the d-dimensional Euclinean space with the inner product $\langle \cdot, \cdot \rangle$ induced the norm $|\cdot|$ and $\mathbb{R}^d \otimes \mathbb{R}^m$ the collection of all $d \times m$ matrixes with real entries, which is endowed with the Hilbert-Schmidt norm $||\cdot||$. $\mathbf{0} \in \mathbb{R}^d$ denotes the zero vector. For a matrix A, A^* denotes the transpose of A. Concerning a square matrix A, A^{-1} means the inverse of A provided that $\det A \neq 0$. For $p \in \mathbb{N}$, let Θ be an open bounded convex subset of \mathbb{R}^p , and $\overline{\Theta}$ the closure of Θ . For r > 0 and $r \in \mathbb{R}^p$, $r \in \mathbb{N}$, represents the closed ball centered at $r \in \mathbb{R}^p$, and the radius $r \in \mathbb{R}^p$, $r \in \mathbb{N}$ denotes Dirac's delta measure or unit mass at the point $r \in \mathbb{N}$. For a real

number a>0, $\lfloor a\rfloor$ stands for the integer part of a. For a random variable ξ , \mathscr{L}_{ξ} denotes its law. For a fixed finite number $r_0>0$, $\mathscr{C}:=C([-r_0,0];\mathbb{R}^d)$ means the family of all continuous functions $f:[-r_0,0]\to\mathbb{R}^d$, which is a Polish (i.e., separable, complete metric) space under the uniform norm $\|f\|_{\infty}:=\sup_{r_0\leq\theta\leq0}|f(\theta)|$. Generally speaking, $r_0>0$ is named as the length of memory. For a continuous map $f:[-r_0,\infty)\to\mathbb{R}^d$ and $t\geq0$, let $f_t\in\mathscr{C}$ be such that $f_t(\theta)=f(t+\theta)$ for $\theta\in[-r_0,0]$. In general, $(f_t)_{t\geq0}$ is called the window (or segment) process of $(f(t))_{t\geq-r_0}$. $\mathcal{P}_2(\mathscr{C})$ stands for the space of all probability measures on \mathscr{C} with the finite second-order moment, i.e., $\mu(\|\cdot\|_{\infty}^2):=\int_{\mathscr{C}}\|\zeta\|_{\infty}^2\mu(\mathrm{d}\zeta)<\infty$ for $\mu\in\mathcal{P}_2(\mathscr{C})$. Define the Wasserstein distance \mathbb{W}_2 on $\mathcal{P}_2(\mathscr{C})$ by

$$\mathbb{W}_2(\mu,\nu) = \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left(\int_{\mathscr{C}} \int_{\mathscr{C}} \|\zeta_1 - \zeta_2\|_{\infty}^2 \pi(\mathrm{d}\zeta_1,\mathrm{d}\zeta_2) \right)^{1/2}, \quad \mu,\nu \in \mathcal{P}_2(\mathscr{C}),$$

where $\mathcal{C}(\mu,\nu)$ signifies the collection of all probability measures on $\mathscr{C} \times \mathscr{C}$ with marginals μ and ν (i.e., $\pi \in \mathcal{C}(\mu,\nu)$ such that $\pi(\cdot,\mathscr{C}) = \mu(\cdot)$ and $\pi(\mathscr{C},\cdot) = \nu(\cdot)$), respectively. Under the distance \mathbb{W}_2 , $\mathcal{P}_2(\mathscr{C})$ is a Polish space; see, e.g., [2, Lemma 5.3 & Theorem 5.4]. Let $(B(t))_{t\geq 0}$ be an m-dimensional Brownian motion defined on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual condition (i.e., \mathscr{F}_0 contains all \mathbb{P} -null sets and $\mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$).

Through all the paper, we fix the time horizon T > 0. For the scale parameter $\varepsilon \in (0,1)$, we consider a path-distribution dependent SDE on $(\mathbb{R}^d, \langle \cdot, \cdot \rangle, |\cdot|)$ in the form

$$(2.1) dX^{\varepsilon}(t) = b(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, \theta)dt + \varepsilon \, \sigma(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dB(t), \quad t \in (0, T], \quad X_0^{\varepsilon} = \xi \in \mathscr{C},$$

where $b: \mathscr{C} \times \mathcal{P}_2(\mathscr{C}) \times \Theta \to \mathbb{R}^d$ and $\sigma: \mathscr{C} \times \mathcal{P}_2(\mathscr{C}) \to \mathbb{R}^d \times \mathbb{R}^m$. In (2.1), we assume that the drift b and the diffusion σ are known apart from the parameter $\theta \in \Theta$ and we stipulate that $\theta_0 \in \Theta$ is the true value of $\theta \in \Theta$.

For any $\zeta_1, \zeta_2 \in \mathscr{C}$ and $\mu, \nu \in \mathcal{P}_2(\mathscr{C})$, we assume that

(A1) There exist $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\sup_{\theta \in \overline{\Theta}} |b(\zeta_1, \mu, \theta) - b(\zeta_2, \nu, \theta)|^2 \le \alpha_1 \|\zeta_1 - \zeta_2\|_{\infty}^2 + \alpha_2 \mathbb{W}_2(\mu, \nu)^2,$$

and

$$\|\sigma(\zeta_1,\mu) - \sigma(\zeta_2,\nu)\|^2 \le \beta_1 \|\zeta_1 - \zeta_2\|_{\infty}^2 + \beta_2 \mathbb{W}_2(\mu,\nu)^2;$$

(A2) For each random variable $\zeta \in \mathscr{C}$ with $\mathscr{L}_{\zeta} \in \mathcal{P}_{2}(\mathscr{C})$, $(\sigma \sigma^{*})(\zeta, \mathscr{L}_{\zeta})$ is invertible, and there exists an $L_{1} > 0$ such that

$$\|(\sigma\sigma^*)^{-1}(\zeta_1,\mu) - (\sigma\sigma^*)^{-1}(\zeta_2,\nu)\| \le L_1 \{\|\zeta_1 - \zeta_2\|_{\infty} + \mathbb{W}_2(\mu,\nu)\};$$

(A3) For the initial value $X_0^{\varepsilon} = \xi$, there exists an $L_2 > 0$ such that

$$|\xi(t) - \xi(s)| \le L_2|t - s|, \quad t, s \in [-r_0, 0].$$

We further assume that

(B1) There exists $K_1 > 0$ such that

$$\sup_{\theta \in \overline{\Theta}} \| (\nabla_{\theta} b)(\zeta_1, \mu, \theta) - (\nabla_{\theta} b)(\zeta_2, \nu, \theta) \| \le K_1 \Big\{ \| \zeta_1 - \zeta_2 \|_{\infty} + \mathbb{W}_2(\mu, \nu) \Big\},$$

where $(\nabla_{\theta}b)$ means the gradient operator w.r.t. the third spatial variable.

(B2) There exists $K_2 > 0$ such that

$$\sup_{\theta \in \overline{\Theta}} \| (\nabla_{\theta} (\nabla_{\theta} b^*))(\zeta_1, \mu, \theta) - (\nabla_{\theta} (\nabla_{\theta} b^*))(\zeta_2, \nu, \theta) \| \le K_2 \Big\{ \| \zeta_1 - \zeta_2 \|_{\infty} + \mathbb{W}_2(\mu, \nu) \Big\}.$$

Before we move forward, let's give some remarks. Under (A1), (2.1) admits a unique strong solution $(X^{\varepsilon}(t))_{t\in[-r_0,T]}$; see, for instance, [9, Theorem 3.1]. For more details on existence and uniqueness of strong solutions to distribution-dependent SDEs, we would like to refer to, e.g., [4, 24, 35] and references within. As far as existence and uniqueness of weak solutions are concerned, please consult, e.g., [11, 17, 34] for reference. (B1) and (B2) are imposed merely to discuss the asymptotic distribution of LSE constructed below; see Theorem 4.1. (A3) is put to analyze continuity of the window process associated with (2.4); see Lemma 3.3 for more details. Obviously, (A2) holds provided that $\sigma(\cdot, \cdot) \equiv \sigma \in \mathbb{R}^d \otimes \mathbb{R}^m$, a constant matrix, such that $\sigma\sigma^*$ is invertible. Moreover, for the scalar setting of (2.1), (A2) is also true in case of $\sigma(x, \mu) = 1 + |x|$ for any $x \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$.

Without loss of generality, we assume the stepsize $\delta = \frac{T}{n} = \frac{r_0}{M}$ for some integers $n, M \in \mathbb{N}$ sufficiently large. Suppose that the solution process $(X^{\varepsilon}(t))_{t \in [-r_0,T]}$ is observed at regularly spaced time points $t_k = k\delta$ for $k = 0, 1, \dots, n$. In this paper, our goal is to investigate the LSE on the parameter $\theta \in \Theta$ based on the sampling data $(X^{\varepsilon}(t_k)_{k=0}^n)$ with small dispersion ε and large sample size n (i.e., small step size δ).

The discrete-time Euler-Maruyama (EM) scheme corresponding to (2.1) admits the form

$$(2.2) Y^{\varepsilon}(t_{k}) = Y^{\varepsilon}(t_{k-1}) + b(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta)\delta + \varepsilon \, \sigma(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}) \triangle B_{k}, \quad k \ge 1,$$

and $Y^{\varepsilon}(t)=X^{\varepsilon}(t)=\xi(t), t\in[-r_0,0].$ In (2.2), $\widehat{Y}_{k\delta}^{\varepsilon}=\{\widehat{Y}_{k\delta}^{\varepsilon}(s):-r_0\leq s\leq 0\}$ is a $\mathscr C$ -valued random variable defined as follows: for any $s\in[-(i+1)\delta,-i\delta],\ i=1,\cdots,M-1$,

(2.3)
$$\widehat{Y}_{k\delta}^{\varepsilon}(s) = Y^{\varepsilon}((k-i)\delta) + \frac{s+i\delta}{\delta} \{ Y^{\varepsilon}((k-i)\delta) - Y^{\varepsilon}((k-i-1)\delta) \},$$

i.e., $\widehat{Y}_{k\delta}^{\varepsilon}$ is the linear interpolation of $Y^{\varepsilon}((k-M)\delta)$, $Y^{\varepsilon}((k-(M-1))\delta)$, \cdots , $Y^{\varepsilon}((k-1)\delta)$, $Y^{\varepsilon}(k\delta)$, and $\Delta B_k := B(k\delta) - B((k-1)\delta)$, the increment of Brownian motion. Motivated by [20, 21, 29], for our present setting we construct the following contrast function

(2.4)
$$\Psi_{n,\varepsilon}(\theta) = \varepsilon^{-2} \delta^{-1} \sum_{k=1}^{n} P_k^*(\theta) \Lambda_{k-1}^{-1} P_k(\theta),$$

where

$$(2.5) P_k(\theta) := Y^{\varepsilon}(t_k) - Y^{\varepsilon}(t_{k-1}) - b(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta)\delta \quad \text{ and } \quad \Lambda_k := (\sigma\sigma^*)(\widehat{Y}_{t_k}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_k}^{\varepsilon}})$$

for $k=1,\dots,n$. To achieve the LSE of $\theta\in\Theta$, it suffices to choose an argument $\widehat{\theta}_{n,\varepsilon}\in\Theta$ such that

(2.6)
$$\Psi_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon}) = \min_{\theta \in \Theta} \Psi_{n,\varepsilon}(\theta).$$

Next, we write $\widehat{\theta}_{n,\varepsilon} \in \Theta$ satisfying (2.6) by

$$\widehat{\theta}_{n,\varepsilon} = \arg\min_{\theta \in \Theta} \Psi_{n,\varepsilon}(\theta).$$

Set

$$\Phi_{n,\varepsilon}(\theta) := \varepsilon^2 (\Psi_{n,\varepsilon}(\theta) - \Psi_{n,\varepsilon}(\theta_0)).$$

It follows from (2.6) that

(2.7)
$$\Phi_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon}) = \min_{\theta \in \Theta} \Phi_{n,\varepsilon}(\theta).$$

Likewise, we reformulate $\widehat{\theta}_{n,\varepsilon} \in \Theta$ ensuring (2.7) to hold true as

(2.8)
$$\widehat{\theta}_{n,\varepsilon} = \arg\min_{\theta \in \Theta} \Phi_{n,\varepsilon}(\theta).$$

Through the whole paper, $\widehat{\theta}_{n,\varepsilon}$ such that (2.8) holds is named as the LSE of $\theta \in \Theta$. Before we end this section, we give some remarks.

Remark 2.1. If $\sigma(\cdot,\cdot) \in \mathbb{R}^d \otimes \mathbb{R}^d$ is invertible, (2.6) can be rewritten as

$$\frac{\triangle B_k}{\sqrt{\delta}} = \frac{1}{\varepsilon^{-1}\sqrt{\delta}} \sigma^{-1}(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}) P_k(\theta).$$

Then, we can design the contrast function $\Psi_{n,\varepsilon}(\cdot)$ as

$$\Psi_{n,\varepsilon}(\theta) = \varepsilon^{-2} \delta^{-1} |\sigma^{-1}(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathcal{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}) P_k(\theta)|^2$$

$$= \varepsilon^{-2} \delta^{-1} P_k^*(\theta) ((\sigma^{-1})^* \sigma^{-1}) (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathcal{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}) P_k(\theta)$$

$$= \varepsilon^{-2} \delta^{-1} P_k^*(\theta) \Lambda_{k-1}^{-1} P_k(\theta).$$

Motivated by the invertible setup above, we establish the contrast function for the setting that the diffusion $\sigma(\cdot,\cdot)$ need not to be invertible; see (2.4) for further details. On the other hand, if $b(\cdot,\cdot,\theta)$ is explicit w.r.t. the parameter θ , then the LSE $\widehat{\theta}_{n,\varepsilon}$ can indeed be obtained by Fermat's theorem.

Remark 2.2. Formally, the contrast function $\Psi_{n,\varepsilon}$ can be defined as in (2.4) with $\widehat{Y}_{t_k}^{\varepsilon}$ being replaced by $X_{t_k}^{\varepsilon}$ in (2.6). Nevertheless, $X_{t_k}^{\varepsilon}$ cannot be available provided that $(X^{\varepsilon}(t))_{t\in[0,T]}$ is observed only at the points $t=k\delta$. So, in our paper, we approximate the window process $X_{t_k}^{\varepsilon}$ via the linear interpolation; see (2.3) for more details.

Remark 2.3. We remark that our contrast function is established on the basis of EM scheme. With regard to path-distribution dependent SDEs, if the global Lipschitz condition (A1) is replaced by the monotone condition, then the contrast function (2.4) will no longer work due to the fact that the EM numerical solution will explode in finite time. So, for such case, we need to establish the LSE for the unknown parameter based on the new contrast function, which will be reported in our forthcoming paper.

3 The consistency of LSE

First of all, let's consider the following deterministic ordinary differential equation

(3.1)
$$dX(t) = b(X_t^0, \mathcal{L}_{X_t^0}, \theta_0) dt, \quad t > 0, \quad X_0^0 = \xi \in \mathcal{C}.$$

Under (A1), (3.1) possesses a unique solution $(X^0(t))_{t\geq -r_0}$. Herein, it is worth pointing out that (2.1) and (3.1) share the same initial datum. For the sake of notation brevity, for a random variable $\zeta \in \mathscr{C}$ with $\mathscr{L}_{\zeta} \in \mathscr{P}_2(\mathscr{C})$, let

(3.2)
$$\Lambda(\zeta, \theta, \theta_0) = b(\zeta, \mathcal{L}_{\zeta}, \theta_0) - b(\zeta, \mathcal{L}_{\zeta}, \theta) \quad \text{and} \quad \widehat{\sigma}(\zeta) = (\sigma \sigma^*)^{-1}(\zeta, \mathcal{L}_{\zeta}).$$

Set, for any $\theta \in \Theta$,

(3.3)
$$\Xi(\theta) := \int_0^T \Lambda^*(X_t^0, \theta, \theta_0) \widehat{\sigma}(X_t^0) \Lambda(X_t^0, \theta, \theta_0) dt,$$

where (X_t) is the segment process generated by the solution (X(t)) to (3.1).

Our first main result, which is concerned with the consistency of the LSE of $\theta \in \Theta$, is stated as below.

Theorem 3.1. Let (A1)-(A3) hold and assume further $\Xi(\theta) > 0$ for any $\theta \in \overline{\Theta}$. Then

$$\widehat{\theta}_{n,\varepsilon} \to \theta_0$$
 in probability as $\varepsilon \to 0$ and $n \to \infty$.

The proof of Theorem 3.1 is based on several auxiliary lemmas below.

Lemma 3.2. Under (A1), for any p > 0, there exists $C_{p,T} > 0$ such that

(3.4)
$$\sup_{0 \le t \le T} \mathbb{E} \|Y_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^{p} \le C_{p,T} (1 + \|\xi\|_{\infty}^{p}),$$

and

(3.5)
$$\mathbb{E}\left(\sup_{-r_0 \le t \le T} |X^{\varepsilon}(t)|^p\right) \le C_{p,T}(1 + \|\xi\|_{\infty}^p).$$

Proof. By Hölder's inequality, it is sufficient to show that (3.4) and (3.5) holds, respectively, for any $p \ge 2$. Herein, we only focus on the argument of (3.4) since (3.5) can be done in a similar way. Define the continuous-time EM scheme associated with (2.1)

with $\widetilde{Y}^{\varepsilon}(t) = X^{\varepsilon}(t) = \xi(t)$ for $t \in [-r_0, 0]$, where $\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}(\cdot)$ is defined as in (2.3). It is straightforward to check $\widetilde{Y}^{\varepsilon}(k\delta) = Y^{\varepsilon}(k\delta)$ for any integer $k \in [-M, n]$. For any $t \in [0, T]$, a direct calculation shows from (2.3) that

$$\begin{aligned} \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty} &= \sup_{-r_0 \leq v \leq 0} |\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}(v)| \\ (3.7) &= \max_{k=0,\cdots,M-1} \sup_{-(k+1)\delta \leq v \leq -k\delta} \left| \frac{(k+1)\delta + v}{\delta} Y^{\varepsilon}((\lfloor t/\delta \rfloor - k)\delta) - \frac{k\delta + v}{\delta} Y^{\varepsilon}((\lfloor t/\delta \rfloor - k - 1)\delta) \right| \\ &\leq \max_{k=0,\cdots,M-1} \sup_{-(k+1)\delta \leq v \leq -k\delta} \left(|\widetilde{Y}^{\varepsilon}(\lfloor t/\delta \rfloor \delta - k\delta)| + |\widetilde{Y}^{\varepsilon}(\lfloor t/\delta \rfloor \delta - (k+1)\delta)| \right) \\ &\leq 2 \sup_{-r_0 \leq s \leq t} |\widetilde{Y}^{\varepsilon}(s)|, \end{aligned}$$

where in the first inequality we have used the facts that $\widetilde{Y}^{\varepsilon}(k\delta) = Y^{\varepsilon}(k\delta)$ for any integer $k \in [-M, n]$ and that, for any $v \in [-(k+1)\delta, -k\delta]$,

$$\frac{(k+1)\delta + v}{\delta} \in [0,1]$$
 and $\frac{k\delta + v}{\delta} \in [-1,0].$

From (A1), one has, for any $\zeta \in \mathscr{C}$ and $\mu \in \mathcal{P}_2(\mathscr{C})$,

$$|b(\zeta, \mu, \theta)|^2 \le 2 \left\{ \alpha_1 \|\zeta\|_{\infty}^2 + \alpha_2 \mathbb{W}_2(\mu, \delta_{\zeta_0})^2 + |b(\zeta_0, \delta_{\zeta_0}, \theta)|^2 \right\},$$

and

(3.9)
$$\|\sigma(\zeta,\mu)\|^2 \le 2 \left\{ \beta_1 \|\zeta\|_{\infty}^2 + \beta_2 \mathbb{W}_2(\mu,\delta_{\zeta_0})^2 + \|\sigma(\zeta_0,\delta_{\zeta_0})\|^2 \right\},$$

where $\zeta_0(s) = \mathbf{0} \in \mathbb{R}^d$ for any $s \in [-r_0, 0]$. For any $p \geq 2$, by Hölder's inequality and Burkhold-Davis-Gundy's (BDG's for brevity) inequality (see, e.g., [22, Theorem 7.3, P.40]), we deduce from (3.8) and (3.9) that

$$\begin{split} &\Gamma(t):=1+\mathbb{E}\Big(\sup_{-r_0\leq s\leq t}|\widehat{Y}^{\varepsilon}(s)|^p\Big)\\ &\leq 1+c\,\|\xi\|_{\infty}^p+c\,t^{p-1}\int_0^t\mathbb{E}|b(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon},\mathscr{L}_{\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}},\theta)|^p\mathrm{d}s+c\,\mathbb{E}\Big(\int_0^t\|\sigma(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon},\mathscr{L}_{\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}})\|^2\mathrm{d}s\Big)^{p/2}\\ &\leq 1+c\,\|\xi\|_{\infty}^p+c(t^{p-1}+t^{\frac{p-2}{2}})\int_0^t\{\mathbb{E}|b(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon},\mathscr{L}_{\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}},\theta)|^p+\mathbb{E}\|\sigma(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon},\mathscr{L}_{\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}})\|^p\}\mathrm{d}s\\ &\leq 1+c\,\|\xi\|_{\infty}^p+c(t^{p-1}+t^{\frac{p-2}{2}})\int_0^t\{1+\mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}\|_{\infty}^p+\mathbb{W}_2(\mathscr{L}_{\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}},\delta_{\zeta_0})^p\}\mathrm{d}s\\ &\leq 1+c\,\|\xi\|_{\infty}^p+c(t^{p-1}+t^{\frac{p-2}{2}})\int_0^t\{1+\mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}\|_{\infty}^p\}\mathrm{d}s. \end{split}$$

This, together with (3.7), leads to

$$\Gamma(t) \le 1 + c \|\xi\|_{\infty}^p + c(t^{p-1} + t^{\frac{p-2}{2}}) \int_0^t \Gamma(s) ds.$$

Then, the desired assertion (3.4) follows from Gronwall's inequality and (3.7).

Lemma 3.3. Let (A1) be satisfied. Then, there is a constant $C_T > 0$ such that

(3.10)
$$\sup_{0 \le t \le T} \mathbb{E} \|X_t^{\varepsilon} - X_t^0\|_{\infty}^2 \le C_T \varepsilon^2.$$

Proof. Note that

$$\mathbb{E}||X_t^{\varepsilon} - X_t^0||_{\infty}^2 \le \mathbb{E}\Big(\sup_{0 \le s \le t} |X^{\varepsilon}(s) - X^0(s)|^2\Big) =: A(t, \varepsilon),$$

where we have used $X_0^{\varepsilon} = X_0^0 = \xi$. By Hölder's inequality, Doob's submartingale inequality as well as Itô's isometry, we obtain from (A1) and (3.9) that

$$\begin{split} A(t,\varepsilon) &\leq 2\,t\,\int_{0}^{t} \mathbb{E}|b(X_{s}^{\varepsilon},\mathcal{L}_{X_{s}^{\varepsilon}},\theta_{0}) - b(X_{s}^{0},\mathcal{L}_{X_{s}^{0}},\theta_{0})|^{2}\mathrm{d}s + 2\,\varepsilon^{2}\,\mathbb{E}\Big(\sup_{0\leq s\leq t}\Big|\int_{0}^{s}\sigma(X_{u}^{\varepsilon},\mathcal{L}_{X_{u}^{\varepsilon}})\mathrm{d}B(u)\Big|^{2}\Big) \\ &\leq 2\,t\,\int_{0}^{t} \mathbb{E}|b(X_{s}^{\varepsilon},\mathcal{L}_{X_{s}^{\varepsilon}},\theta_{0}) - b(X_{s}^{0},\mathcal{L}_{X_{s}^{0}},\theta_{0})|^{2}\mathrm{d}s + 8\,\varepsilon^{2}\int_{0}^{t} \mathbb{E}\|\sigma(X_{s}^{\varepsilon},\mathcal{L}_{X_{s}^{\varepsilon}})\|^{2}\mathrm{d}s \\ &\leq 2\,t\,\int_{0}^{t} \{\alpha_{1}\mathbb{E}\|X_{s}^{\varepsilon} - X_{s}^{0}\|_{\infty}^{2} + \alpha_{2}\mathbb{W}_{2}(\mathcal{L}_{X_{s}^{\varepsilon}},\mathcal{L}_{X_{s}^{0}})^{2}\}\mathrm{d}s \\ &\quad + c\,\varepsilon^{2}\int_{0}^{t} \{1 + \mathbb{E}\|X_{s}^{\varepsilon}\|_{\infty}^{2} + \mathbb{W}_{2}(\mathcal{L}_{X_{s}^{\varepsilon}},\delta_{\zeta_{0}})^{2}\}\mathrm{d}s \\ &\leq c\,t\,\int_{0}^{t} \mathbb{E}\|X_{s}^{\varepsilon} - X_{s}^{0}\|_{\infty}^{2}\mathrm{d}s + c\,\varepsilon^{2}\int_{0}^{t} \{1 + \mathbb{E}\|X_{s}^{\varepsilon}\|_{\infty}^{2}\}\mathrm{d}s \\ &\leq c\,t\,\int_{0}^{t} A(s,\varepsilon)\mathrm{d}s + c(1 + C_{2,T})\,\varepsilon^{2}t, \end{split}$$

where we have used (3.5) in the last display. As a result, (3.10) holds true by Gronwall's inequality.

Lemma 3.4. Assume that (A1) and (A3) hold. Then, for any $\beta \in (0,1)$, there exists $c_{\beta} > 0$ such that

(3.11)
$$\sup_{0 \le t \le T} \mathbb{E} \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_t^0\|_{\infty}^2 \le c \,\varepsilon^2 + c_{\beta} \delta^{\beta}.$$

Proof. Due to (3.10), for any $t \in [0, T]$,

$$(3.12) \qquad \mathbb{E}\|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{2} \leq 3\{\mathbb{E}\|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - \widetilde{Y}_{t}^{\varepsilon}\|_{\infty}^{2} + \mathbb{E}\|\widetilde{Y}_{t}^{\varepsilon} - X_{t}^{\varepsilon}\|_{\infty}^{2} + \mathbb{E}\|X_{t}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{2}\} \\ \leq c\{\varepsilon^{2} + \mathbb{E}\|\widehat{Y}_{t/\delta \rfloor \delta}^{\varepsilon} - \widetilde{Y}_{t}^{\varepsilon}\|_{\infty}^{2} + \mathbb{E}\|\widetilde{Y}_{t}^{\varepsilon} - X_{t}^{\varepsilon}\|_{\infty}^{2}\}.$$

Next, exploiting Hölder's inequality, Doob's submartingale inequality and Itô's isometry, we derive from (A1) and $X_0^{\varepsilon} = \widetilde{Y}_0^{\varepsilon} = \xi$ that

$$\begin{split} \mathbb{E}\|X_t^{\varepsilon} - \widetilde{Y}_t^{\varepsilon}\|_{\infty}^2 &\leq \mathbb{E}\Big(\sup_{0 \leq s \leq t} |X^{\varepsilon}(s) - \widetilde{Y}^{\varepsilon}(s)|^2\Big) \\ &\leq 2\,t\int_0^t \mathbb{E}|b(X_s^{\varepsilon}, \mathcal{L}_{X_s^{\varepsilon}}, \theta) - b(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}}, \theta)|^2 \mathrm{d}s \\ &\quad + 8\,\varepsilon^2 \int_0^t \mathbb{E}\|\sigma(X_s^{\varepsilon}, \mathcal{L}_{X_s^{\varepsilon}}) - \sigma(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}})\|^2 \mathrm{d}s \\ &\leq 2\,t\int_0^t \{\alpha_1 \mathbb{E}\|X_s^{\varepsilon} - \widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^2 + \alpha_2 \mathbb{W}_2(\mathcal{L}_{X_s^{\varepsilon}}, \mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}})^2\} \mathrm{d}s \\ &\quad + 8\,\varepsilon^2 \int_0^t \{\beta_1 \mathbb{E}\|X_s^{\varepsilon} - \widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^2 + \beta_2 \mathbb{W}_2(\mathcal{L}_{X_s^{\varepsilon}}, \mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}})^2\} \mathrm{d}s. \end{split}$$

Consequently, we obtain from $\varepsilon \in (0,1)$ that

$$\begin{split} \mathbb{E}\|X_t^{\varepsilon} - \widetilde{Y}_t^{\varepsilon}\|_{\infty}^2 &\leq c\left(1+t\right) \int_0^t \mathbb{E}\|X_s^{\varepsilon} - \widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^2 \mathrm{d}s \\ &\leq c\left(1+t\right) \int_0^t \mathbb{E}\|X_s^{\varepsilon} - \widetilde{Y}_s^{\varepsilon}\|_{\infty}^2 \mathrm{d}s + c\left(1+t\right) \int_0^t \mathbb{E}\|\widetilde{Y}_s^{\varepsilon} - \widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^2 \mathrm{d}s. \end{split}$$

Thus, Gronwall's inequality yields that

$$(3.13) \mathbb{E}\|X_t^{\varepsilon} - \widetilde{Y}_t^{\varepsilon}\|_{\infty}^2 \le c \sup_{0 < t < T} \mathbb{E}\|\widetilde{Y}_t^{\varepsilon} - \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^2.$$

Substituting (3.13) into (3.12) gives that

$$(3.14) \qquad \mathbb{E}\|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{2} \leq c \Big\{ \varepsilon^{2} + \sup_{t \in [0,T]} \mathbb{E}\|\widetilde{Y}_{t}^{\varepsilon} - \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^{2} \Big\}.$$

So, to achieve (3.11), it remains to show that, for any $\beta \in (0,1)$, there exists $c_{\beta} > 0$ such that

(3.15)
$$\sup_{t \in [0,T]} \mathbb{E} \|\widetilde{Y}_t^{\varepsilon} - \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^2 \le c_{\beta} \delta^{\beta}.$$

For any $t \in [0, T)$, there exists an integer $k_0 \in [0, n-1]$ such that $t \in [k_0 \delta, (k_0 + 1)\delta)$ so that $\lfloor t/\delta \rfloor = k_0$. By Hölder's inequality, for any $\beta \in (0, 1)$,

$$\begin{split} \mathbb{E} \| \widetilde{Y}_{t}^{\varepsilon} - \widehat{Y}_{k_{0}\delta}^{\varepsilon} \|_{\infty}^{2} &= \mathbb{E} \Big(\sup_{-r_{0} \leq v \leq 0} |\widetilde{Y}^{\varepsilon}(t+v) - \widehat{Y}_{k_{0}\delta}^{\varepsilon}(v)|^{2} \Big) \\ &\leq \Big(\mathbb{E} \Big(\sup_{-r_{0} \leq v \leq 0} |\widetilde{Y}^{\varepsilon}(t+v) - \widehat{Y}_{k_{0}\delta}^{\varepsilon}(v)|^{\frac{2}{1-\beta}} \Big) \Big)^{1-\beta} \\ &\leq M^{1-\beta} \max_{k=0,\cdots,M-1} \Big(\mathbb{E} \Big(\sup_{-(k+1)\delta \leq v \leq -k\delta} |\widetilde{Y}^{\varepsilon}(t+v) - \widehat{Y}_{k_{0}\delta}^{\varepsilon}(v)|^{\frac{2}{1-\beta}} \Big) \Big)^{1-\beta}, \end{split}$$

where M > 0 is an integer such that $r_0 = M\delta$. For any $v \in [-(k+1)\delta, -k\delta]$ with $k = 0, \dots, M-1$, it follows from (2.3) that

$$\widetilde{Y}^{\varepsilon}(t+v) - \widehat{Y}_{k_0\delta}^{\varepsilon}(v) = \frac{(k+1)\delta + v}{\delta} \Big(\widetilde{Y}^{\varepsilon}(t+v) - Y^{\varepsilon}((k_0-k)\delta) \Big) - \frac{k\delta + v}{\delta} \Big(\widetilde{Y}^{\varepsilon}(t+v) - Y^{\varepsilon}((k_0-k-1)\delta) \Big).$$

As a consequence, we deduce that

$$(3.16) \begin{split} \mathbb{E} \|\widetilde{Y}_{t}^{\varepsilon} - \widehat{Y}_{k_{0}\delta}^{\varepsilon}\|_{\infty}^{2} \\ &\leq c \, M^{1-\beta} \max_{k=0,\cdots,M-1} \left(\mathbb{E} \left(\sup_{(k_{0}-k-1)\delta \leq s \leq (k_{0}-k+1)\delta} |\widetilde{Y}^{\varepsilon}(s) - Y^{\varepsilon}((k_{0}-k)\delta)|^{\frac{2}{1-\beta}} \right) \right)^{1-\beta} \\ &+ c \, M^{1-\beta} \max_{k=0,\cdots,M-1} \left(\mathbb{E} \left(\sup_{(k_{0}-k-1)\delta \leq s \leq (k_{0}-k+1)\delta} |\widetilde{Y}^{\varepsilon}(s) - Y^{\varepsilon}((k_{0}-k-1)\delta)|^{\frac{2}{1-\beta}} \right) \right)^{1-\beta} \\ &=: A_{1}(\varepsilon,\delta) + A_{2}(\varepsilon,\delta). \end{split}$$

For any $t \in [l\delta, (l+1)\delta]$ with $l = 0, 1, \dots, n-1$, we have

$$\mathbb{E}\Big(\sup_{l\delta \leq s \leq t} |\widetilde{Y}^{\varepsilon}(s) - \widetilde{Y}^{\varepsilon}(l\delta)|^{\frac{2}{1-\beta}}\Big) \leq c \left\{\delta^{\frac{2}{1-\beta}} \mathbb{E}|b(\widehat{Y}^{\varepsilon}_{l\delta}, \mathcal{L}_{\widehat{Y}^{\varepsilon}_{l\delta}}, \theta)|^{\frac{2}{1-\beta}} + \mathbb{E}\|\sigma(\widehat{Y}^{\varepsilon}_{l\delta}, \mathcal{L}_{\widehat{Y}^{\varepsilon}_{l\delta}})\|^{\frac{2}{1-\beta}} \mathbb{E}\Big(\sup_{l\delta \leq s \leq t} |B(s) - B(l\delta)|^{\frac{2}{1-\beta}}\Big)\right\} \\
= c \left\{\delta^{\frac{2}{1-\beta}} \mathbb{E}|b(\widehat{Y}^{\varepsilon}_{l\delta}, \mathcal{L}_{\widehat{Y}^{\varepsilon}_{l\delta}}, \theta)|^{\frac{2}{1-\beta}} + \mathbb{E}\|\sigma(\widehat{Y}^{\varepsilon}_{l\delta}, \mathcal{L}_{\widehat{Y}^{\varepsilon}_{l\delta}})\|^{\frac{2}{1-\beta}} \mathbb{E}\Big(\sup_{0 \leq s \leq t-l\delta} |B(s)|^{\frac{2}{1-\beta}}\Big)\right\}$$

where we have used the fact that $\widehat{Y}_{l\delta}^{\varepsilon}$ is independent of $B(t) - B(l\delta)$ for any $t \in [l\delta, (l+1)\delta]$ in the first inequality and the independent increment property of Brownian motion in the last display.

Let $(e_i)_{1 \leq i \leq m}$ be the standard orthogonal basis of \mathbb{R}^m . Note that $B_i(t) := \langle B(t), e_i \rangle$ is a scalar Brownian motion and

$$\mathbb{P}\Big(\sup_{0 \le s \le t} B_i(s) \ge x\Big) = 2\mathbb{P}(B_i(t) \ge x),$$

see, for instance, [13, Theorem 3.15]. Whence, for any p > 1, we deduce that

$$\begin{split} \mathbb{E} \Big(\sup_{0 \le s \le t} |B(s)|^p \Big) &= \int_0^\infty \mathbb{P} \Big(\sup_{0 \le s \le t} |B(s)|^p \ge x \Big) \mathrm{d}x \\ &= 2 \sum_{i=1}^m \int_0^\infty \mathbb{P} \Big(B_i(t) \ge x^{1/p} / m^{1/2} \Big) \mathrm{d}x \\ &= \frac{2}{\sqrt{2\pi t}} \sum_{i=1}^m \int_0^\infty \mathrm{d}x \int_{\frac{x^{\frac{1}{p}}}{m^{\frac{1}{2}}}}^\infty \mathrm{e}^{-\frac{y^2}{2t}} \mathrm{d}y \\ &\le \frac{2m^{1/2}}{\sqrt{2\pi t}} \sum_{i=1}^m \int_0^\infty x^{-\frac{1}{p}} \mathrm{d}x \int_{\frac{x^{\frac{1}{p}}}{m^{\frac{1}{2}}}}^\infty y \mathrm{e}^{-\frac{y^2}{2t}} \mathrm{d}y \\ &= \frac{2m^{1/2}t}{\sqrt{2\pi t}} \sum_{i=1}^m \int_0^\infty x^{-\frac{1}{p}} \mathrm{e}^{-\frac{x^{2/p}}{2mt}} \mathrm{d}x \\ &\le c \, t^{\frac{p}{2}}, \end{split}$$

where in the last step we have utilized the Gamma function

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha - 1} dx, \quad \alpha > 0.$$

This, combining (3.8) with (3.9) and (3.17), yields that, for any $t \in [l\delta, (l+1)\delta]$,

$$\mathbb{E}\left(\sup_{l\delta \leq s \leq t} |\widetilde{Y}^{\varepsilon}(s) - \widetilde{Y}^{\varepsilon}(l\delta)|^{\frac{2}{1-\beta}}\right) \leq c\delta^{\frac{1}{1-\beta}} \{\mathbb{E}|b(\widehat{Y}^{\varepsilon}_{l\delta}, \mathcal{L}_{\widehat{Y}^{\varepsilon}_{l\delta}}, \theta)|^{\frac{2}{1-\beta}} + \mathbb{E}\|\sigma(\widehat{Y}^{\varepsilon}_{l\delta}, \mathcal{L}_{\widehat{Y}^{\varepsilon}_{l\delta}})\|^{\frac{2}{1-\beta}}\}$$

$$\leq c\delta^{\frac{1}{1-\beta}} \{1 + \mathbb{E}\|\widehat{Y}^{\varepsilon}_{l\delta}\|_{\infty}^{\frac{2}{1-\beta}} + \mathbb{W}_{2}(\mathcal{L}_{\widehat{Y}^{\varepsilon}_{l\delta}}, \delta_{\zeta_{0}})^{\frac{2}{1-\beta}}\}$$

$$\leq c\delta^{\frac{1}{1-\beta}} \{1 + \mathbb{E}\|\widehat{Y}^{\varepsilon}_{l\delta}\|_{\infty}^{\frac{2}{1-\beta}}\}$$

$$\leq c\delta^{\frac{1}{1-\beta}},$$

where in the last procedure we have exploited (3.4).

In the sequel, we divide three cases to show the estimates on $A_1(\varepsilon, \delta)$ and $A_2(\varepsilon, \delta)$. Case 1: $k \ge k_0 + 1$. With regard to such case, $(k_0 - k + 1)\delta \in [-r_0, 0]$. We infer from (A1) and (3.16), in addition to $M\delta = r_0$, that

$$A_1(\varepsilon,\delta) + A_2(\varepsilon,\delta) \le cM^{1-\beta}\delta = c\,r_0^{1-\beta}\delta^{\beta}.$$

Case 2: $k_0 = k$. For this case, $t \in [k\delta, (k+1)\delta)$. Again, one gets from (3.16) that

$$\begin{split} &A_{1}(\varepsilon,\delta) + A_{2}(\varepsilon,\delta) \\ &\leq c\,M^{1-\beta} \max_{k=0,\cdots,M-1} \left(\mathbb{E}\Big(\sup_{-\delta \leq s \leq \delta} |\widetilde{Y}^{\varepsilon}(s) - \widetilde{Y}^{\varepsilon}(0)|^{\frac{2}{1-\beta}} \Big) \Big)^{1-\beta} \\ &+ c\,M^{1-\beta} \max_{k=0,\cdots,M-1} \left(\mathbb{E}\Big(\sup_{-\delta \leq s \leq \delta} |\widetilde{Y}^{\varepsilon}(s) - \widetilde{Y}^{\varepsilon}(-\delta)|^{\frac{2}{1-\beta}} \right) \right)^{1-\beta}, \end{split}$$

where we have employed $Y^{\varepsilon}(t) = \widetilde{Y}(t), t \in [-r_0, 0]$. This, besides (A3) and (3.18), implies that

$$A_{1}(\varepsilon,\delta) + A_{2}(\varepsilon,\delta) \leq c\delta^{\beta} + c M^{1-\beta} \max_{k=0,\cdots,M-1} \left(\mathbb{E} \left(\sup_{-\delta \leq s \leq \delta} |\widetilde{Y}^{\varepsilon}(s) - \widetilde{Y}^{\varepsilon}(0)|^{\frac{2}{1-\beta}} \right) \right)^{1-\beta}$$

$$\leq c\delta^{\beta} + c M^{1-\beta} \max_{k=0,\cdots,M-1} \left(\mathbb{E} \left(\sup_{0 \leq s \leq \delta} |\widetilde{Y}^{\varepsilon}(s) - \widetilde{Y}^{\varepsilon}(0)|^{\frac{2}{1-\beta}} \right) \right)^{1-\beta}$$

$$\leq c\delta^{\beta} + c M^{1-\beta} \delta$$

$$\leq c\delta^{\beta}.$$

Case 3: $k \le k_0 - 1$. Also, by making use of (3.18), it follows that

$$\begin{split} &A_{1}(\varepsilon,\delta) + A_{2}(\varepsilon,\delta) \\ &\leq c\,M^{1-\beta}\max_{k=0,\cdots,M-1} \left(\mathbb{E}\Big(\sup_{(k_{0}-k-1)\delta\leq s\leq (k_{0}-k+1)\delta} |\widetilde{Y}^{\varepsilon}(s)-\widetilde{Y}^{\varepsilon}((k_{0}-k-1)\delta)|^{\frac{2}{1-\beta}}\Big)\Big)^{1-\beta} \\ &+ c\,M^{1-\beta}\max_{k=0,\cdots,M-1} \left(\mathbb{E}|\widetilde{Y}^{\varepsilon}((k_{0}-k-1)\delta)-\widetilde{Y}^{\varepsilon}((k_{0}-k)\delta)|^{\frac{2}{1-\beta}}\right)^{1-\beta} \\ &\leq c\,M^{1-\beta}\max_{k=0,\cdots,M-1} \left(\mathbb{E}\Big(\sup_{(k_{0}-k-1)\delta\leq s\leq (k_{0}-k)\delta} |\widetilde{Y}^{\varepsilon}(s)-\widetilde{Y}^{\varepsilon}((k_{0}-k-1)\delta)|^{\frac{2}{1-\beta}}\right)\Big)^{1-\beta} \\ &+ c\,M^{1-\beta}\max_{k=0,\cdots,M-1} \left(\mathbb{E}\Big(\sup_{(k_{0}-k)\delta\leq s\leq (k_{0}-k+1)\delta} |\widetilde{Y}^{\varepsilon}(s)-\widetilde{Y}^{\varepsilon}((k_{0}-k)\delta)|^{\frac{2}{1-\beta}}\right)\Big)^{1-\beta} \\ &+ c\,M^{1-\beta}\max_{k=0,\cdots,M-1} \left(\mathbb{E}|\widetilde{Y}^{\varepsilon}((k_{0}-k-1)\delta)-\widetilde{Y}^{\varepsilon}((k_{0}-k)\delta)|^{\frac{2}{1-\beta}}\right)^{1-\beta} \\ &\leq c\delta^{\beta}. \end{split}$$

By summing up the three cases above, (3.15) holds true.

Lemma 3.5. Let (A1)-(A3) hold. Then,

(3.19)
$$\delta \sum_{k=1}^{n} \Lambda^* (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_0) \widehat{\sigma} (\widehat{Y}_{t_{k-1}}^{\varepsilon}) \Lambda (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_0) \\ \rightarrow \Xi(\theta) := \int_0^T \Lambda^* (X_s^0, \theta, \theta_0) \widehat{\sigma} (X_s^0) \Lambda (X_s^0, \theta, \theta_0) \mathrm{d}s$$

in L^1 as $\varepsilon \to 0$ and $\delta \to 0$ (i.e., $n \to \infty$), in which $\Lambda(\cdot)$ and $\widehat{\sigma}(\cdot)$ are introduced in (3.2).

Proof. It is straightforward to see that

$$\begin{split} &\delta \sum_{k=1}^{n} \Lambda^{*}(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) \Lambda(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) - \int_{0}^{T} \Lambda^{*}(X_{s}^{0}, \theta, \theta_{0}) \widehat{\sigma}(X_{s}^{0}) \Lambda(X_{s}^{0}, \theta, \theta_{0}) \mathrm{d}s \\ &= \int_{0}^{T} \left\{ \Lambda^{*}(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \theta, \theta_{0}) \widehat{\sigma}(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}) \Lambda(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \theta, \theta_{0}) - \Lambda^{*}(X_{s}^{0}, \theta, \theta_{0}) \widehat{\sigma}(X_{s}^{0}) \Lambda(X_{s}^{0}, \theta, \theta_{0}) \right\} \mathrm{d}s \\ &= \int_{0}^{T} \left(\Lambda(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \theta, \theta_{0}) - \Lambda(X_{s}^{0}, \theta, \theta_{0}) \right)^{*} \widehat{\sigma}(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}) \Lambda(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \theta, \theta_{0}) \mathrm{d}s \\ &+ \int_{0}^{T} \Lambda^{*}(X_{s}^{0}, \theta, \theta_{0}) \Big(\widehat{\sigma}(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}) - \widehat{\sigma}(X_{s}^{0}) \Big) \Lambda(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \theta, \theta_{0}) \mathrm{d}s \\ &+ \int_{0}^{T} \Lambda^{*}(X_{s}^{0}, \theta, \theta_{0}) \widehat{\sigma}(X_{s}^{0}) \Big(\Lambda(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \theta, \theta_{0}) - \Lambda(X_{s}^{0}, \theta, \theta_{0}) \Big) \mathrm{d}s \\ &=: J_{1}(\varepsilon, \delta) + J_{2}(\varepsilon, \delta) + J_{3}(\varepsilon, \delta). \end{split}$$

Next, for any random variables $\zeta_1, \zeta_2 \in \mathscr{C}$ with $\mathscr{L}_{\zeta_1}, \mathscr{L}_{\zeta_2} \in \mathcal{P}_2(\mathscr{C})$, observe from $(\mathbf{A1})$ that

$$(3.20) |\Lambda(\zeta_{1}, \theta, \theta_{0}) - \Lambda(\zeta_{2}, \theta, \theta_{0})| \leq |b(\zeta_{1}, \mathcal{L}_{\zeta_{1}}, \theta_{0}) - b(\zeta_{2}, \mathcal{L}_{\zeta_{2}}, \theta_{0})| + |b(\zeta_{1}, \mathcal{L}_{\zeta_{1}}, \theta) - b(\zeta_{2}, \mathcal{L}_{\zeta_{2}}, \theta)| \\ \leq c \Big\{ \|\zeta_{1} - \zeta_{2}\|_{\infty} + \mathbb{W}_{2}(\mathcal{L}_{\zeta_{1}}, \mathcal{L}_{\zeta_{2}}) \Big\}.$$

For a random variable $\zeta \in \mathscr{C}$ with $\mathscr{L}_{\zeta} \in \mathcal{P}_{2}(\mathscr{C})$, employing $(\mathbf{A2})$ gives that

Consequently, combining (3.8) with (3.20) and (3.21), we deduce that

$$|J_{1}(\varepsilon,\delta)| + |J_{3}(\varepsilon,\delta)|$$

$$\leq c \int_{0}^{T} \left\{ \|\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon} - X_{s}^{0}\|_{\infty} + \mathbb{W}_{2}(\mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}}, \mathcal{L}_{X_{s}^{0}}) \right\}$$

$$\times \left\{ 1 + \|X_{s}^{0}\|_{\infty} + \|\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}\|_{\infty} + \mathbb{W}_{2}(\mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}}, \delta_{\zeta_{0}}) \right\}^{2} ds$$

$$\leq c \int_{0}^{T} \left\{ \|\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon} - X_{s}^{0}\|_{\infty} + \sqrt{\mathbb{E}\|\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon} - X_{s}^{0}\|_{\infty}^{2}} \right\}$$

$$\times \left\{ 1 + \|X_{s}^{0}\|_{\infty}^{2} + \|\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^{2} + \mathbb{E}\|\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^{2} \right\} ds.$$

This, together with (3.4) and (3.11) as well as Hölder's inequality, implies that

$$(3.22) \qquad \mathbb{E}|J_{1}(\varepsilon,\delta)| + \mathbb{E}|J_{3}(\varepsilon,\delta)| \\ \leq c \int_{0}^{T} \sqrt{\mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon} - X_{s}^{0}\|_{\infty}^{2}} \left\{1 + \|X_{s}^{0}\|_{\infty}^{4} + \mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}\|_{\infty}^{4}\right\} ds} \\ \to 0$$

as $\varepsilon \to 0$ and $\delta \to 0$. Next, making use of (A2) and (3.8), we derive that

$$|J_{2}(\varepsilon,\delta)| \leq c \int_{0}^{T} (1 + ||X_{s}^{0}||_{\infty}) (1 + ||\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}||_{\infty} + \mathbb{W}_{2}(\mathscr{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}}, \delta_{\zeta_{0}})) \times (||\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon} - X_{s}^{0}||_{\infty} + \sqrt{\mathbb{E}||\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon} - X_{s}^{0}||_{\infty}^{2}}) ds.$$

Again, using (3.4) and (3.11) and utilizing Hölder's inequality gives that

$$(3.23) \qquad \mathbb{E}|J_2(\varepsilon,\delta)| \le c \int_0^T \sqrt{\mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon} - X_s^0\|_{\infty}^2} \Big\{ 1 + \mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}\|_{\infty}^2 \Big\} ds \\ \to 0$$

whenever $\varepsilon \to 0$ and $\delta \to 0$. Hence, (3.19) follows immediately from (3.22) and (3.23).

Lemma 3.6. Let (A1)-(A3) hold. Then,

(3.24)
$$\sum_{k=1}^{n} \Lambda^*(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_0) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_k(\theta_0) \longrightarrow 0$$

in L^1 as $\varepsilon \to 0$, where P_k is introduced in (2.5).

Proof. Note that

$$\begin{split} \Upsilon(\varepsilon,\delta) &:= \sum_{k=1}^n \Lambda^*(\widehat{Y}_{t_{k-1}}^\varepsilon,\theta,\theta_0) \widehat{\sigma}(Y_{t_{k-1}}^\varepsilon) P_k(\theta_0) \\ &= \varepsilon \sum_{k=1}^n \Lambda^*(\widehat{Y}_{t_{k-1}}^\varepsilon,\theta,\theta_0) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^\varepsilon) \sigma(Y_{t_{k-1}}^\varepsilon,\mathscr{L}_{\widehat{Y}_{t_{k-1}}^\varepsilon}) (B(t_k) - B(t_{k-1})) \\ &= \varepsilon \int_0^T \Lambda^*(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^\varepsilon,\theta,\theta_0) \widehat{\sigma}(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^\varepsilon) \sigma(\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^\varepsilon,\mathscr{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^\varepsilon}) \mathrm{d}B(s). \end{split}$$

Employing Hölder's inequality and Itô's isometry and taking (3.8), (3.9) and (3.21) into account, we find that

$$(3.25) \qquad \mathbb{E}|\Upsilon(\varepsilon,\delta)| \leq \varepsilon \Big(\int_0^T \mathbb{E}\|\Lambda^*(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon},\theta,\theta_0)\widehat{\sigma}(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon})\sigma(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon},\mathcal{L}_{\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}})\|^2 ds\Big)^{1/2}$$

$$\leq c\varepsilon \Big(\int_0^T \Big\{1+\mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}\|_{\infty}^6 + \mathbb{W}_2(\mathcal{L}_{\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}},\delta_{\zeta_0})^6\Big\} ds\Big)^{1/2}$$

$$\leq c\varepsilon \Big(\int_0^T \Big\{1+\mathbb{E}\|\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}\|_{\infty}^6\Big\} ds\Big)^{1/2}$$

$$\leq c\varepsilon,$$

where we have applied (3.4) in the last step. Therefore, (3.24) is now available from (3.25).

To make the content self-contained, we cite [33, Theorem 5.9] as the following lemma.

Lemma 3.7. Let $(M_n)_{n\geq 1}$ be random functions and M a fixed function of θ such that, for any $\varepsilon > 0$,

$$\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \longrightarrow 0 \quad \text{in probability}$$

and $\sup_{|\theta-\theta_0|\geq\varepsilon} M(\theta) < M(\theta_0)$. Then, any sequence of estimators $\widehat{\theta}_n$ with $M_n(\widehat{\theta}_n) \geq M_n(\theta_0)$ converges in probability to θ_0 .

With Lemmas 3.5-3.7 in hand, we are in the position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. From (2.4), we infer that

$$\Phi_{n,\varepsilon}(\theta) = \delta^{-1} \sum_{k=1}^{n} \left\{ P_{k}^{*}(\theta) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_{k}(\theta) - P_{k}^{*}(\theta_{0}) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_{k}(\theta_{0}) \right\} \\
= \delta^{-1} \sum_{k=1}^{n} \left\{ \left(P_{k}(\theta_{0}) + \Lambda(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) \delta \right)^{*} \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) \left(P_{k}(\theta_{0}) + \Lambda(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) \delta \right) \right. \\
\left. - P_{k}^{*}(\theta_{0}) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_{k}(\theta_{0}) \right\} \\
= 2 \sum_{k=1}^{n} \Lambda^{*}(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_{k}(\theta_{0}) + \delta \sum_{k=1}^{n} \Lambda^{*}(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) \Lambda(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) \\
=: \Phi_{n,\varepsilon}^{(1)}(\theta) + \Phi_{n,\varepsilon}^{(2)}(\theta).$$

In terms of Lemmas 3.5 and 3.6, we deduce from Chebyshev's inequality that

$$\sup_{\theta \in \Theta} |-\Phi_{n,\varepsilon}(\theta) - (-\Xi(\theta))| \to 0 \quad \text{in probability,}$$

where $\Xi(\cdot)$ is defined as in (3.19). On the other hand, for any $\kappa > 0$, notice that

$$\sup_{|\theta - \theta_0| > \kappa} (-\Xi(\theta)) < -\Xi(\theta_0) = 0$$

due to $\Xi(\cdot) > 0$. Moreover, according to the notion of $\widehat{\theta}_{n,\varepsilon}$, one has $-\Phi_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon}) \ge -\Phi_{n,\varepsilon}(\theta_0) = 0$. As far as our present model is concerned, all of the assumptions in Lemma 3.7 with $M_n(\cdot) = -\Phi_{n,\varepsilon}(\cdot)$ and $M(\cdot) = -\Xi(\cdot)$ are fulfilled. As a consequence, we conclude that $\widehat{\theta}_{n,\varepsilon} \to \theta_0$ in probability as $\varepsilon \to 0$ and $n \to \infty$, as required.

4 The asymptotic distribution of LSE

In this section, to begin, we recall some materials on derivatives for matrix-valued functions and introduce some notation. For a differentiable mapping $V = (V_1, \dots, V_d)^* : \mathbb{R}^p \to \mathbb{R}^d$, its gradient operator $(\nabla_x V)(x) \in \mathbb{R}^d \otimes \mathbb{R}^p$ w.r.t. the argument $x = (x_1, \dots, x_p)^* \in \mathbb{R}^p$ is given by

$$(4.1) \qquad (\nabla_x V)(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} V_1(x) & \frac{\partial}{\partial x_2} V_1(x) & \cdots & \frac{\partial}{\partial x_p} V_1(x) \\ \frac{\partial}{\partial x_1} V_2(x) & \frac{\partial}{\partial x_2} V_2(x) & \cdots & \frac{\partial}{\partial x_p} V_2(x) \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x_1} V_d(x) & \frac{\partial}{\partial x_2} V_d(x) & \cdots & \frac{\partial}{\partial x_p} V_d(x) \end{pmatrix}.$$

If $V = (V_1, \dots, V_d) : \mathbb{R}^p \to (\mathbb{R}^d)^*$ (i.e., the *d*-dimensional raw vector) is differentiable, its gradient operator $(\nabla_x V)(x) \in \mathbb{R}^p \otimes \mathbb{R}^d$ w.r.t. the argument $x = (x_1, \dots, x_p)^* \in \mathbb{R}^p$ reads as follows

(4.2)
$$(\nabla_x V)(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} V_1(x) & \frac{\partial}{\partial x_1} V_2(x) & \cdots & \frac{\partial}{\partial x_1} V_d(x) \\ \frac{\partial}{\partial x_2} V_1(x) & \frac{\partial}{\partial x_2} V_2(x) & \cdots & \frac{\partial}{\partial x_2} V_d(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_p} V_1(x) & \frac{\partial}{\partial x_p} V_2(x) & \cdots & \frac{\partial}{\partial x_p} V_d(x) \end{pmatrix}.$$

So, from (4.1) and (4.2), one has $\nabla_x V^*(x) = (\nabla_x V)^*(x)$ for a differentiable function $V : \mathbb{R}^p \to \mathbb{R}^d$. Let $V = (V_{ij})_{p \times d} : \mathbb{R} \to \mathbb{R}^p \otimes \mathbb{R}^d$ be differentiable. Then, the derivative $\frac{\partial}{\partial x} V(x) \in \mathbb{R}^p \otimes \mathbb{R}^d$ of the matrix-valued mapping V w.r.t. the scalar argument $x \in \mathbb{R}$ enjoys the form

(4.3)
$$\frac{\partial}{\partial x}V(x) = \begin{pmatrix} \frac{\partial}{\partial x}V_{11}(x) & \frac{\partial}{\partial x}V_{12}(x) & \cdots & \frac{\partial}{\partial x}V_{1d}(x) \\ \frac{\partial}{\partial x}V_{21}(x) & \frac{\partial}{\partial x}V_{22}(x) & \cdots & \frac{\partial}{\partial x}V_{2d}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x}V_{p1}(x) & \frac{\partial}{\partial x}V_{p2}(x) & \cdots & \frac{\partial}{\partial x}V_{pd}(x) \end{pmatrix}.$$

For a differentiable function $V=(V_{ij})_{p\times d}:\mathbb{R}^p\to\mathbb{R}^p\otimes\mathbb{R}^d$, the gradient operator, denoted by $\nabla_x V(x)\in\mathbb{R}^p\otimes\mathbb{R}^{pd}$, of V(x) w.r.t. the variable $x=(x_1,\cdots,x_p)^*\in\mathbb{R}^p$ is formulated as

$$(\nabla_x V)(x) = \left(\frac{\partial}{\partial x_1} V(x), \frac{\partial}{\partial x_2} V(x), \cdots, \frac{\partial}{\partial x_n} V(x)\right),$$

where $\frac{\partial}{\partial x_i}V(x)$ is defined as in (4.3). Moreover, for a differentiable function $V=(V_{ij})_{p\times d}:\mathbb{R}^p\to\mathbb{R}^d$, we have

(4.4)
$$(\nabla_x^{(2)}V^*)(x) := (\nabla_x(\nabla_x V^*))(x) = (\nabla_x(\nabla_x V)^*)(x).$$

For $A = (A_1, A_2, \dots, A_p) \in \mathbb{R}^p \otimes \mathbb{R}^{pd}$ with $A_k \in \mathbb{R}^p \otimes \mathbb{R}^d$, $k = 1, \dots, p$, and $B \in \mathbb{R}^d$, let's define $A \circ B \in \mathbb{R}^p \otimes \mathbb{R}^p$ by

$$A \circ B = (A_1B, A_2B, \cdots, A_nB).$$

Set, for any $\theta \in \Theta$,

(4.5)
$$I(\theta) := \int_0^T (\nabla_\theta b)^* (X_s^0, \mathscr{L}_{X_s^0}, \theta) \widehat{\sigma}(X_s^0) (\nabla_\theta b) (X_s^0, \mathscr{L}_{X_s^0}, \theta) \mathrm{d}s,$$

and, for any random variable $\zeta \in \mathscr{C}$ with $\mathscr{L}_{\zeta} \in \mathcal{P}_2(\mathscr{C})$,

(4.6)
$$\Upsilon(\zeta, \theta_0) := (\nabla_{\theta} b)^*(\zeta, \mathcal{L}_{\zeta}, \theta_0) \widehat{\sigma}(\zeta) \sigma(\zeta, \mathcal{L}_{\zeta}).$$

Furthermore, we set

$$(4.7) K(\theta) := -2 \int_0^T \left\{ (\nabla_{\theta}^{(2)} b^*)(X_s^0, \mathcal{L}_{X_s^0}, \theta) \circ \left(\widehat{\sigma}(X_s^0) \Lambda(X_s^0, \theta, \theta_0) \right) \right\} \mathrm{d}s, \theta \in \Theta.$$

Another main result in this paper is presented as below, which reveals the asymptotic distribution of $\widehat{\theta}_{n,\varepsilon}$.

Theorem 4.1. Let the assumptions of Theorem 3.1 hold and suppose further that $(\mathbf{A2})$ and $(\mathbf{A3})$ hold and that $I(\cdot)$ and $K(\cdot)$ defined in (4.5) and (4.7), respectively, are continuous. Then,

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta_0)\to I^{-1}(\theta_0)\int_0^T \Upsilon(X_s^0,\theta_0)\mathrm{d}B(s)$$
 in probability

as $\varepsilon \to 0$ and $n \to \infty$, where $I(\cdot)$ and $\Upsilon(\cdot)$ are given in (4.5) and (4.6), respectively.

Now, we provide an example to demonstrate our main results.

Example 4.2. Let $\theta = (\theta^{(1)}, \theta^{(2)})^* \in \Theta_0 := (c_1, c_2) \times (c_3, c_4) \subset \mathbb{R}^2$ for some $c_1 < c_2$ and $c_3 < c_4$. For any $\varepsilon \in (0, 1)$, consider the following scalar path-distribution dependent SDE

(4.8)
$$dX^{\varepsilon}(t) = \theta^{(1)} + \theta^{(2)} \int_{\mathscr{C}} b_0(X_t^{\varepsilon}, \zeta) \mathscr{L}_{X_t^{\varepsilon}}(d\zeta) + \varepsilon (1 + |X^{\varepsilon}(t)|) dB(t), \quad t \in (0, T]$$

with the initial value $X_0^{\varepsilon} = \xi$, where $\theta \in \Theta_0$ is an unknown parameter with the true value $\theta_0 = (\theta_0^{(1)}, \theta_0^{(2)}) \in \Theta_0$, and $b_0 : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ satisfy the global Lipschitz condition, i.e., there exists a constant K > 0 such that

$$(4.9) |b_0(\zeta_1, \zeta_2) - b(\zeta_1', \zeta_2')| \le K\{|\zeta_1 - \zeta_1'| + |\zeta_2 - \zeta_2'|\}, \zeta_1, \zeta_2, \zeta_1', \zeta_2' \in \mathscr{C}.$$

For any $\zeta \in \mathcal{C}$, $\mu \in \mathcal{P}_2(\mathcal{C})$ and $\theta = (\theta^{(1)}, \theta^{(2)})^*$, set

$$b(\zeta,\mu,\theta) := \theta^{(1)} + \theta^{(2)} \int_{\mathscr{C}} b_0(\zeta,\zeta') \mu(\mathrm{d}\zeta') \quad \text{and} \quad \sigma(\zeta,\mu) := 1 + |\zeta(0)|.$$

Then, (4.8) can be reformulated as (2.1). By a direct calculation, it follows from (4.9) that, for any $\mu, \nu \in \mathcal{P}_2(\mathscr{C})$ and $\zeta_1, \zeta_2 \in \mathscr{C}$,

$$|b(\zeta_{1}, \mu, \theta) - b(\zeta_{2}, \nu, \theta)| = |\theta^{(2)}| \cdot \left| \int_{\mathscr{C}} b_{0}(\zeta_{1}, \zeta) \mu(d\zeta) - \int_{\mathscr{C}} b_{0}(\zeta_{2}, \zeta') \nu(d\zeta') \right|$$

$$\leq |\theta^{(2)}| \int_{\mathscr{C}} \int_{\mathscr{C}} |b_{0}(\zeta_{1}, \zeta) - b_{0}(\zeta_{2}, \zeta')| \pi(d\zeta, d\zeta')$$

$$\leq K|\theta^{(2)}| \int_{\mathscr{C}} \int_{\mathscr{C}} \{|\zeta_{1} - \zeta_{2}| + |\zeta - \zeta'|\} \pi(d\zeta, d\zeta')$$

$$\leq K(|c_{3}| \vee |c_{4}|) \{|\zeta_{1} - \zeta_{2}| + \mathbb{W}_{1}(\mu, \nu)\}$$

$$\leq K(|c_{3}| \vee |c_{4}|) \{|\zeta_{1} - \zeta_{2}| + \mathbb{W}_{2}(\mu, \nu)\},$$

in which $\pi \in \mathcal{C}(\mu, \nu)$. On the other hand, for any $x, y \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, one has

$$|\sigma(x,\mu) - \sigma(y,\nu)| \le |x-y|.$$

Hence, the assumption (A1) holds for (4.8). Next, for any $x, y \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, we have

$$|\sigma^{-2}(x,\mu) - \sigma^{-2}(y,\nu)| = \left|\frac{1}{(1+|x|)^2} - \frac{1}{(1+|y|)^2}\right| \le 4|x-y|.$$

So, (A2) is fulfilled. Furthermore, observe that

(4.11)
$$(\nabla_{\theta}b)(\zeta,\mu,\theta) = \left(1, \int_{\mathscr{C}} b_0(\zeta,\zeta')\mu(\mathrm{d}\zeta')\right)^* \quad \text{and} \quad (\nabla_{\theta}(\nabla_{\theta}b))(\zeta,\mu,\theta) = \mathbf{0}_{2\times 2},$$

where $\mathbf{0}_{2\times 2}$ stands for the 2 × 2-zero matrix. Thus, (4.10) yields that both (**B1**) and (**B2**) hold. We further assume that the initial value is global Lipschitz, i.e., there exists an L > 0 such that

$$|\xi(t) - \xi(s)| \le L|t - s|, \quad t, s \in [-r_0, 0].$$

As a consequence, concerning (4.8), the assumptions (A1)-(A3) and (B1)-(B2) hold, respectively.

The discrete-time EM scheme associated with (4.8) is given by

$$(4.12) Y^{\varepsilon}(t_k) = Y^{\varepsilon}(t_{k-1}) + \left(\theta^{(1)} + \theta^{(2)} \int_{\mathscr{L}} b_0(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \zeta) \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}(\mathrm{d}\zeta)\right) \delta + \varepsilon (1 + |Y^{\varepsilon}(t_{k-1})|) \triangle B_k, \quad k \ge 1,$$

with $Y^{\varepsilon}(t) = X^{\varepsilon}(t) = \xi(t), t \in [-r_0, 0]$, where $(\widehat{Y}_{t_k}^{\varepsilon})$ is defined as in (2.3). According to (2.4), the contrast function admits the form below

$$\Psi_{n,\varepsilon}(\theta) = \varepsilon^{-2} \delta^{-1} \sum_{k=1}^{n} \frac{1}{(1+|Y^{\varepsilon}(t_{k-1})|)^2} \Big| Y^{\varepsilon}(t_k) - Y^{\varepsilon}(t_{k-1}) - \Big(\theta^{(1)} + \theta^{(2)} \int_{\mathscr{C}} b_0(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \zeta) \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}(\mathrm{d}\zeta) \Big) \delta \Big|^2.$$

Observe that

$$\begin{split} \frac{\partial}{\partial \theta^{(1)}} \Psi_{n,\varepsilon}(\theta) &= -2 \, \varepsilon^{-2} \sum_{k=1}^{n} \frac{1}{(1+|Y^{\varepsilon}(t_{k-1})|)^{2}} \Big\{ Y^{\varepsilon}(t_{k}) - Y^{\varepsilon}(t_{k-1}) \\ &- \Big(\theta^{(1)} + \theta^{(2)} \int_{\mathscr{C}} b_{0}(\widehat{Y}^{\varepsilon}_{t_{k-1}}, \zeta) \mathscr{L}_{\widehat{Y}^{\varepsilon}_{t_{k-1}}}(\mathrm{d}\zeta) \Big) \delta \Big\}, \end{split}$$

and

$$\begin{split} \frac{\partial}{\partial \theta^{(2)}} \Psi_{n,\varepsilon}(\theta) &= -2 \,\varepsilon^{-2} \sum_{k=1}^{n} \frac{1}{(1+|Y^{\varepsilon}(t_{k-1})|)^{2}} \Big\{ Y^{\varepsilon}(t_{k}) - Y^{\varepsilon}(t_{k-1}) \\ &- \Big(\theta^{(1)} + \theta^{(2)} \int_{\mathscr{C}} b_{0}(\widehat{Y}^{\varepsilon}_{t_{k-1}}, \zeta) \mathscr{L}_{\widehat{Y}^{\varepsilon}_{t_{k-1}}}(\mathrm{d}\zeta) \Big) \delta \Big\} \int_{\mathscr{C}} b_{0}(\widehat{Y}^{\varepsilon}_{t_{k-1}}, \zeta) \mathscr{L}_{\widehat{Y}^{\varepsilon}_{t_{k-1}}}(\mathrm{d}\zeta). \end{split}$$

Subsequently, solving the equation below

$$\frac{\partial}{\partial \theta^{(1)}} \Psi_{n,\varepsilon}(\theta) = \frac{\partial}{\partial \theta^{(2)}} \Psi_{n,\varepsilon}(\theta) = 0,$$

we obtain the LSE $\widehat{\theta}_{n,\varepsilon} = (\widehat{\theta}_{n,\varepsilon}^{(1)}, \widehat{\theta}_{n,\varepsilon}^{(2)})^*$ of the unknown parameter $\theta = (\theta^{(1)}, \theta^{(2)})^* \in \Theta_0$ possesses the formula

$$\widehat{\theta}_{n,\varepsilon}^{(1)} = \frac{A_2 A_5 - A_3 A_4}{\delta(A_1 A_5 - A_4^2)}$$
 and $\widehat{\theta}_{n,\varepsilon}^{(2)} = \frac{A_1 A_3 - A_2 A_4}{\delta(A_1 A_5 - A_4^2)}$,

where

$$A_1 := \sum_{k=1}^{n} \frac{1}{(1 + |Y^{\varepsilon}(t_{k-1})|)^2}, \qquad A_2 := \sum_{k=1}^{n} \frac{Y^{\varepsilon}(t_k) - Y^{\varepsilon}(t_{k-1})}{(1 + |Y^{\varepsilon}(t_{k-1})|)^2},$$

$$A_3 := \sum_{k=1}^n \frac{(Y^{\varepsilon}(t_k) - Y^{\varepsilon}(t_{k-1})) \int_{\mathscr{C}} b_0(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \zeta) \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}(\mathrm{d}\zeta)}{(1 + |Y^{\varepsilon}(t_{k-1})|)^2}, \quad A_4 := \sum_{k=1}^n \frac{\int_{\mathscr{C}} b_0(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \zeta) \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}(\mathrm{d}\zeta)}{(1 + |Y^{\varepsilon}(t_{k-1})|)^2},$$

and

$$A_5 := \sum_{k=1}^n \frac{\left(\int_{\mathscr{C}} b_0(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \zeta) \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}(\mathrm{d}\zeta)\right)^2}{(1 + |Y^{\varepsilon}(t_{k-1})|)^2}.$$

In terms of Theorem 3.1, $\widehat{\theta}_{n,\varepsilon} \to \theta$ in probability as $\varepsilon \to 0$ and $n \to \infty$. Next, from (4.11), it follows that

$$I(\theta_0) = \int_0^T \frac{1}{(1+|X_s^0|)^2} \begin{pmatrix} 1 & b_0(X_s^0, X_s^0) \\ b_0(X_s^0, X_s^0) & b_0(X_s^0, X_s^0)^2 \end{pmatrix} ds,$$

and, for $\zeta \in \mathscr{C}$,

$$\int_0^T \Upsilon(X_s^0, \theta_0) dB(s) = \int_0^T \frac{1}{1 + |X^0(s)|} \begin{pmatrix} 1 \\ b_0(X_s^0, X_s^0) \end{pmatrix} dB(s).$$

At last, according to Theorem 4.1, we conclude that

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta_0)\to I^{-1}(\theta_0)\int_0^T \Upsilon(X_s^0,\theta_0)\mathrm{d}B(s)$$
 in probability

as $\varepsilon \to 0$ and $n \to \infty$ provided that $I(\cdot)$ is positive definite.

Before we proceed to complete the proof of Theorem 4.1, let's prepare the lemmas below.

Lemma 4.3. Assume that (A1)- (A3) and (B1)- (B2) hold. Then,

(4.13)
$$\int_0^T \Upsilon(\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}, \theta_0) dB(t) \longrightarrow \int_0^T \Upsilon(X_t^0, \theta_0) dB(t) \quad \text{in probability}$$

as $\varepsilon \to 0$ and $\delta \to 0$. Moreover,

(4.14)
$$\varepsilon^{-1}(\nabla_{\theta}\Phi_{n,\varepsilon})(\theta_0) \to -2\int_0^T \Upsilon(X_s^0,\theta_0) dB(s) \quad \text{in probability}$$

whenever $\varepsilon \to 0$ and $\delta \to 0$.

Proof. We first claim that

(4.15)
$$\int_0^T \|\Upsilon(\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}, \theta_0) - \Upsilon(X_t^0, \theta_0)\|^2 dt \to 0 \quad \text{in probability}$$

as $\varepsilon \to 0$ and $\delta \to 0$. For any $\kappa > 0$ and $\rho > 0$, by the aid of (4.15) and by making use of [5, Theorem 2.6, P.63], we have

$$\mathbb{P}\Big(\Big|\int_0^T (\Upsilon(\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}, \theta_0) - \Upsilon(X_t^0, \theta_0)) dB(t)\Big| \ge \kappa\Big)$$

$$\le \mathbb{P}\Big(\int_0^T \|\Upsilon(\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}, \theta_0) - \Upsilon(X_t^0, \theta_0)\|^2 dt \ge \kappa^2 \rho\Big) + \rho.$$

Thus, (4.13) follows from (4.15) and the arbitrariness of ρ . So, in what follows, it remains to show that (4.15) holds true. Observe that

$$\begin{split} &\Upsilon(\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}, \theta_{0}) - \Upsilon(X^{0}_{t}, \theta_{0}) \\ &= \{ (\nabla_{\theta}b)^{*} (\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}, \mathscr{L}_{\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}}, \theta_{0}) - (\nabla_{\theta}b)^{*} (X^{0}_{t}, \mathscr{L}_{X^{0}_{t}}, \theta_{0}) \} \widehat{\sigma}(\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}) \sigma(\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}, \mathscr{L}_{\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}}) \\ &+ (\nabla_{\theta}b)^{*} (X^{0}_{t}, \mathscr{L}_{X^{0}_{t}}, \theta_{0}) \{ \widehat{\sigma}(\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}) - \widehat{\sigma}(X^{0}_{t}) \} \sigma(\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}, \mathscr{L}_{\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}}) \\ &+ (\nabla_{\theta}b)^{*} (X^{0}_{t}, \mathscr{L}_{X^{0}_{t}}, \theta_{0}) \widehat{\sigma}(X^{0}_{t}) \{ \sigma(\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}, \mathscr{L}_{\widehat{Y}^{\varepsilon}_{\lfloor t/\delta \rfloor \delta}}) - \sigma(X^{0}_{t}, \mathscr{L}_{X^{0}_{t}}) \\ &+ (\Sigma_{1}(t, \varepsilon, \delta) + \Sigma_{2}(t, \varepsilon, \delta) + \Sigma_{3}(t, \varepsilon, \delta). \end{split}$$

From (B1), (3.9), and (3.21), it follows that

$$\int_{0}^{T} (\|\Sigma_{1}(t,\varepsilon,\delta)\|^{2} + \|\Sigma_{2}(t,\varepsilon,\delta)\|^{2}) dt$$

$$\leq c \int_{0}^{T} (1 + \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^{4}) \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{2} dt + \widehat{\Pi}(\varepsilon,\delta)$$

$$\leq c \int_{0}^{T} (1 + \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{4} + \|X_{t}^{0}\|_{\infty}^{4}) \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{2} ds + \widehat{\Pi}(\varepsilon,\delta)$$

$$\leq c \int_{0}^{T} (1 + \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{4}) \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{2} dt + \widehat{\Pi}(\varepsilon,\delta),$$

where

$$\widehat{\Pi}(\varepsilon,\delta) := c \int_0^T (1 + \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^4) \mathbb{E} \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_t^0\|_{\infty}^2 dt.$$

For any $\rho > 0$, one gets from (4.16) that

$$\mathbb{P}\Big(\int_{0}^{T} (\|\Sigma_{1}(t,\varepsilon,\delta)\|^{2} + \|\Sigma_{2}(t,\varepsilon,\delta)\|^{2}) dt \ge \rho\Big) \\
\leq \mathbb{P}(\widehat{\Pi}(\varepsilon,\delta) \ge \rho/2) + \mathbb{P}\Big(c\int_{0}^{T} (1 + \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{4}) \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_{t}^{0}\|_{\infty}^{2} dt \ge \frac{\rho}{2}\Big).$$

By the Chebyshev inequality, in addition to (3.4) and (3.11),

$$\mathbb{P}(\widehat{\Pi}(\varepsilon,\delta) \ge \rho/2) \le \frac{c}{\rho} \int_0^T (1 + \mathbb{E} \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon}\|_{\infty}^4) \mathbb{E} \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_t^0\|_{\infty}^2 ds$$

$$\longrightarrow 0$$

as $\varepsilon \to 0$ and $\delta \to 0$. Also, for any K > 0, by Chebyshev's inequality, besides (3.4),

$$\begin{split} & \mathbb{P} \Big(c \int_0^T (1 + \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty^4) \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty^2 \mathrm{d}t \geq \frac{\rho}{2} \Big) \\ & \leq \mathbb{P} \Big(c (1 + K^4) \int_0^T \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty^2 \mathrm{d}t \geq \frac{\rho}{4} \Big) \\ & + \mathbb{P} \Big(c \int_0^T (1 + \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon \|_\infty^8) \mathbf{1}_{\{\| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty \geq K\}} \mathrm{d}t \geq \frac{\rho}{4} \Big) \\ & \leq \frac{c (1 + K^4)}{\rho} \int_0^T \mathbb{E} \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty^2 \mathrm{d}t \\ & + \frac{c}{\rho} \int_0^T \mathbb{E} ((1 + \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon \|_\infty^8) \mathbf{1}_{\{\| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty \geq K\}}) \mathrm{d}t \\ & \leq \frac{c (1 + K^4)}{\rho} \int_0^T \mathbb{E} \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty^2 \mathrm{d}t \\ & + \frac{c}{\rho} \int_0^T \Big(1 + \mathbb{E} \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon \|_\infty^{16} \Big)^{1/2} \Big(\mathbb{P} (\| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty \geq K) \Big)^{1/2} \mathrm{d}t \\ & \leq \frac{c (1 + K^4)}{\rho} \int_0^T \mathbb{E} \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty^2 \mathrm{d}t + \frac{c}{\rho K} \int_0^T \Big(\mathbb{E} \| \widehat{Y}_{\lfloor t/\delta \rfloor \delta}^\varepsilon - X_t^0 \|_\infty^2 \Big)^{1/2} \mathrm{d}t. \end{split}$$

This, together with (3.11), leads to

(4.17)
$$\int_0^T (\|\Sigma_1(t,\varepsilon,\delta)\|^2 + \|\Sigma_2(t,\varepsilon,\delta)\|^2) dt \longrightarrow 0 \quad \text{in probability}$$

as $\varepsilon \to 0$ and $\delta \to 0$. Furthermore, (A1), (3.21) as well as (B1) imply that

(4.18)
$$\int_0^T \mathbb{E} \|\Sigma_3(t, \varepsilon, \delta)\|^2 dt \le c \int_0^T \mathbb{E} \|\widehat{Y}_{\lfloor t/\delta \rfloor \delta}^{\varepsilon} - X_t^0\|_{\infty}^2 dt \longrightarrow 0$$

as $\varepsilon \to 0$ and $\delta \to 0$. As a result, (4.15) follows from (4.17), (4.18) and Chebyshev's inequality. For any $\theta \in \Theta$ and random variable $\zeta \in \mathscr{C}$ with $\mathcal{P}_2(\mathscr{C})$, note from (3.2) that

$$(\nabla_{\theta}\Lambda)(\zeta,\theta,\theta_0) = -(\nabla_{\theta}b)(\zeta,\mathcal{L}_{\zeta},\theta).$$

A straightforward calculation shows that

$$(\nabla_{\theta} \Phi_{n,\varepsilon})(\theta) = 2 \sum_{k=1}^{n} (\nabla_{\theta} \Lambda)^* (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_0) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) \Big\{ P_k(\theta_0) + \delta \Lambda(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_0) \Big\}$$
$$= -2 \sum_{k=1}^{n} (\nabla_{\theta} b)^* (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_k(\theta).$$

Therefore, one has

$$\varepsilon^{-1}(\nabla_{\theta}\Phi_{n,\varepsilon})(\theta_0) = -2\int_0^T \Upsilon(\widehat{Y}_{\lfloor s/\delta\rfloor\delta}^{\varepsilon}, \theta_0) dB(s).$$

Subsequently, (4.14) follows from (4.13) immediately.

Lemma 4.4. Under the assumptions of Theorem 4.1,

(4.19)
$$(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta) \longrightarrow K_0(\theta) := K(\theta) + 2I(\theta)$$
 in probability

as $\varepsilon \to 0, n \to \infty$, where $(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon}), I(\theta), K(\theta)$ are defined as in (4.4), (4.5), and (4.7), respectively. *Proof.* By the chain rule, we infer from (4.4) that

$$\begin{split} (\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta) &= -2 \sum_{k=1}^{n} (\nabla_{\theta}^{(2)} b^{*}) (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta) \circ \left(\widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_{k}(\theta) \right) \\ &- 2 \sum_{k=1}^{n} (\nabla_{\theta} b)^{*} (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) (\nabla_{\theta} P_{k})(\theta) \\ &= -2 \sum_{k=1}^{n} (\nabla_{\theta}^{(2)} b^{*}) (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta) \circ \left(\widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) P_{k}(\theta_{0}) \right) \\ &- 2 \delta \sum_{k=1}^{n} \left\{ (\nabla_{\theta}^{(2)} b^{*}) (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta) \circ \left(\widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) \Lambda(\widehat{Y}_{t_{k-1}}^{\varepsilon}, \theta, \theta_{0}) \right) \\ &- (\nabla_{\theta} b)^{*} (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta) \widehat{\sigma}(\widehat{Y}_{t_{k-1}}^{\varepsilon}) (\nabla_{\theta} b) (\widehat{Y}_{t_{k-1}}^{\varepsilon}, \mathscr{L}_{\widehat{Y}_{t_{k-1}}^{\varepsilon}}, \theta) \right\} \\ &=: \Theta_{1}(\varepsilon, \delta) + \Theta_{2}(\varepsilon, \delta). \end{split}$$

Taking (B2) into consideration and mimicking the argument of Lemma 3.6, we obtain that

$$\Theta_1(\varepsilon, \delta) \to 0$$
 in probability as $\varepsilon \to 0$, $\delta \to 0$.

Observe that

$$\Theta_{2}(\varepsilon,\delta) = -2 \int_{0}^{T} \left\{ (\nabla_{\theta}^{(2)} b^{*}) (\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}}, \theta) \circ \left(\widehat{\sigma} (\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}) \Lambda (\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \theta, \theta_{0}) \right) ds \right. \\
\left. + 2 \int_{0}^{T} \left\{ (\nabla_{\theta} b)^{*} (\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}}, \theta) \widehat{\sigma} (\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}) (\nabla_{\theta} b) (\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}, \mathcal{L}_{\widehat{Y}_{\lfloor s/\delta \rfloor \delta}^{\varepsilon}}, \theta) ds \right. \\
=: \Psi_{1}(\varepsilon, \delta) + \Psi_{2}(\varepsilon, \delta).$$

Carrying out an analogous argument to derive Lemma 3.5, we infer that

(4.20)
$$\Psi_1(\varepsilon,\delta) \to K(\theta)$$
 in probability as $\varepsilon \to 0, \delta \to 0$

by taking (B2) into account, and that

(4.21)
$$\Psi_2(\varepsilon, \delta) \to 2I(\theta)$$
 in probability as $\varepsilon \to 0, \delta \to 0$

by using (B1). Thus, the desired assertion follows from (4.20) and (4.21) immediately.

Now we start to finish the argument of Theorem 4.1 on the basis of the previous lemmas.

Proof of Theorem 4.1. The original idea on the proof of Theorem 4.1 is taken from [31]. To make the content self-contained, we herein provide a sketch of the proof. In terms of Theorem 3.1, there exists a sequence $\eta_{n,\varepsilon} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$ such that $\widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0) \subset \Theta$, \mathbb{P} -a.s. By the Taylor expansion, one has

$$(4.22) \qquad (\nabla_{\theta} \Phi_{n,\varepsilon})(\widehat{\theta}_{n,\varepsilon}) = (\nabla_{\theta} \Phi_{n,\varepsilon})(\theta_0) + D_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon} - \theta_0), \quad \widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0)$$

with

$$D_{n,\varepsilon} := \int_0^1 (\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon}) (\theta_0 + u(\widehat{\theta}_{n,\varepsilon} - \theta_0)) du, \quad \widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0).$$

Observe that, for $\widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0)$,

$$\begin{split} \|D_{n,\varepsilon} - K_{0}(\theta_{0})\| &\leq \|D_{n,\varepsilon} - (\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0})\| + \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0}) - K_{0}(\theta_{0})\| \\ &\leq \int_{0}^{1} \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0} + u(\widehat{\theta}_{n,\varepsilon} - \theta_{0})) - (\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0})\| du \\ &+ \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0}) - K_{0}(\theta_{0})\| \\ &\leq \sup_{\theta \in B_{\eta n,\varepsilon}(\theta_{0})} \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta) - (\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0})\| + \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0}) - K_{0}(\theta_{0})\| \\ &\leq \sup_{\theta \in B_{\eta n,\varepsilon}(\theta_{0})} \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta) - K_{0}(\theta)\| + \sup_{\theta \in B_{\eta n,\varepsilon}(\theta_{0})} \|K_{0}(\theta) - K_{0}(\theta_{0})\| \\ &+ 2\|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta_{0}) - K_{0}(\theta_{0})\|, \end{split}$$

in which $K_0(\cdot)$ is introduced in (4.19). This, together with Lemma 4.4 and continuity of $K_0(\cdot)$, gives that

$$(4.23) D_{n,\varepsilon} \to K_0(\theta_0) \text{in probability}$$

as $\varepsilon \to 0$ and $n \to \infty$. By following the exact line of [21, Theorem 2.2], we can deduce that $D_{n,\varepsilon}$ is invertible on the set

$$\Gamma_{n,\varepsilon} := \left\{ \sup_{\theta \in B_{n_n,\varepsilon}(\theta_0)} \| (\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta) - K_0(\theta_0) \| \le \frac{\alpha}{2}, \ \widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0) \right\}$$

for some constant $\alpha > 0$. Let

$$\mathcal{D}_{n,\varepsilon} = \{D_{n,\varepsilon} \text{ is invertible }, \widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0)\}.$$

By virtue of Lemma 4.4, one has

(4.24)
$$\lim_{\varepsilon \to 0, n \to \infty} \mathbb{P}\left(\sup_{\theta \in B_{n_n,\varepsilon}(\theta_0)} \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta) - K_0(\theta_0)\| \le \frac{\alpha}{2}\right) = 1.$$

On the other hand, recall that

(4.25)
$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0)\right) = 1.$$

By the fundamental fact: for any events $A, B, \mathbb{P}(AB) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$, we observe that

$$(4.26) 1 \ge \mathbb{P}(\Gamma_{n,\varepsilon}) \ge \mathbb{P}\left(\sup_{\theta \in B_{\eta_{n,\varepsilon}}(\theta_0)} \|(\nabla_{\theta}^{(2)} \Phi_{n,\varepsilon})(\theta) - K_0(\theta_0)\| \le \frac{\alpha}{2}\right) \\ + \mathbb{P}\left(\widehat{\theta}_{n,\varepsilon} \in B_{\eta_{n,\varepsilon}}(\theta_0)\right) - 1.$$

Thus, taking advantage of (4.24), (4.25) as well as (4.26), we deduce from Sandwich theorem that

$$(4.27) \mathbb{P}(\mathcal{D}_{n,\varepsilon}) \ge \mathbb{P}(\Gamma_{n,\varepsilon}) \to 1$$

as $\varepsilon \to 0$ and $n \to \infty$. Set

$$U_{n,\varepsilon} := D_{n,\varepsilon} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}} + I_{p \times p} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}^c}$$

where $I_{p\times p}$ is a $p\times p$ identity matrix. For $S_{n,\varepsilon}:=\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta_0)$, we deduce from (4.22) that

$$S_{n,\varepsilon} = S_{n,\varepsilon} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}} + S_{n,\varepsilon} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}^{c}}$$

$$= U_{n,\varepsilon}^{-1} D_{n,\varepsilon} S_{n,\varepsilon} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}} + S_{n,\varepsilon} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}^{c}}$$

$$= \varepsilon^{-1} U_{n,\varepsilon}^{-1} \{ (\nabla_{\theta} \Phi_{n,\varepsilon}) (\widehat{\theta}_{n,\varepsilon}) - (\nabla_{\theta} \Phi_{n,\varepsilon}) (\theta_{0}) \} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}} + S_{n,\varepsilon} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}^{c}}$$

$$= -\varepsilon^{-1} U_{n,\varepsilon}^{-1} (\nabla_{\theta} \Phi_{n,\varepsilon}) (\theta_{0}) \mathbf{1}_{\mathscr{D}_{n,\varepsilon}} + S_{n,\varepsilon} \mathbf{1}_{\mathscr{D}_{n,\varepsilon}^{c}}$$

$$\to I^{-1}(\theta_{0}) \int_{0}^{T} \Upsilon(X_{s}^{0}, \theta_{0}) dB(s),$$

as $\varepsilon \to 0$ and $n \to \infty$, where in the forth identity we dropped the term $(\nabla_{\theta} \Phi_{n,\varepsilon})(\widehat{\theta}_{n,\varepsilon})$ according to the notion of LSE and Fermat's lemma, and the last display follows from Lemma 4.3, (4.23) as well as (4.27) and by noting $K_0(\theta_0) = 2I(\theta_0)$. We therefore complete the proof.

References

- [1] Bishwal, J. P. N., Parameter Estimation in Stochastic Differential Equations, Springer, Berlin, 2008.
- [2] Chen, M.-F., From Markov Chains to Non-Equilibrium Particle Systems., Second Ed., World Scientific Publishing Co., Inc., River Edge, NJ, 2004.
- [3] Dorogovcev, A. Ja., The consistency of an estimate of a parameter of a stochastic differential equation, *Theory Probab. Math. Statist.*, **10** (1976), 73–82.
- [4] Dos Reis, G., Salkeld, W., Tugaut, J., Freidlin-Wentzell LDPs in path space for McKean-Vlasov equations and the Functional Iterated Logarithm Law, arXiv:1708.04961.
- [5] Friedman, A., Stochastic Differential Equations and Applications, Vol 1, Academic Press, INC, 1975.
- [6] Gloter, A., Sørensen, M., Estimation for stochastic differential equations with a small diffusion coefficient, Stochastic Process. Appl., 119 (2009), 679–699.
- [7] Hu, Y., Long, H., Least squares estimator for Ornstein-Uhlenbeck processes driven by α-stable motions, Stochastic Process. Appl., 119 (2009), 2465–2480.
- [8] Huang, X., Liu, C., Wang, F.-Y., Order preservation for path-distribution dependent SDEs, arXiv:1710.08569.
- [9] Huang, X., Röckner, M., Wang, F.-Y., Nonlinear Fokker-Planck equations for probability measures on path space and path-distribution dependent SDEs, arXiv:1709.00556.
- [10] Ikeda, N., Watanabe, S. Stochastic Differential Equations and Diffusion Processes, Second Ed., North-Holland Mathemntilcal Library, 24, Amsterdam, 1989.
- [11] Jourdain, B., Méléard, S., Propagation of chaos and fluctuations for a moderate model with smooth initial data, Ann. Inst. H. Poincaré Probab. Statist., 34 (1998), 727–766.
- [12] Kasonga, R. A., The consistency of a nonlinear least squares estimator for diffusion processes, *Stochastic Process. Appl.*, **30** (1988), 263–275.
- [13] Klebaner, F. C., Introduction to Stochastic Calculus with Applications, Third Ed., Imperial College Press, London, 2012.
- [14] Kunitomo, N., Takahashi, A., The asymptotic expansion approach to the valuation of interest rate contingent claims, Math. Finance, 11 (2001), 117–151.
- [15] Kutoyants, Yu. A., Statistical Inference for Ergodic Diffusion Processes, Springer-Verlag, London, Berlin, Heidelberg, 2004.
- [16] Li, J., Min, H., Weak solutions of mean-field stochastic differential equations, Stoch. Anal. Appl., 35 (2017), 542–568.
- [17] Li, J., Min, H., Weak solutions of mean-field stochastic differential equations and application to zerosum stochastic differential games, SIAM J. Control Optim., 54 (2016), 1826–1858.
- [18] Li, J., Wu, J.-L., On drift parameter estimation for mean-reversion type stochastic differential equations with discrete observations, Adv. Difference Equ., 2016, Paper No. 90, 23 pp. (https://doi.org/10.1186/s13662-016-0819-1)
- [19] Liptser, R. S., Shiryaev, A. N., Statistics of Random Processes: II Applications, Second Edition, Springer-Verlag, Berlin, Heidelberg, New York, 2001.
- [20] Long, H., Ma, C., Shimizu, Y., Least squares estimators for stochastic differential equations driven by small Lévy noises, Stochastic Process. Appl., 127 (2017), 1475–1495.
- [21] Long, H., Shimizu, Y., Sun, W., Least squares estimators for discretely observed stochastic processes driven by small Lévy noises, J. Multivariate Anal., 116 (2013), 422–439.
- [22] Mao, X., Stochastic Differential Equations and Applications, Second Ed., Horwood Publishing Limited, Chichester, 2008.
- [23] Masuda, H., Simple estimators for non-linear Markovian trend from sampled data: I. ergodic cases. MHF Preprint Series 2005-7, Kyushu University, 2005.
- [24] Mishura, Yu. S., Veretennikov, A. Yu., Existence and uniqueness theorems for solutions of McKean– Vlasov stochastic equations, arXiv:1603.02212v4.
- [25] Øksendal, B., Stochastic Differential Equations. An Introduction with Applications, Sixth Ed., Springer-Verlag, Berlin, Heidelberg, New York, 2003.

- [26] Prakasa Rao, B. L. S., Statistical Inference for Diffusion Type Processes, Kendall's Library of Statistics, 8. Edward Arnold, London, Oxford University Press, New York, 1999.
- [27] Protter, P., Stochastic Integrations and Differential Equations, Second edition. Applications of Mathematics (New York), 21. Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin, 2004.
- [28] Shimizu, Y., Yoshida, N., Estimation of parameters for diffusion processes with jumps from discrete observations, Stat. Inference Stoch. Process., 9 (2006), 227–277.
- [29] Sørensen, M., Uchida, M., Small diffusion asymptotics for discretely sampled stochastic differential equations, *Bernoulli*, 9 (2003), 1051–1069.
- [30] Takahashi, A., Yoshida, N., An asymptotic expansion scheme for optimal investment problems, Stat. Inference Stoch. Process., 7 (2004), 153–188.
- [31] Uchida, M., Estimation for discretely observed small diffusions based on approximate martingale estimating functions, *Scand. J. Statist.*, **31** (2004), 553–566.
- [32] Uchida, M., Approximate martingale estimating functions for stochastic differential equations with small noises, *Stochastic Process. Appl.*, **118** (2008), 1706–1721.
- [33] van der Vaart, A. W., Asymptotic Statistics, Cambridge Series in Statistical and Probabilistic Mathematics, 3, Cambridge University Press, Cambridge, 1998.
- [34] Veretennikov, A. Yu., On ergodic measures for McKean-Vlasov stochastic equations, In Monte Carlo and Quasi-Monte Carlo Methods 2004, 471–486, Springer, Berlin, 2006.
- [35] Wang, F.-Y., Distribution-Dependent SDEs for Landau Type Equations, arXiv:1606.05843.
- [36] Wen, J., Wang, X., Mao, S., Xiao, X., Maximum likelihood estimation of McKean-Vlasov stochastic differential equation and its application, Appl. Math. Comput., 274 (2016), 237–246.
- [37] Yoshida, N., Asymptotic expansion for statistics related to small diffusions, J. Japan Statist. Soc., 22 (1992), 139–159.