# Express the number of spanning trees in term of degrees* 

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#### Abstract

It is well-known that the number of spanning trees, denoted by $\tau(G)$, in a connected multi-graph $G$ can be calculated by the Matrix-Tree Theorem and Tutte's deletioncontraction formula. In this short note, we find an alternate method to compute $\tau(G)$ by degrees of vertices.


Keywords: spanning tree; degree; graph polynomial
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## 1 Introduction

In this article, we consider loopless and undirected multi-graphs. For a graph $G$, let $V(G), E(G)$ and $\mathcal{T}(G)$ be the set of vertices, the set of edges and the set of spanning trees in $G$ respectively, and let $\tau(G)=|\mathcal{T}(G)|$. For any $u \in V(G)$, let $E_{G}(u)$ (or simply $E(u)$ ) denote the set of edges in $G$ that are incident with $u$, and let $d_{G}(u)$ (or simply $d(u)$ ) be the degree of $u$ in $G$, i.e., $d_{G}(u)=\left|E_{G}(u)\right|$. For any $S \subseteq V(G)$, if $S \neq \emptyset$, let $G[S]$ be the subgraph of $G$ induced by $S$, and if $S \neq V$, let $G-S=G[V \backslash S]$.

The study of spanning trees plays an important role in graph theory. The number of spanning trees $\tau(G)$ is a key parameter in Tutte polynomials, and it has a close relation with some other parameters. Given a multi-graph $G, \tau(G)=0$ if and only if $G$ is

[^0]disconnected. When $G$ is connected, $\tau(G)$ can be computed by some different methods, such as Kirchhoff's Matrix-Tree Theorem [8, 9, Tutte's deletion-contraction formula [13, etc. In some special cases, $\tau(G)$ can be computed directly by explicit formulas. The most famous one is Cayley's formula, i.e., $\tau\left(K_{n}\right)=n^{n-2}$ for complete graphs [2]. This formula has been extended to $\tau\left(K_{n_{1}, n_{2}, \cdots, n_{k}}\right)=n^{k-2} \prod_{i=1}^{k}\left(n-n_{i}\right)^{n_{i}-1}$ for any complete $k$-particle graph $K_{n_{1}, n_{2}, \cdots, n_{k}}$, where $n=n_{1}+n_{2}+\cdots+n_{k}$ [1]. It is also known that $\tau\left(Q_{n}\right)=2^{2^{n}-n-1} \prod_{k=2}^{n} k^{\binom{n}{k}}$ for the $n$-dimensional hypercube graph $Q_{n}$ [7]. For the line graph $G=L(H)$ of an arbitrary connected graph $H$, a relation between $\tau(G)$ and spanning trees in $H$ was also established [3]. More works on $\tau(G)$ can be found in [5, 6, 10, 11, 15].

In the following is an upper bounds for $\tau(G)$ due to Thomassen [12].
Theorem 1 ([12]). Let $G=(V, E)$ be a multi-graph and $u$ be any vertex in $G$. Then

$$
\tau(G) \leq \prod_{v \in V-\{u\}} d(v) .
$$

For any multi-graph $G$ and any vertex $u$ in $G$, let $\mathcal{N S T}_{u}(G)$ be the set of non-spanning subtrees $T$ of $G$ such that $u \in V(T)$ and $G-V(T)$ has no isolated vertices. In this article, we find the following formula expressing $\tau(G)$ in terms of degrees. It shows how far is Thomassen's upper bound from $\tau(G)$ exactly.

Theorem 2. For a multi-graph $G=(V, E)$ and a vertex $u$ in $G$,

$$
\begin{equation*}
\tau(G)=\prod_{v \in V-\{u\}} d(v)-\sum_{T \in \mathcal{N S} \mathcal{T} \mathcal{T}_{u}(G)} \prod_{v \in V-V(T)} d_{G-V(T)}(v) . \tag{1}
\end{equation*}
$$

Theorem 2 can be proved by some different approaches. In this note, we shall prove Theorem 3 in Section 3 from which Theorem 2 follows directly. In Section 2, we introduce a polynomial $F(G, \omega)$ of a graph $G$ by assigning a variable $y_{i}$ to each edge $e_{i}$ in $G$. This polynomial will be applied in Section 3 for proving Theorem 3 by a method inspired by Wang algebra [4, 14]. In Section[4, we apply Theorem[2]to compute $\tau(G)$ for some graphs.

## 2 A polynomial $F(G, \omega)$

For any positive integer $n$, let $[n]=\{1,2, \cdots, n\}$. Let $G=(V, E)$ be a loopless and connected multi-graph with $V=\left\{v_{i}: i \in[n]\right\}$ and $E=\left\{e_{j}: j \in[m]\right\}$. Assume that $\omega$ is a weight function on $E(G)$ defined by $\omega\left(e_{j}\right)=y_{j}$ for each $j \in[m]$, where $y_{1}, y_{2}, \cdots, y_{m}$ are considered as indeterminates. Define a polynomial $F(G, \omega)$ as follows:

$$
\begin{equation*}
F(G, \omega)=\prod_{i \in[n]} \sum_{e_{j} \in E\left(v_{i}\right)} \omega\left(e_{j}\right)=\prod_{i \in[n]} \sum_{e_{j} \in E\left(v_{i}\right)} y_{j}, \quad \text { when } V \neq \emptyset ; \tag{2}
\end{equation*}
$$

[^1]and $F(G, \omega)=1$ when $V=\emptyset$. Clearly, $F(G, \omega)=0$ whenever $d\left(v_{i}\right)=0$ for some $v_{i} \in V$. If $y_{i}=1$ for all $i \in[m]$, then $F(G, \omega)=\prod_{1 \leq i \leq n} d_{G}\left(v_{i}\right)$.

The expansion of $F(G, \omega)$ can be applied to study some structures of $G$, such as the minimum edge coverings, maximum matchings, perfect matchings, and spanning trees, and hence the edge covering number $\rho(G)$, the matching number $\nu(G)$ and the number of spanning trees $\tau(G)$. Let $\mathscr{F}(G, \omega)$ denote the set of terms in the expansion of $F(G, \omega)$. Note that each term in $\mathscr{F}(G, \omega)$ is in the form $y_{i_{1}}^{2} y_{i_{2}}^{2} \cdots y_{i_{r}}^{2} y_{j_{1}} \cdots y_{j_{k}}$, where $k+2 r=$ $n$ and $i_{1}, i_{2}, \cdots, i_{r}, j_{1}, \cdots, j_{k}$ are pairwise distinct. Each term $y_{i_{1}}^{2} y_{i_{2}}^{2} \cdots y_{i_{r}}^{2} y_{j_{1}} \cdots y_{j_{k}}$ in $\mathscr{F}(G, \omega)$ corresponds to an edge cover $\left\{e_{i_{1}}, e_{i_{2}}, \cdots, e_{i_{r}}\right\} \cup\left\{e_{j_{1}}, e_{j_{2}}, \cdots, e_{j_{k}}\right\}$ of $G$, where $\left\{e_{i_{1}}, e_{i_{2}}, \cdots, e_{i_{r}}\right\}$ is a matching of $G$. In particular, if $y_{i_{1}}^{2} y_{i_{2}}^{2} \cdots y_{i_{r}}^{2}$ is a term in $\mathscr{F}(G, \omega)$, then $n=2 r$ and it corresponds to a perfect matching $\left\{e_{i_{1}}, e_{i_{2}}, \cdots, e_{i_{r}}\right\}$ of $G$. Thus, $\rho(G)$ is the minimum value of $k+r$ among all terms $y_{i_{1}}^{2} y_{i_{2}}^{2} \cdots y_{i_{r}}^{2} y_{j_{1}} \cdots y_{j_{k}}$ in $\mathscr{F}(G, \omega)$, and $\nu(G)$ is the maximum value of $r$ among all terms $y_{i_{1}}^{2} y_{i_{2}}^{2} \cdots y_{i_{r}}^{2} y_{j_{1}} \cdots y_{j_{k}}$ in $\mathscr{F}(G, \omega)$.


Figure 1: A multi-graph

For example, if $G$ is the multi-graph in Figure [1, then

$$
\begin{equation*}
F(G, \omega)=\left(y_{1}+y_{3}+y_{4}+y_{6}\right)\left(y_{1}+y_{2}\right)\left(y_{2}+y_{3}+y_{4}+y_{5}\right)\left(y_{5}+y_{6}\right), \tag{3}
\end{equation*}
$$

and the expansion of $F(G, \omega)$ contains terms $y_{1}^{2} y_{5}^{2}$ and $y_{2}^{2} y_{6}^{2}$, which correspond to the two perfect matchings in $G: M_{1}=\left\{e_{1}, e_{5}\right\}$ and $M_{2}=\left\{e_{2}, e_{6}\right\}$.

In the next section, we shall apply $F(G, \omega)$ to study $\tau(G)$.

## 3 An identity associated with spanning trees

In this section, we assume that $G=(V, E)$ is a loopless connected multi-graph, where $V=\left\{v_{i}: i \in[n]\right\}, n \geq 2$, and $E=\left\{e_{j}: j \in[m]\right\}$. Let $\omega$ be a weight function on $E$.

We first establish two lemmas which will be applied to prove the main result in this section.

Let $\vec{G}$ denote the digraph obtained from $G$ by replacing each edge $e_{i}$ in $G$ by two arcs which are incident the same pair of ends of $e_{i}$ and have opposite directions. Assume that the weight function $\omega$ is extended to the arc set $A(\vec{G})$ such that $\omega(a)=\omega\left(e_{i}\right)$ for each $a \in A(\vec{G})$ if $a$ is obtained from $e_{i}$ by assigning a direction.

For a digraph $D$ and a vertex $v$ in $D$, let $i d_{D}(v)$ denote the in-degree of $v$ in $D$. If $i d_{D}(v)=0$, then $v$ is called a source of $D$.

Let $\mathbb{D}^{*}$ denote the family of spanning subdigraphs $D$ of $\vec{G}$ with $i d_{D}\left(v_{n}\right)=0$ and $i d_{D}\left(v_{i}\right)=1$ for each $i \in[n-1]$.

For any subdigraph $D$ of $\vec{G}$, let $\omega(D)=\prod_{a \in A(D)} \omega(a)$ if $A(D) \neq \emptyset$ and $\omega(D)=1$ otherwise.

Lemma 1. Let $G=(V, E)$ be a loopless connected multi-graph, where $V=\left\{v_{i}: i \in[n]\right\}$, $n \geq 2$ and $E=\left\{e_{j}: j \in[m]\right\}$, and let $\omega$ be a weight function on $E$. The following holds:

$$
\begin{equation*}
\prod_{i=1}^{n-1} \sum_{e_{j} \in E\left(v_{i}\right)} \omega\left(e_{j}\right)=\sum_{D \in \mathbb{D}^{*}} \omega(D) . \tag{4}
\end{equation*}
$$

Proof. Let $\Pi$ be the set of mappings $\pi:[n-1] \rightarrow[m]$ such that $e_{\pi(i)} \in E\left(v_{i}\right)$ for each $i \in[n-1]$. Observe that

$$
\begin{equation*}
\prod_{i=1}^{n-1} \sum_{e_{j} \in E\left(v_{i}\right)} \omega\left(e_{j}\right)=\sum_{\pi \in \Pi} \prod_{1 \leq i \leq n-1} \omega\left(e_{\pi(i)}\right) \tag{5}
\end{equation*}
$$

For any $\pi \in \Pi$, $\left(e_{\pi(1)}, e_{\pi(2)}, \cdots, e_{\pi(n-1)}\right)$ is a list of $n-1$ edges in $G$, where each edge $e_{\pi(i)}$ is incident to $v_{i}$. Let $f(\pi)$ denote the spanning subdigraph $D$ of $\vec{G}$ that can be obtained by converting each edge $e_{\pi(i)}$ into an arc with $v_{i}$ as its head. Observe that $f(\pi)$ is a digraph in $\mathbb{D}^{*}$ and, if $D=f(\pi)$, then

$$
\begin{equation*}
\prod_{1 \leq i \leq n-1} \omega\left(e_{\pi(i)}\right)=\prod_{a \in A(D)} \omega(a)=\omega(D) . \tag{6}
\end{equation*}
$$

It is obvious that $f$ is a bijection from $\Pi$ to $\mathbb{D}^{*}$. Thus, (4) follows from (5) and (6) and the lemma holds.

For any $U \subseteq V$ with $U \neq \emptyset$, let $\mathbb{D}[U]$ denote the family of subdigraphs $D$ of $\vec{G}$ with vertex set $U$ and $i d_{D}\left(v_{i}\right)=1$ for each $v_{i} \in U$. Note that $\mathbb{D}[V]$ is different from $\mathbb{D}^{*}$, although both are spanning subdigraphs of $\vec{G}$. The following lemma can be proved similarly.

Lemma 2. Let $G=(V, E)$ be a loopless connected multi-graph, where $V=\left\{v_{i}: i \in[n]\right\}$, $n \geq 2$ and $E=\left\{e_{j}: j \in[m]\right\}$, and let $\omega$ be a weight function on $E$. For any $U \subseteq V(G)$ with $U \neq \emptyset$,

$$
\begin{equation*}
F(G[U], \omega)=\sum_{D \in \mathbb{D}[U]} \omega(D) . \tag{7}
\end{equation*}
$$

Recall that $\mathcal{T}(G)$ is the set of spanning trees in $G$. For any $T \in \mathcal{T}(G)$, let $\tau(T, \omega)=1$ when $|V(G)|=1$, and let

$$
\begin{equation*}
\tau(T, \omega)=\prod_{e_{i} \in E(T)} \omega\left(e_{i}\right), \quad \text { when }|V(G)| \geq 2 . \tag{8}
\end{equation*}
$$

Now we define another function $\tau(G, \omega)$ :

$$
\begin{equation*}
\tau(G, \omega)=\sum_{T \in \mathcal{T}(G)} \tau(T, \omega) . \tag{9}
\end{equation*}
$$

Thus $\tau(G, \omega)=0$ whenever $\mathcal{T}(G)=\emptyset$ (i.e., $G$ is disconnected). Clearly, when $G$ is connected, every term in the expansion of $\tau(G, \omega)$ corresponds to a spanning tree in $G$, and $\tau(G, \omega)=\tau(G)$ whenever $\omega\left(e_{j}\right)=1$ for all $j \in[m]$.

Recall that for any $u \in V(G), \mathcal{N S T}_{u}(G)$ denotes the set of non-spanning subtrees $T$ of $G$ such that $u \in V(T)$ and $G-V(T)$ has no isolated vertices. We are now going to prove the following identity on $\tau(G, \omega)$ from which Theorem 2 follows directly.

Theorem 3. Let $G=(V, E)$ be a loopless connected multi-graph, where $V=\left\{v_{i}: i \in[n]\right\}$, $n \geq 2$ and $E=\left\{e_{j}: j \in[m]\right\}$. Assume that $\omega$ is a weight function on $E$. Then,

$$
\begin{equation*}
\prod_{i=1}^{n-1} \sum_{e_{j} \in E_{G}\left(v_{i}\right)} \omega\left(e_{j}\right)=\tau(G, \omega)+\sum_{T_{0} \in \mathcal{N S} \mathcal{S}_{v_{n}}(G)} \tau\left(T_{0}, \omega\right) F\left(G-V\left(T_{0}\right), \omega\right) \tag{10}
\end{equation*}
$$

Proof. A digraph is called a directed tree if its underlying graph is a tree. A directed tree with a unique source is called a rooted directed tree and the unique source is its root. We are now going to establish the following claims.
Claim 1: For any weakly connected diraph $Q$ with vertices $u_{0}, u_{1}, \cdots, u_{k}$, if $i d_{Q}\left(u_{0}\right)=0$ and $i d_{Q}\left(u_{i}\right)=1$ for all $i \in[k]$, then $Q$ is a directed rooted tree with root $u_{0}$.
$Q$ is a directed tree as its underlying graph is connected and has exactly $k$ edges and $k+1$ vertices. Then the claim holds as $u_{0}$ is the only source in $Q$.

Recall that $\mathbb{D}^{*}$ is the family of spanning subdigraphs $D$ of $\vec{G}$ such that $i d_{D}\left(v_{n}\right)=0$ and $i d_{D}\left(v_{i}\right)=1$ for each $i \in[n-1]$. For any $D \in \mathbb{D}^{*}$, let $D_{v_{n}}$ denote the component (i.e., a weakly connected component) of $D$ that contains vertex $v_{n}$.
Claim 2: For any $D \in \mathbb{D}^{*}, D_{v_{n}}$ is a rooted directed tree with root $v_{n}$.
If $V\left(D_{v_{n}}\right)=\left\{v_{n}\right\}$, the claim is trivial. Now, without loss of generality, assume that $V\left(D_{v_{n}}\right)=\left\{v_{i}: i \in[k]\right\} \cup\left\{v_{n}\right\}$, where $1 \leq k \leq n-1$. As $D_{v_{n}}$ is weakly connected and $\left|V\left(D_{v_{n}}\right)\right|=k+1$, we have $\left|A\left(D_{v_{n}}\right)\right| \geq k$.

It is known that $D$ has exactly $n-1$ arcs and $i d_{D}\left(v_{i}\right)=1$ for all $i \in[n-1]$. Assume that $a_{i}$ is the arc in $D$ with head $v_{i}$ for each $i \in[n-1]$. Thus, $A(D)=\left\{a_{i}: i \in[n-1]\right\}$. As $V\left(D_{v_{n}}\right)=\left\{v_{i}: i \in[k]\right\} \cup\left\{v_{n}\right\}$, we have $A\left(D_{v_{n}}\right) \subseteq\left\{a_{i}: i \in[k]\right\}$. Since $\left|A\left(D_{v_{n}}\right)\right| \geq k$, $A\left(D_{v_{n}}\right)=\left\{a_{i}: i \in[k]\right\}$ holds.

Thus, $D_{v_{n}}$ is weakly connected with a source $v_{n}$ and $a_{i}$ is the only arc in $D_{v_{n}}$ with head $v_{i}$ for all $i \in[k]$. Claim 2 then follows from Claim 1 .

Claim 3: For each subtree $T$ of $G$ with $v_{n} \in V(T)$, there is exactly one rooted directed tree, denoted by $\vec{T}$, with the following properties:
(i) $T$ is the underlying graph of $\vec{T}$; and
(ii) $i d_{\vec{T}}\left(v_{n}\right)=0$ and $i d_{\vec{T}}\left(v_{i}\right)=1$ for each $v_{i} \in V(T) \backslash\left\{v_{n}\right\}$.

Claim 3 is obvious, as such a directed tree $\vec{T}$ can only be obtained by assigning directions to edges in $T$ so that each $v_{n}-v_{i}$ path in $T$ becomes a directed $v_{n}-v_{i}$ path (i.e., a path from $v_{n}$ to $v_{i}$ ) in $\vec{T}$. Observe that $\omega(T)=\omega(\vec{T})$ for each subtree $T$ of $G$.

Recall that $\mathcal{N S}_{v_{n}}(G)$ is the set of non-spanning subtrees $T$ of $G$ such that $v_{n} \in V(T)$ and $G-V(T)$ has no isolated vertices. Let $\mathcal{N S T}_{v_{n}}(\vec{G})=\left\{\vec{T}: T \in \mathcal{N S} \mathcal{T}_{v_{n}}(G)\right\}$.

By Claim 2, for each $D \in \mathbb{D}^{*}$, if $D$ is not weakly connected, then, the unlderlying graph $T$ of $D_{v_{n}}$ is a non-spanning tree. Furthermore, by the definition of $\mathbb{D}^{*}$, each vertex $v_{i}$, where $i \in[n-1]$, is the head of some arc in $\mathbb{D}^{*}$ and thus is not isolated in $G-V(T)$, implying that $D_{v_{n}}=\vec{T} \in \mathcal{N S} \mathcal{T}_{v_{n}}(\vec{G})$.

For any $\vec{T} \in \mathcal{N S T}_{v_{n}}(\vec{G})$, let $\mathbb{D}^{*}(\vec{T})$ denote the set of $D \in \mathbb{D}^{*}$ such that $D_{v_{n}}$ is the directed tree $\vec{T}$. Thus, by the definition of $\mathbb{D}[U]$ for $U \subseteq V$, for any $T \in \mathcal{N S} \mathcal{T}_{v_{n}}(G)$,

$$
\begin{equation*}
\mathbb{D}^{*}(\vec{T})=\{\vec{T} \cup Q: Q \in \mathbb{D}[V(G) \backslash V(T)]\}, \tag{11}
\end{equation*}
$$

where $\vec{T} \cup Q$ denotes the spanning digraph of $\vec{G}$ with arc set $A(\vec{T}) \cup A(Q)$.
Let $\mathbb{D}_{0}^{*}$ denote the family of $D \in \mathbb{D}^{*}$ such that $D$ is weakly connected. By Claim 2, $D$ is a rooted directed tree for each $D \in \mathbb{D}_{0}^{*}$. Actually, $\mathbb{D}_{0}^{*}=\{\vec{T}: T \in \mathcal{T}(G)\}$. As $D_{v_{n}}$ belongs to $\mathcal{N S T}_{v_{n}}(\vec{G})$ for each $D \in \mathbb{D}^{*} \backslash \mathbb{D}_{0}^{*}$, by (11),

$$
\begin{equation*}
\mathbb{D}^{*} \backslash \mathbb{D}_{0}^{*}=\bigcup_{T \in \mathcal{N S} \mathcal{T}_{v_{n}}(G)} \mathbb{D}^{*}(\vec{T})=\bigcup_{T \in \mathcal{N S} \mathcal{T}_{v_{n}}(G)}\{\vec{T} \cup Q: Q \in \mathbb{D}[V(G) \backslash V(T)]\} \tag{12}
\end{equation*}
$$

By Lemmas 1 and 2 and (12),

$$
\begin{align*}
\prod_{i=1}^{n-1} \sum_{e_{j} \in E_{G}\left(v_{i}\right)} \omega\left(e_{j}\right) & =\sum_{D \in \mathbb{D}_{0}^{*}} \omega(D)+\sum_{D \in \mathbb{D}^{*} \backslash \mathbb{D}_{0}^{*}} \omega(D) \\
& =\sum_{T \in \mathcal{T}(G)} \omega(\vec{T})+\sum_{T \in \mathcal{N S} \mathcal{T}_{v_{n}}(G)} \sum_{Q \in \mathbb{D}[V(G) \backslash V(T)]} \omega(\vec{T}) \omega(Q) \\
& =\sum_{T \in \mathcal{T}(G)} \omega(T)+\sum_{T \in \mathcal{N S \mathcal { S }} v_{v_{n}}(G)} \omega(T) \sum_{Q \in \mathbb{D}[V(G) \backslash V(T)]} \omega(Q) \\
& =\sum_{T \in \mathcal{T}(G)} \omega(T)+\sum_{T \in \mathcal{N S \mathcal { S }} v_{v_{n}}(G)} \omega(T) F(G-V(T), \omega) \tag{13}
\end{align*}
$$

Thus Theorem 3 is proved.

Observe that Theorem 2 follows directly from Theorem 3 by taking $u=v_{n}$ and $y_{j}=1$ for all $j \in[m]$.

## 4 Application

In the last section, we give some examples of applying Theorem 2 to determine spanning numbers of graphs.

Let $G$ be a connected multi-graph with $u \in V(G)$. For $1 \leq i \leq|V(G)|-2$, let $\mathscr{C}_{i}(G, u)$ (or simply $\mathscr{C}_{i}(u)$ ) be the set of connected induced subgraphs $G[S]$, where $u \in S \subset V(G)$, such that $|S|=i$ and $G-S$ has no isolated vertices. Clearly, $\left|\mathscr{C}_{1}(u)\right| \leq 1$ and $\left|\mathscr{C}_{2}(u)\right| \leq$ $\left|N_{G}(u)\right|$, where $N_{G}(u)$ is the set of neighbors of $u$ in $G$.

Observe that expression (11) in Theorem 2 is equivalent to the following one:

$$
\begin{equation*}
\tau(G)=\prod_{v \in V(G)-\{u\}} d_{G}(v)-\sum_{i=1}^{|V(G)|-2} \sum_{H \in \mathscr{C}_{i}(u)}\left(\tau(H) \prod_{v \in V(G)-V(H)} d_{G-V(H)}(v)\right) \tag{14}
\end{equation*}
$$

Now we apply (14) to determine $\tau\left(W_{4}\right), \tau\left(W_{4}^{\prime}\right)$ and $\tau\left(W_{5}^{\prime}\right)$, where $W_{4}$ is the wheel of order 5 and $W_{4}^{\prime}$ and $W_{5}^{\prime}$ are multi-graphs which can be obtained from $W_{4}$ and $W_{5}$ respectively by adding new edges parallel to edges incident with the central vertex, as shown in Figure 2 (b) and (c).

(a) $W_{4}$

(b) $W_{4}^{\prime}$

(c) $W_{5}^{\prime}$

Figure 2: Graphs $W_{4}, W_{4}^{\prime}$ and $W_{5}^{\prime}$

Let $u$ be the central vertex in $W_{4}$ as shown in Figure 2 (a). By (14), we have

$$
\begin{equation*}
\tau\left(W_{4}\right)=3^{4}-2^{4}-4 \times 1 \times\left(2 \times 1^{2}\right)-4 \times 3 \times 1^{2}=45 \tag{15}
\end{equation*}
$$

The above equality follows from the fact that $\left|\mathscr{C}_{1}(u)\right|=1,\left|\mathscr{C}_{2}(u)\right|=\left|\mathscr{C}_{3}(u)\right|=4, \tau(H)=1$ and $G-V(H) \cong C_{4}$ for $H \in \mathscr{C}_{1}, \tau(H)=i^{i-2}$ and $G-V(H)$ is a path of length $5-i$ for each $H \in \mathscr{C}_{i}(u)$ and $i=2,3$. Again, taking $u$ to be the central vertex in $W_{4}^{\prime}$, we have

$$
\begin{equation*}
\tau\left(W_{4}^{\prime}\right)=4^{4}-2^{4}-4 \times 2 \times\left(2 \times 1^{2}\right)-4 \times 8 \times 1^{2}=192 \tag{16}
\end{equation*}
$$

The above equality follows from the fact that $\left|\mathscr{C}_{1}(u)\right|=1,\left|\mathscr{C}_{2}(u)\right|=\left|\mathscr{C}_{3}(u)\right|=4, \tau(H)=1$ and $G-V(H) \cong C_{4}$ for $H \in \mathscr{C}_{1}, \tau(H)=2$ for each $H \in \mathscr{C}_{2}(u), \tau(H)=8$ for each $H \in \mathscr{C}_{3}(u)$, and $G-V(H)$ is a path of length $5-i$ for each $H \in \mathscr{C}_{i}(u)$ and $i=2,3$.

Similarly, taking $u$ to be the central vertex in $W_{5}^{\prime}$, we have

$$
\begin{equation*}
\tau\left(W_{5}^{\prime}\right)=4^{5}-2^{5}-5 \times 2 \times\left(2 \times 2 \times 1^{2}\right)-5 \times 8 \times 2-5\left(4 \times 3^{2}-2-2 \times 2\right)=722 \tag{17}
\end{equation*}
$$

The above equality follows from the fact that $\left|\mathscr{C}_{1}(u)\right|=1,\left|\mathscr{C}_{i}(u)\right|=5$ for $i=2,3,4$, $\tau(H)=1$ and $G-V(H) \cong C_{5}$ for $H \in \mathscr{C}_{1}, \tau(H)=2$ for each $H \in \mathscr{C}_{2}(u), \tau(H)=8$ for each $H \in \mathscr{C}_{3}(u), \tau(H)=4 \times 3^{2}-2-2=32$ for each $H \in \mathscr{C}_{4}(u)$, and $G-V(H)$ is a path of length $6-i$ for each $H \in \mathscr{C}_{i}(u)$ and $i=2,3,4$.

Our examples above show that as an alternative method of computing spanning trees in small graphs by hand, applying Theorem 2 is sometimes not less efficient than other methods.

Another potential usefulness of this formula is, maybe for some graph classes, we can use Theorem 2 to obtain a better upper bound for the number of spanning trees than Theorem 1. Corollary 1 below is an example.

Corollary 1. Let $G$ be a graph with degree sequence $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. Then

$$
\tau(G) \leq \prod_{i=1}^{n-1} d_{i}-\prod_{i=1}^{n-1}\left(d_{i}-1\right)
$$

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[^1]:    ${ }^{1}$ Wang algebra assumes that $x+x=0, x \cdot x=0$ and $x y=y x$ for any variables $x$ and $y$.

