Express the number of spanning trees in term of degrees^{*}

Fengming Dong[†]

National Institute of Education, Nanyang Technological University, Singapore Jun Ge[‡]

School of Mathematical Sciences, Sichuan Normal University, Chengdu, P. R. China Zhangdong Ouyang[§]

Department of Mathematics, Hunan First Normal University, Changsha, P.R. China

Abstract

It is well-known that the number of spanning trees, denoted by $\tau(G)$, in a connected multi-graph G can be calculated by the Matrix-Tree Theorem and Tutte's deletioncontraction formula. In this short note, we find an alternate method to compute $\tau(G)$ by degrees of vertices.

Keywords: spanning tree; degree; graph polynomial

Mathematics Subject Classification (2010): 05C30, 05C05

1 Introduction

In this article, we consider loopless and undirected multi-graphs. For a graph G, let V(G), E(G) and $\mathcal{T}(G)$ be the set of vertices, the set of edges and the set of spanning trees in G respectively, and let $\tau(G) = |\mathcal{T}(G)|$. For any $u \in V(G)$, let $E_G(u)$ (or simply E(u)) denote the set of edges in G that are incident with u, and let $d_G(u)$ (or simply d(u)) be the degree of u in G, i.e., $d_G(u) = |E_G(u)|$. For any $S \subseteq V(G)$, if $S \neq \emptyset$, let G[S] be the subgraph of G induced by S, and if $S \neq V$, let $G - S = G[V \setminus S]$.

The study of spanning trees plays an important role in graph theory. The number of spanning trees $\tau(G)$ is a key parameter in Tutte polynomials, and it has a close relation with some other parameters. Given a multi-graph G, $\tau(G) = 0$ if and only if G is

^{*}The work was supported by the National Natural Science Foundation of China (No. 11701401) and the Scientific Research Fund of Hunan Provincial Education Department of China (No. 18A432).

[†]Corresponding author. Email: fengming.dong@nie.edu.sg and donggraph@163.com.

[‡]Email: mathsgejun@163.com.

[§]oymath@163.com.

disconnected. When G is connected, $\tau(G)$ can be computed by some different methods, such as Kirchhoff's Matrix-Tree Theorem [8, 9], Tutte's deletion-contraction formula [13], etc. In some special cases, $\tau(G)$ can be computed directly by explicit formulas. The most famous one is Cayley's formula, i.e., $\tau(K_n) = n^{n-2}$ for complete graphs [2]. This formula has been extended to $\tau(K_{n_1,n_2,\dots,n_k}) = n^{k-2} \prod_{i=1}^k (n-n_i)^{n_i-1}$ for any complete k-particle graph K_{n_1,n_2,\dots,n_k} , where $n = n_1 + n_2 + \dots + n_k$ [1]. It is also known that $\tau(Q_n) = 2^{2^n - n - 1} \prod_{k=2}^n k^{\binom{n}{k}}$ for the *n*-dimensional hypercube graph Q_n [7]. For the line graph G = L(H) of an arbitrary connected graph H, a relation between $\tau(G)$ and spanning trees in H was also established [3]. More works on $\tau(G)$ can be found in [5, 6, 10, 11, 15].

In the following is an upper bounds for $\tau(G)$ due to Thomassen [12].

Theorem 1 ([12]). Let G = (V, E) be a multi-graph and u be any vertex in G. Then

$$\tau(G) \le \prod_{v \in V - \{u\}} d(v)$$

For any multi-graph G and any vertex u in G, let $\mathcal{NST}_u(G)$ be the set of non-spanning subtrees T of G such that $u \in V(T)$ and G - V(T) has no isolated vertices. In this article, we find the following formula expressing $\tau(G)$ in terms of degrees. It shows how far is Thomassen's upper bound from $\tau(G)$ exactly.

Theorem 2. For a multi-graph G = (V, E) and a vertex u in G,

$$\tau(G) = \prod_{v \in V - \{u\}} d(v) - \sum_{T \in \mathcal{NST}_u(G)} \prod_{v \in V - V(T)} d_{G - V(T)}(v).$$
(1)

Theorem 2 can be proved by some different approaches. In this note, we shall prove Theorem 3 in Section 3 from which Theorem 2 follows directly. In Section 2, we introduce a polynomial $F(G, \omega)$ of a graph G by assigning a variable y_i to each edge e_i in G. This polynomial will be applied in Section 3 for proving Theorem 3 by a method inspired by Wang algebra [4, 14]¹. In Section 4, we apply Theorem 2 to compute $\tau(G)$ for some graphs.

2 A polynomial $F(G, \omega)$

For any positive integer n, let $[n] = \{1, 2, \dots, n\}$. Let G = (V, E) be a loopless and connected multi-graph with $V = \{v_i : i \in [n]\}$ and $E = \{e_j : j \in [m]\}$. Assume that ω is a weight function on E(G) defined by $\omega(e_j) = y_j$ for each $j \in [m]$, where y_1, y_2, \dots, y_m are considered as indeterminates. Define a polynomial $F(G, \omega)$ as follows:

$$F(G,\omega) = \prod_{i \in [n]} \sum_{e_j \in E(v_i)} \omega(e_j) = \prod_{i \in [n]} \sum_{e_j \in E(v_i)} y_j, \quad \text{when } V \neq \emptyset;$$
(2)

¹Wang algebra assumes that x + x = 0, $x \cdot x = 0$ and xy = yx for any variables x and y.

and $F(G, \omega) = 1$ when $V = \emptyset$. Clearly, $F(G, \omega) = 0$ whenever $d(v_i) = 0$ for some $v_i \in V$. If $y_i = 1$ for all $i \in [m]$, then $F(G, \omega) = \prod_{1 \le i \le n} d_G(v_i)$.

The expansion of $F(G, \omega)$ can be applied to study some structures of G, such as the minimum edge coverings, maximum matchings, perfect matchings, and spanning trees, and hence the edge covering number $\rho(G)$, the matching number $\nu(G)$ and the number of spanning trees $\tau(G)$. Let $\mathscr{F}(G, \omega)$ denote the set of terms in the expansion of $F(G, \omega)$. Note that each term in $\mathscr{F}(G, \omega)$ is in the form $y_{i_1}^2 y_{i_2}^2 \cdots y_{i_r}^2 y_{j_1} \cdots y_{j_k}$, where k + 2r = n and $i_1, i_2, \cdots, i_r, j_1, \cdots, j_k$ are pairwise distinct. Each term $y_{i_1}^2 y_{i_2}^2 \cdots y_{i_r}^2 y_{j_1} \cdots y_{j_k}$ in $\mathscr{F}(G, \omega)$ corresponds to an edge cover $\{e_{i_1}, e_{i_2}, \cdots, e_{i_r}\} \cup \{e_{j_1}, e_{j_2}, \cdots, e_{j_k}\}$ of G, where $\{e_{i_1}, e_{i_2}, \cdots, e_{i_r}\}$ is a matching of G. In particular, if $y_{i_1}^2 y_{i_2}^2 \cdots y_{i_r}^2$ is a term in $\mathscr{F}(G, \omega)$, then n = 2r and it corresponds to a perfect matching $\{e_{i_1}, e_{i_2}, \cdots, e_{i_r}\}$ of G. Thus, $\rho(G)$ is the minimum value of k + r among all terms $y_{i_1}^2 y_{i_2}^2 \cdots y_{i_r}^2 y_{j_1} \cdots y_{j_k}$ in $\mathscr{F}(G, \omega)$.



Figure 1: A multi-graph

For example, if G is the multi-graph in Figure 1, then

$$F(G,\omega) = (y_1 + y_3 + y_4 + y_6)(y_1 + y_2)(y_2 + y_3 + y_4 + y_5)(y_5 + y_6),$$
(3)

and the expansion of $F(G, \omega)$ contains terms $y_1^2 y_5^2$ and $y_2^2 y_6^2$, which correspond to the two perfect matchings in G: $M_1 = \{e_1, e_5\}$ and $M_2 = \{e_2, e_6\}$.

In the next section, we shall apply $F(G, \omega)$ to study $\tau(G)$.

3 An identity associated with spanning trees

In this section, we assume that G = (V, E) is a loopless connected multi-graph, where $V = \{v_i : i \in [n]\}, n \ge 2$, and $E = \{e_j : j \in [m]\}$. Let ω be a weight function on E.

We first establish two lemmas which will be applied to prove the main result in this section.

Let \overrightarrow{G} denote the digraph obtained from G by replacing each edge e_i in G by two arcs which are incident the same pair of ends of e_i and have opposite directions. Assume that the weight function ω is extended to the arc set $A(\overrightarrow{G})$ such that $\omega(a) = \omega(e_i)$ for each $a \in A(\overrightarrow{G})$ if a is obtained from e_i by assigning a direction.

For a digraph D and a vertex v in D, let $id_D(v)$ denote the in-degree of v in D. If $id_D(v) = 0$, then v is called a *source* of D.

Let \mathbb{D}^* denote the family of spanning subdigraphs D of \overrightarrow{G} with $id_D(v_n) = 0$ and $id_D(v_i) = 1$ for each $i \in [n-1]$.

For any subdigraph D of \overrightarrow{G} , let $\omega(D) = \prod_{a \in A(D)} \omega(a)$ if $A(D) \neq \emptyset$ and $\omega(D) = 1$ otherwise.

Lemma 1. Let G = (V, E) be a loopless connected multi-graph, where $V = \{v_i : i \in [n]\}$, $n \ge 2$ and $E = \{e_j : j \in [m]\}$, and let ω be a weight function on E. The following holds:

$$\prod_{i=1}^{n-1} \sum_{e_j \in E(v_i)} \omega(e_j) = \sum_{D \in \mathbb{D}^*} \omega(D).$$
(4)

Proof. Let Π be the set of mappings $\pi : [n-1] \to [m]$ such that $e_{\pi(i)} \in E(v_i)$ for each $i \in [n-1]$. Observe that

$$\prod_{i=1}^{n-1} \sum_{e_j \in E(v_i)} \omega(e_j) = \sum_{\pi \in \Pi} \prod_{1 \le i \le n-1} \omega(e_{\pi(i)}).$$
(5)

For any $\pi \in \Pi$, $(e_{\pi(1)}, e_{\pi(2)}, \cdots, e_{\pi(n-1)})$ is a list of n-1 edges in G, where each edge $e_{\pi(i)}$ is incident to v_i . Let $f(\pi)$ denote the spanning subdigraph D of \overrightarrow{G} that can be obtained by converting each edge $e_{\pi(i)}$ into an arc with v_i as its head. Observe that $f(\pi)$ is a digraph in \mathbb{D}^* and, if $D = f(\pi)$, then

$$\prod_{1 \le i \le n-1} \omega(e_{\pi(i)}) = \prod_{a \in A(D)} \omega(a) = \omega(D).$$
(6)

It is obvious that f is a bijection from Π to \mathbb{D}^* . Thus, (4) follows from (5) and (6) and the lemma holds.

For any $U \subseteq V$ with $U \neq \emptyset$, let $\mathbb{D}[U]$ denote the family of subdigraphs D of \overrightarrow{G} with vertex set U and $id_D(v_i) = 1$ for each $v_i \in U$. Note that $\mathbb{D}[V]$ is different from \mathbb{D}^* , although both are spanning subdigraphs of \overrightarrow{G} . The following lemma can be proved similarly.

Lemma 2. Let G = (V, E) be a loopless connected multi-graph, where $V = \{v_i : i \in [n]\}$, $n \ge 2$ and $E = \{e_j : j \in [m]\}$, and let ω be a weight function on E. For any $U \subseteq V(G)$ with $U \neq \emptyset$,

$$F(G[U],\omega) = \sum_{D \in \mathbb{D}[U]} \omega(D).$$
(7)

Recall that $\mathcal{T}(G)$ is the set of spanning trees in G. For any $T \in \mathcal{T}(G)$, let $\tau(T, \omega) = 1$ when |V(G)| = 1, and let

$$\tau(T,\omega) = \prod_{e_i \in E(T)} \omega(e_i), \quad \text{when } |V(G)| \ge 2.$$
(8)

Now we define another function $\tau(G, \omega)$:

$$\tau(G,\omega) = \sum_{T \in \mathcal{T}(G)} \tau(T,\omega).$$
(9)

Thus $\tau(G, \omega) = 0$ whenever $\mathcal{T}(G) = \emptyset$ (i.e., G is disconnected). Clearly, when G is connected, every term in the expansion of $\tau(G, \omega)$ corresponds to a spanning tree in G, and $\tau(G, \omega) = \tau(G)$ whenever $\omega(e_j) = 1$ for all $j \in [m]$.

Recall that for any $u \in V(G)$, $\mathcal{NST}_u(G)$ denotes the set of non-spanning subtrees Tof G such that $u \in V(T)$ and G - V(T) has no isolated vertices. We are now going to prove the following identity on $\tau(G, \omega)$ from which Theorem 2 follows directly.

Theorem 3. Let G = (V, E) be a loopless connected multi-graph, where $V = \{v_i : i \in [n]\}$, $n \ge 2$ and $E = \{e_j : j \in [m]\}$. Assume that ω is a weight function on E. Then,

$$\prod_{i=1}^{n-1} \sum_{e_j \in E_G(v_i)} \omega(e_j) = \tau(G, \omega) + \sum_{T_0 \in \mathcal{NST}_{v_n}(G)} \tau(T_0, \omega) F(G - V(T_0), \omega).$$
(10)

Proof. A digraph is called a *directed tree* if its underlying graph is a tree. A directed tree with a unique source is called a *rooted directed tree* and the unique source is its root. We are now going to establish the following claims.

Claim 1: For any weakly connected diraph Q with vertices u_0, u_1, \dots, u_k , if $id_Q(u_0) = 0$ and $id_Q(u_i) = 1$ for all $i \in [k]$, then Q is a directed rooted tree with root u_0 .

Q is a directed tree as its underlying graph is connected and has exactly k edges and k+1 vertices. Then the claim holds as u_0 is the only source in Q.

Recall that \mathbb{D}^* is the family of spanning subdigraphs D of \overrightarrow{G} such that $id_D(v_n) = 0$ and $id_D(v_i) = 1$ for each $i \in [n-1]$. For any $D \in \mathbb{D}^*$, let D_{v_n} denote the component (i.e., a weakly connected component) of D that contains vertex v_n .

Claim 2: For any $D \in \mathbb{D}^*$, D_{v_n} is a rooted directed tree with root v_n .

If $V(D_{v_n}) = \{v_n\}$, the claim is trivial. Now, without loss of generality, assume that $V(D_{v_n}) = \{v_i : i \in [k]\} \cup \{v_n\}$, where $1 \le k \le n-1$. As D_{v_n} is weakly connected and $|V(D_{v_n})| = k+1$, we have $|A(D_{v_n})| \ge k$.

It is known that D has exactly n-1 arcs and $id_D(v_i) = 1$ for all $i \in [n-1]$. Assume that a_i is the arc in D with head v_i for each $i \in [n-1]$. Thus, $A(D) = \{a_i : i \in [n-1]\}$. As $V(D_{v_n}) = \{v_i : i \in [k]\} \cup \{v_n\}$, we have $A(D_{v_n}) \subseteq \{a_i : i \in [k]\}$. Since $|A(D_{v_n})| \ge k$, $A(D_{v_n}) = \{a_i : i \in [k]\}$ holds. Thus, D_{v_n} is weakly connected with a source v_n and a_i is the only arc in D_{v_n} with head v_i for all $i \in [k]$. Claim 2 then follows from Claim 1.

Claim 3: For each subtree T of G with $v_n \in V(T)$, there is exactly one rooted directed tree, denoted by \overrightarrow{T} , with the following properties:

- (i) T is the underlying graph of \overrightarrow{T} ; and
- (ii) $id_{\overrightarrow{T}}(v_n) = 0$ and $id_{\overrightarrow{T}}(v_i) = 1$ for each $v_i \in V(T) \setminus \{v_n\}$.

Claim 3 is obvious, as such a directed tree \overrightarrow{T} can only be obtained by assigning directions to edges in T so that each $v_n - v_i$ path in T becomes a directed $v_n - v_i$ path (i.e., a path from v_n to v_i) in \overrightarrow{T} . Observe that $\omega(T) = \omega(\overrightarrow{T})$ for each subtree T of G.

Recall that $\mathcal{NST}_{v_n}(G)$ is the set of non-spanning subtrees T of G such that $v_n \in V(T)$ and G - V(T) has no isolated vertices. Let $\mathcal{NST}_{v_n}(\overrightarrow{G}) = \{\overrightarrow{T} : T \in \mathcal{NST}_{v_n}(G)\}.$

By Claim 2, for each $D \in \mathbb{D}^*$, if D is not weakly connected, then, the unlderlying graph T of D_{v_n} is a non-spanning tree. Furthermore, by the definition of \mathbb{D}^* , each vertex v_i , where $i \in [n-1]$, is the head of some arc in \mathbb{D}^* and thus is not isolated in G - V(T), implying that $D_{v_n} = \overrightarrow{T} \in \mathcal{NST}_{v_n}(\overrightarrow{G})$.

For any $\overrightarrow{T} \in \mathcal{NST}_{v_n}(\overrightarrow{G})$, let $\mathbb{D}^*(\overrightarrow{T})$ denote the set of $D \in \mathbb{D}^*$ such that D_{v_n} is the directed tree \overrightarrow{T} . Thus, by the definition of $\mathbb{D}[U]$ for $U \subseteq V$, for any $T \in \mathcal{NST}_{v_n}(G)$,

$$\mathbb{D}^*(\overrightarrow{T}) = \{ \overrightarrow{T} \cup Q : Q \in \mathbb{D}[V(G) \setminus V(T)] \},$$
(11)

where $\overrightarrow{T} \cup Q$ denotes the spanning digraph of \overrightarrow{G} with arc set $A(\overrightarrow{T}) \cup A(Q)$.

Let \mathbb{D}_0^* denote the family of $D \in \mathbb{D}^*$ such that D is weakly connected. By Claim 2, D is a rooted directed tree for each $D \in \mathbb{D}_0^*$. Actually, $\mathbb{D}_0^* = \{\overrightarrow{T} : T \in \mathcal{T}(G)\}$. As D_{v_n} belongs to $\mathcal{NST}_{v_n}(\overrightarrow{G})$ for each $D \in \mathbb{D}^* \setminus \mathbb{D}_0^*$, by (11),

$$\mathbb{D}^* \setminus \mathbb{D}_0^* = \bigcup_{T \in \mathcal{NST}_{v_n}(G)} \mathbb{D}^*(\overrightarrow{T}) = \bigcup_{T \in \mathcal{NST}_{v_n}(G)} \{ \overrightarrow{T} \cup Q : Q \in \mathbb{D}[V(G) \setminus V(T)] \}.$$
(12)

By Lemmas 1 and 2 and (12),

$$\prod_{i=1}^{n-1} \sum_{e_j \in E_G(v_i)} \omega(e_j) = \sum_{D \in \mathbb{D}_0^*} \omega(D) + \sum_{D \in \mathbb{D}^* \setminus \mathbb{D}_0^*} \omega(D)$$

$$= \sum_{T \in \mathcal{T}(G)} \omega(\vec{T}) + \sum_{T \in \mathcal{NST}_{v_n}(G)} \sum_{Q \in \mathbb{D}[V(G) \setminus V(T)]} \omega(\vec{T}) \omega(Q)$$

$$= \sum_{T \in \mathcal{T}(G)} \omega(T) + \sum_{T \in \mathcal{NST}_{v_n}(G)} \omega(T) \sum_{Q \in \mathbb{D}[V(G) \setminus V(T)]} \omega(Q)$$

$$= \sum_{T \in \mathcal{T}(G)} \omega(T) + \sum_{T \in \mathcal{NST}_{v_n}(G)} \omega(T) F(G - V(T), \omega). \quad (13)$$

Thus Theorem 3 is proved.

Observe that Theorem 2 follows directly from Theorem 3 by taking $u = v_n$ and $y_j = 1$ for all $j \in [m]$.

4 Application

In the last section, we give some examples of applying Theorem 2 to determine spanning numbers of graphs.

Let G be a connected multi-graph with $u \in V(G)$. For $1 \leq i \leq |V(G)| - 2$, let $\mathscr{C}_i(G, u)$ (or simply $\mathscr{C}_i(u)$) be the set of connected induced subgraphs G[S], where $u \in S \subset V(G)$, such that |S| = i and G - S has no isolated vertices. Clearly, $|\mathscr{C}_1(u)| \leq 1$ and $|\mathscr{C}_2(u)| \leq |N_G(u)|$, where $N_G(u)$ is the set of neighbors of u in G.

Observe that expression (1) in Theorem 2 is equivalent to the following one:

$$\tau(G) = \prod_{v \in V(G) - \{u\}} d_G(v) - \sum_{i=1}^{|V(G)|-2} \sum_{H \in \mathscr{C}_i(u)} \left(\tau(H) \prod_{v \in V(G) - V(H)} d_{G-V(H)}(v) \right).$$
(14)

Now we apply (14) to determine $\tau(W_4)$, $\tau(W'_4)$ and $\tau(W'_5)$, where W_4 is the wheel of order 5 and W'_4 and W'_5 are multi-graphs which can be obtained from W_4 and W_5 respectively by adding new edges parallel to edges incident with the central vertex, as shown in Figure 2 (b) and (c).



Figure 2: Graphs W_4 , W'_4 and W'_5

Let u be the central vertex in W_4 as shown in Figure 2 (a). By (14), we have

$$\tau(W_4) = 3^4 - 2^4 - 4 \times 1 \times (2 \times 1^2) - 4 \times 3 \times 1^2 = 45.$$
(15)

The above equality follows from the fact that $|\mathscr{C}_1(u)| = 1$, $|\mathscr{C}_2(u)| = |\mathscr{C}_3(u)| = 4$, $\tau(H) = 1$ and $G - V(H) \cong C_4$ for $H \in \mathscr{C}_1$, $\tau(H) = i^{i-2}$ and G - V(H) is a path of length 5 - i for each $H \in \mathscr{C}_i(u)$ and i = 2, 3. Again, taking u to be the central vertex in W'_4 , we have

$$\tau(W_4') = 4^4 - 2^4 - 4 \times 2 \times (2 \times 1^2) - 4 \times 8 \times 1^2 = 192.$$
(16)

The above equality follows from the fact that $|\mathscr{C}_1(u)| = 1$, $|\mathscr{C}_2(u)| = |\mathscr{C}_3(u)| = 4$, $\tau(H) = 1$ and $G - V(H) \cong C_4$ for $H \in \mathscr{C}_1$, $\tau(H) = 2$ for each $H \in \mathscr{C}_2(u)$, $\tau(H) = 8$ for each $H \in \mathscr{C}_3(u)$, and G - V(H) is a path of length 5 - i for each $H \in \mathscr{C}_i(u)$ and i = 2, 3.

Similarly, taking u to be the central vertex in W'_5 , we have

$$\tau(W_5') = 4^5 - 2^5 - 5 \times 2 \times (2 \times 2 \times 1^2) - 5 \times 8 \times 2 - 5(4 \times 3^2 - 2 - 2 \times 2) = 722.$$
(17)

The above equality follows from the fact that $|\mathscr{C}_1(u)| = 1$, $|\mathscr{C}_i(u)| = 5$ for i = 2, 3, 4, $\tau(H) = 1$ and $G - V(H) \cong C_5$ for $H \in \mathscr{C}_1$, $\tau(H) = 2$ for each $H \in \mathscr{C}_2(u)$, $\tau(H) = 8$ for each $H \in \mathscr{C}_3(u)$, $\tau(H) = 4 \times 3^2 - 2 - 2 = 32$ for each $H \in \mathscr{C}_4(u)$, and G - V(H) is a path of length 6 - i for each $H \in \mathscr{C}_i(u)$ and i = 2, 3, 4.

Our examples above show that as an alternative method of computing spanning trees in small graphs by hand, applying Theorem 2 is sometimes not less efficient than other methods.

Another potential usefulness of this formula is, maybe for some graph classes, we can use Theorem 2 to obtain a better upper bound for the number of spanning trees than Theorem 1. Corollary 1 below is an example.

Corollary 1. Let G be a graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Then

$$\tau(G) \le \prod_{i=1}^{n-1} d_i - \prod_{i=1}^{n-1} (d_i - 1)$$

Acknowledgements

The authors would like to thank the referees for their constructive comments.

References

- T. Austin, The enumeration of point labelled chromatic graphs and tress, Canad. J. Math. 12 (1960), 535–545.
- [2] A. Cayley, A theorem on trees, Quart. J. Pure Appl. Math. 23 (1889), 376–378.
- [3] Fengming Dong and Yan Weigen, Expression for the Number of Spanning Trees of Line Graphs of Arbitrary Connected Graphs, J. Graph Theory 85(1) (2017), 74–93.
- [4] R. J. Duffin, An analysis of the Wang algebra of networks, Trans. Amer. Math. Soc. 93 (1959), 114–131.

- [5] Jun Ge and Fengming Dong, Spanning trees in complete bipartite graphs and resistance distance in nearly complete bipartite graphs, *Discrete Appl. Math.* 283 (2020), 542–554.
- [6] Helin Gong and Xi'an Jin, A simple formula for the number of spanning trees of line graphs, J. Graph Theory 88 (2018), 294–301.
- [7] F. Harary, J.P. Hayes and H.J. Wu, A survey of the theory of hypercube graphs, Computers and Mathematics with Applications 15 (4) (1988), 277–289.
- [8] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847), 497–508. (English transl. IRE Trans. Circuit Theory CT-5 (1958), 4–7.)
- [9] W. Kocay and D.L. Kreher, "The matrix-tree theorem", Graphs, Algorithms and Optimization, Discrete Mathematics and Its Applications, CRC Press, pp. 111–116, 2004.
- [10] L. Lovász, Combinatorial problems and exercises, second edition, North-Holland Publishing Co., Amsterdam, 1993.
- [11] J. W. Moon, The second moment of the complexity of a graph, Mathematika 11 (1964), 95–98.
- [12] C. Thomassen, Spanning trees and orientations of graphs, J. Comb. 1(2) (2010), 101–111.
- [13] W.T. Tutte, Graph-polynomials, Advances in Appl. Math. 32 (1-2) (2004), 5–9.
- [14] K. T. Wang, On a new method of analysis of electrical networks, Academia Sinica: Memoir of the National Research Institute of Engneering Memoirs 2 (1934), 1–11.
- [15] Weigen Yan, Enumeration of spanning trees of middle graphs, Appl. Math. Comput. 307 (2017), 239–243.