The Neighbour Sum Distinguishing Relaxed Edge Colouring

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Abstract

A d-relaxed k-edge colouring is an edge colouring using colours from the set $\{1, \ldots, k\}$ such that each monochromatic set of edges induces a subgraph with maximum degree at most d. A neighbour sum distinguishing d-relaxed k-edge colouring of G is a d-relaxed k-edge colouring such that for each edge $uv \in E(G)$, the sum of colours taken on the edges incident to u is different from the sum of colours taken on the edges incident to v. By $\chi'^{d}_{\sum}(G)$ we denote the smallest value k in such a colouring of G.

In this paper, we prove that $\chi'^{2}_{\sum}(G) \leq 4$ for every connected subcubic graph with at least three vertices. For complete graphs with at least three vertices, we show that $\chi'^{d}_{\sum}(K_n) \leq 4$ if $d \in \{\lceil (n-1)/2 \rceil, \ldots, n-1\}$ and we also determine the exact value of $\chi'^{2}_{\sum}(K_n)$. Finally, we determine the value of $\chi'^{d}_{\sum}(T)$ for any tree T with at least three vertices.

Keywords: neighbour sum distinguish edge colouring, relaxed edge colouring, subcubic graphs
Mathematics Subject Classification: 05C15

1 Introduction

We consider undirected simple graphs and denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. For a vertex v of a graph G, $N_G(v)$ denotes the set of vertices which are adjacent to v, $d_G(v)$ denotes the degree of the vertex v in G, or simply $N(v), d(v)$ whenever the graph G is clear from the context. For undefined notations and terminology, we refer the reader to [2].

A k-*edge colouring* of G is a mapping $\omega : E(G) \longrightarrow \{1, \dots, k\}$. The edge colouring naturally induces a vertex colouring $\sigma_\omega : V(G) \longrightarrow \mathbb{N}$ given by

$$\sigma_\omega(v) = \sum_{u \in N_G(v)} \omega(vu)$$

for every $v \in V(G)$. We say that the edge colouring ω *distinguishes* vertices $v, w \in V(G)$ if $\sigma_\omega(v) \neq \sigma_\omega(w)$. The edge colouring (vertex colouring) is *proper* if adjacent edges (vertices) receive different colours.

The edge colouring which induces a proper colouring of vertices gained a lot of attention, especially 1-2-3 Conjecture, addressed in 2004 by Karoński, Łuczak and Thomason [8]. More precisely, by $\chi^e_\sum(G)$, we denote the smallest value k for which there exists a k-edge colouring ω of G (not necessarily proper) such that $\sigma_\omega(v) \neq \sigma_\omega(u)$ for every edge $uv \in E(G)$.

Conjecture 1. [8](1-2-3 Conjecture) *If G is a connected graph on at least 3 vertices, then $\chi^e_\sum(G) \leq 3$.*

The best known upper bound on $\chi^e_\sum$ is 5 and has been proved by Kalkowski, Karoński and Pfender [7]. Recently Przybyło [11] proved that $\chi^e_\sum(G) \leq 4$ for every d-regular graph with $d \geq 2$ and 1-2-3 Conjecture is true for d-regular graphs with $d \geq 10^8$. 1-2-3 Conjecture inspires a lot of studies on the original conjecture and variants of it. For more information on that topic, we refer to the survey [14].

The version of the edge colouring which distinguishes vertices and in which the edge colouring is proper has been introduced by Flandrin et al. [5]. If the k-edge colouring ω is proper and satisfies that $\sigma_\omega(v) \neq \sigma_\omega(u)$ for every edge $uv \in E(G)$, then we call such colouring the *neighbour sum distinguishing k-edge colouring*. By $\chi'_\sum(G)$, we denote the smallest value k for which G has a neighbour sum distinguishing k-edge proper colouring and we call it the *neighbour sum distinguishing index*.

Flandrin et al. [5] completely determined the neighbour sum distinguishing index for paths, trees, complete graphs and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 2. [5] *If G is a connected graph on at least 3 vertices and $G \neq C_5$, then $\chi'_{\sum}(G) \leq \Delta(G) + 2$.*

The objective of our work is to generalize both problems, by introducing a sum distinguishing edge colouring which lies between them. More precisely, we will allow a vertex to be incident to edges having the same colour, in a limited way. Such a colouring will be called a *d-relaxed k-edge colouring*. Namely, a *d-relaxed k-edge colouring* is a k-edge colouring such that each monochromatic set of edges induces a subgraph with maximum degree at most d. If the d-relaxed k-edge colouring ω satisfies that $\sigma_\omega(v) \neq \sigma_\omega(u)$ for every edge $uv \in E(G)$, then we call such colouring the *neighbour sum distinguishing d-relaxed k-edge colouring*. By $\chi'^{d}_{\sum}(G)$, we denote the smallest value k for which there is a neighbour sum distinguishing d-relaxed k-edge colouring of G. Hence when $d = 1$, we have that $\chi'^{d}_{\sum}(G) = \chi'_{\sum}(G)$ and when $d = \Delta(G)$, we have $\chi'^{d}_{\sum}(G) = \chi^{e}_{\sum}(G)$. In addition, for $1 \leq d \leq \Delta(G)$, the following inequality holds for the three parameters:

$$\chi^{e}_{\sum}(G) \leq \chi'^{d}_{\sum}(G) \leq \chi'_{\sum}(G)$$

Also, a natural lower bound for $\chi'^{d}_{\sum}(G)$ is given by the definition of a d-relaxed edge colouring:

$$\chi'^{d}_{\sum}(G) \geq \left\lceil \frac{\Delta(G)}{d} \right\rceil$$

We will see further that this bound may be tight in some cases.

This paper is organized as follows. In Section 2 we determine the value of $\chi'^{d}_{\sum}(T)$ for any tree T of order at least three and $1 \leq d \leq \Delta(T)$. In Section 3 we consider complete graphs with at least three vertices, we prove that $\chi'^{d}_{\sum}(K_n) \leq 4$ if $d \in \{\lceil (n-1)/2 \rceil, \ldots, n-1\}$ and compute the exact value of the parameter when $d = 2$. In Section 4 we prove that $\chi'^{2}_{\sum}(G) \leq 4$ for every connected subcubic graph of order at least three. It is worth noting that the best known upper bound on the neighbour sum distinguishing index of connected subcubic graphs of order at least three is 6 [6] (*i.e.* $\chi'_{\sum}(G) \leq 6$). In every figure throughout the paper, we will use the following notations: the colours of edges are the numbers next to the edges, the colours of the vertices induced by the edge colouring are boxed next to the vertices.

2 Trees

It is known that $\chi^{e}_{\sum}(T) \leq 2$ for any tree T, according to the following result of Chang et al. [3]:

Theorem 3. [3] *If G is a connected bipartite graph with at least three edges and $\delta(G) = 1$, then $\chi^{e}_{\sum}(G) \leq 2$.*

Furthermore, Flandrin et al. [5] obtain the following result for the neighbour sum distinguishing index of trees:

Theorem 4. [5] *Let T be a tree of order $n \geq 3$ and maximum degree Δ. Then $\chi'_{\sum}(T) = \Delta + 1$ if there are two adjacent vertices of degree Δ, and $\chi'_{\sum}(T) = \Delta$ otherwise.*

In the next result we give the exact value of $\chi'^{d}_{\sum}(T)$ for a tree T and for any d.

Theorem 5. *Let T be a tree of order $n \geq 3$ with the maximum degree Δ and $1 \leq d \leq \Delta$. We have*

$$\chi'^{d}_{\sum}(T) = \begin{cases} \frac{\Delta}{d} + 1, & \text{if } \Delta = 0 \mod d \text{ and there are two adjacent vertices of degree } \Delta \\ \left\lceil \frac{\Delta}{d} \right\rceil, & \text{otherwise} \end{cases}.$$

Proof. First recall that $\chi'^{d}_{\sum}(T) \geq \lceil \Delta/d \rceil$. Furthermore, if $\Delta = 0 \mod d$, then the edges incident to two adjacent vertices of maximum degree cannot be coloured with Δ/d colours, so if $\Delta = 0 \mod d$ and there are two adjacent vertices of degree Δ, then $\chi'^{d}_{\sum}(T) \geq \Delta/d + 1$. We prove by induction on n that ($\chi'^{d}_{\sum}(T) \leq \Delta/d + 1$ if $\Delta = 0 \mod d$ and there are two adjacent vertices of degree Δ) and $\chi'^{d}_{\sum}(T) \leq \lceil \Delta/d \rceil$, otherwise.

Observe that the theorem trivially holds if T is a star $K_{1,n-1}$, hence, in particular, for $n = 3$. Suppose the theorem is true for all trees of order at most $n-1$ and let T be a tree of order n. We may assume that $T \neq K_{1,n-1}$. Let P be a longest path in T and x be an endvertex of P. Since T is not a star $|V(P)| \geq 4$. Let x be chosen such that the only neighbour y of x in P is degree at most $\Delta - 1$, whenever T has only one vertex of degree Δ. Let $T' = T - x$. By our choice of x, $\Delta(T') = \Delta(T) = \Delta$ and y has only one neighbour, say z, of degree ≥ 2. Moreover, for every $v \in V(T') \setminus \{y\}$ we have $d_{T'}(v) = d_T(v)$ and $d_{T'}(y) = d_T(y) - 1$. Let $k = \Delta/d + 1$ if $\Delta = 0 \mod d$ and in T there are two adjacent vertices of degree Δ or $k = \lceil \Delta/d \rceil$, otherwise. Thus, by induction hypothesis, there is a neighbour sum distinguishing d-relaxed k-edge colouring of T'. Let ω be such an edge colouring. Let $F = \{c \in \{1,\dots,k\}$: there are d edges incident to y that are coloured with $c\}$.

If in T' there is a neighbour v of y different from z, then $\sigma_\omega(v) < \sigma_\omega(y)$, so in order to extend the colouring ω to a neighbour sum distinguishing d-relaxed k-edge colouring ω' of T it is sufficient to choose a colour c for yx such that $c \in \{1,\dots,k\} \setminus F$ and $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$. To prove that such a colour exists we consider two cases.

Case 1. $\Delta = 0 \mod d$

If in T there are two adjacent vertices of degree Δ, then in $\{1,\dots,k\} \setminus F$ there are two colours, hence, one of them is proper for the edge yx, *i.e.* if we put this colour on yx, then we obtain the colouring ω' such that $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$. Suppose, now, that in T there are no two adjacent vertices of degree Δ, in this case $k = \Delta/d$. Suppose that $d_T(y) = \Delta$. Then in $\{1,\dots,k\} \setminus F$ there is only one colour. We colour yx with this colour, let ω' be the resultant colouring. By our assumption $d_T(z) < \Delta$ what implies that $\sigma_{\omega'}(z) < \sigma_{\omega'}(y) = (1 + \ldots + k)d$. Suppose that $d_T(y) < \Delta$. If in $\{1,\dots,k\} \setminus F$ there are at least two colours, then we can choose one for yz to obtain a neighbour sum distinguishing d-relaxed k-edge colouring of T. Suppose that in $\{1,\dots,k\} \setminus F$ there is only one colour, say c. Thus for each colour $c' \in \{1,\dots,k\} \setminus \{c'\}$ there are d edges incident to y coloured with c' and there are at most $d-2$ edges incident to y coloured with c. In such a case it must be $d \geq 2$. Arguments $d \geq 2$, $k \geq 2$ and $|\{1,\dots,k\} \setminus F| = 1$ imply that y has at least one pendant vertex w in T' such that $\omega(yw) \neq c$. First we colour yx with c. If $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$, then ω' is a neighbour sum distinguishing d-relaxed k-edge colouring. If $\sigma_{\omega'}(z) = \sigma_{\omega'}(y)$, then we also recolour the edge yw, we put $\omega'(yw) := c$. The resultant edge colouring is neighbour sum distinguishing.

Case 2. $\Delta \neq 0 \mod d$

Thus $k = \lceil \Delta/d \rceil$ and $d \geq 2$. If in $\{1,\dots,k\} \setminus F$ there are two colours, then one of them is proper for the edge yx and we can extend the colouring ω to a neighbour sum distinguishing d-relaxed k-edge colouring of T. Suppose that $|\{1,\dots,k\} \setminus F| = 1$ and $c \in \{1,\dots,k\} \setminus F$. Similarly as in the case 1 we can observe that there are at most $d-2$ edges incident to y and coloured with c and y has at least one pendant vertex w in T' such that $\omega(yw) \neq c$. We colour yx with c. If $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$, then we are done. Otherwise, we also recolour the edge yw, we put $\omega'(yw) := c$. Finally, we obtain a neighbour sum distinguishing d-relaxed k-edge colouring ω'. □

3 Complete graphs

In [3], it was proved that complete graphs verify the 1-2-3 Conjecture. More precisely, we have $\chi^e_\sum(K_n) = 3$ for $n \geq 3$. Flandrin et al. [5] determined the neighbour sum distinguishing index of complete graphs:

Proposition 6. [5] *For every $n \geq 3$*

$$\chi'_\sum(K_n) = \begin{cases} n; & \text{if } n \text{ is odd} \\ n+1; & \text{if } n \text{ is even} \end{cases}.$$

We now consider the neighbour sum distinguishing d-relaxed edge colouring of complete graphs for several cases when $1 < d < \Delta(G)$.

Theorem 7. *Let $n \geq 4$ and $d \in \{\lceil (n-1)/2 \rceil, \ldots, n-1\}$ be two integers. We have $\chi'^d_\sum(K_n) \leq 4$.*

Proof. We prove the result of the theorem for $d = \lceil (n-1)/2 \rceil$, since having a higher value of d only gives more leeway. We will prove that there is a neighbour sum distinguishing d-relaxed 4-edge colouring of K_n.We use an inductive construction to get this colouring. The first case is K_4, and is depicted in Figure 1.

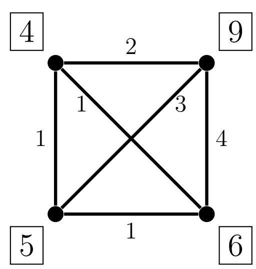


Figure 1: A neighbour sum distinguishing 2-relaxed 4-edge colouring for K_4.

Assume now that K_n has a neighbour sum distinguishing d-relaxed 4-edge colouring.We construct such a colouring of K_{n+1}. Let us call x the vertex we add to K_n.

First, assume that n is even. We order all the vertices of K_n by increasing colours. The first $n/2$ vertices of K_n (those with the smaller colours) are linked to x with an edge coloured with 3. The other vertices are linked to x with an edge coloured with 4.

Now, assume that n is odd. We order all the vertices of K_n by increasing colours. The first $\lceil n/2 \rceil$ vertices of K_n (those with the smaller colours) are linked to x with an edge coloured with 1. The other vertices are linked to x with an edge coloured with 2.

Those two constructions are depicted for K_5 and K_6 in Figure 2.

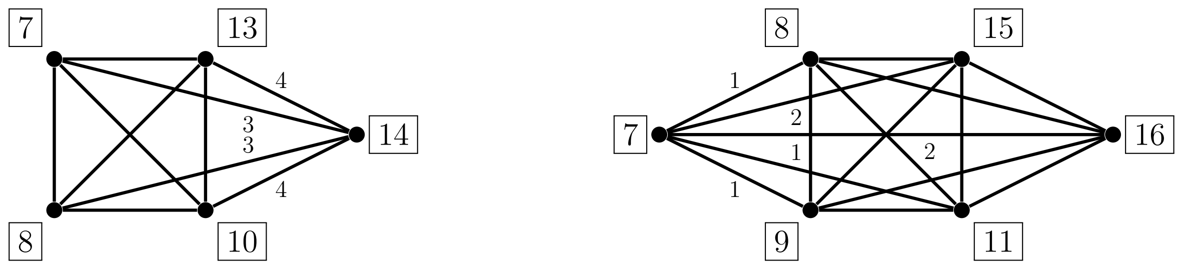


Figure 2: Neighbour sum distinguishing d-relaxed 4-edge colourings for K_5 and K_6 obtained by using our construction (for K_5, we have $d = 2$; for K_6, we have $d = 3$).

First, note that by alternating between those two constructions, the edge colouring will always be d-relaxed, since a vertex will gain an edge coloured with a certain colour once every two steps, and the construction starts from an even n.

Now, we need to verify that the colouring is neighbour sum distinguishing. The two following properties hold:

1. If n is even, then the highest colour is $2+3\frac{n-2}{2}+4\frac{n-2}{2}$ and the smallest colour is $\frac{n}{2}+2\frac{n-2}{2}$.

2. If n is odd, then the highest colour is $3\frac{n-1}{2}+4\frac{n-1}{2}$ and the smallest colour is $3+\frac{n-1}{2}+2\frac{n-3}{2}$.

Indeed, when constructing the new colouring, if n is even then we add the vertex x with colour $3n/2+4n/2$ which is the new highest colour in the graph, and we add 3 to the value of the smallest colour; and if n is odd then we add the vertex x with colour $(n+1)/2+2(n-1)/2$ which is the new smallest colour in the graph, and we add 2 to the value of the highest colour.

Those two properties remaining true during our construction, the vertex x that we add is always distinguished from all the vertices that were already in the graph. Furthermore, the vertices that were in the graph are still neighbour sum distinguished, since we added the smallest value to the smallest colours. Thus, two vertices that had different colours cannot have the same colour in the new colouring, which concludes the proof. $\square$

However, we also prove that this bound of 4 is not necessarily tight:

Observation 8. *For $n \in \{3, \dots, 7\}$ and $d = \lceil (n-1)/2 \rceil$, we have $\chi'^{d}_{\sum}(K_n) = 3$.*

Proof. First, note that a d-relaxed 2-edge colouring would not give us enough possible labels to be neighbour sum distinguishing, so we have $\chi'^{d}_{\sum}(K_n) \geq 3$. Now, to prove the statement, we construct a neighbour sum distinguishing d-relaxed 3-edge colouring of $K_3, \dots, K_7$. This is shown in Figure 3 (the caption indicates how to read the constructions). □

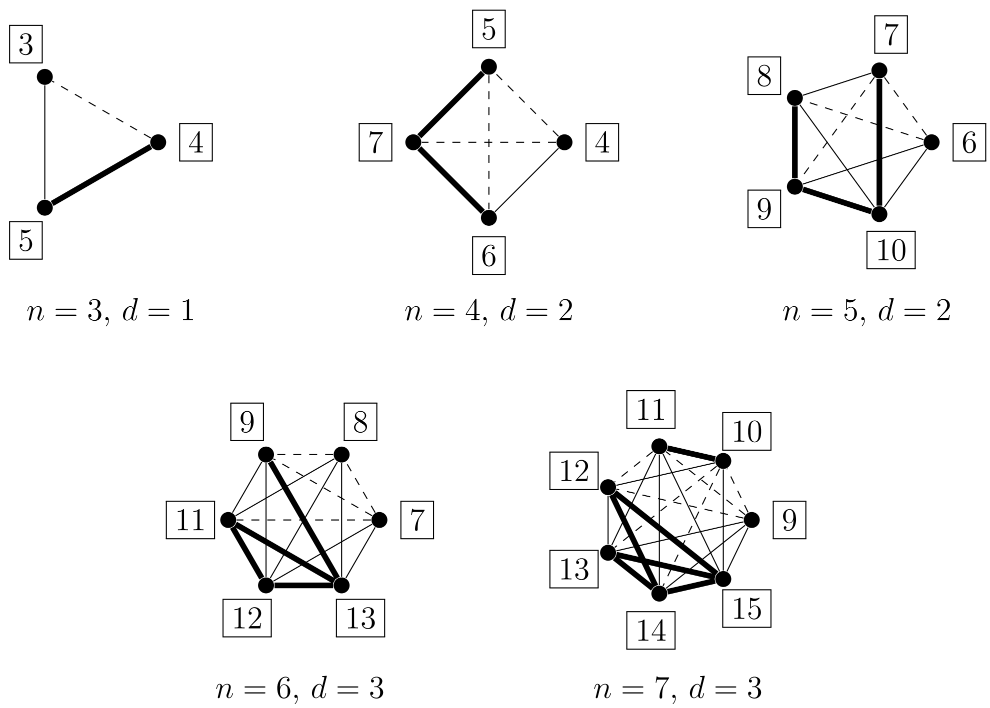


Figure 3: A neighbour sum distinguishing d-relaxed 3-edge colouring of $K_3, \dots, K_7$. Dashed lines indicate that the edges are coloured with 1, normal lines indicate that the edges are coloured with 2, and thick lines indicate that the edges are coloured with 3.

However, note that the constructions shown in Figure 3 were obtained by hand, and there does not seem to be a simple way to construct a neighbour sum distinguishing d-relaxed 3-edge colouring of K_{n+1} from the colouring of K_n. Thus, the exact value of $\chi'^{d}_{\sum}(K_n)$ remains open for $n \geq 8$ and $d \in \{\lceil (n-1)/2 \rceil, \dots, n-1\}$.

We now compute the exact value of the parameter for complete graphs when $d = 2$.

Theorem 9. *Let $n \geq 4$. We have $\chi'^2_{\sum}(K_n) = \lceil \frac{n-1}{2} \rceil + 1$ if $n \neq 3 \bmod 4$ and $\chi'^2_{\sum}(K_n) = \lceil \frac{n-1}{2} \rceil + 2$ otherwise.*

Proof. We first show that for all $n \geq 4$, there exist 2-relaxed distinguishing colourings having this number of colours. There are four cases according to $n \bmod 4$, that all share a common basis that we present now. We denote and order the vertices of K_n by $\{x_{-1}, \dots, x_{-\lceil n/2 \rceil}, x_1, \dots, x_{\lfloor n/2 \rfloor}\}$. Given a vertex x_i of K_n, the vertex $x_i + 1$ denotes its successor in the above ordering. In addition, we will consider this ordering in a cyclic way, *i.e.* $x_{-\lceil n/2 \rceil} + 1 = x_1$ and $x_{\lfloor n/2 \rfloor} + 1 = x_{-1}$.

The following algorithm labels all the edges of K_n to provide a 2-relaxed colouring (but not distinguishing yet):

1. Label the edge $x_{-\lceil n/2 \rceil} - k, x_1 + k$ with colour 1 for all $k \in \{0, \dots, \lfloor n/2 \rfloor - 1\}$.
2. Label the edge $x_{-\lceil n/2 \rceil} - k, x_2 + k$ with colour 1 for all $k \in \{0, \dots, \lceil n/2 \rceil - 2\}$.
3. For each edge $x_i x_j$ coloured 1, label the edge $x_i + c - 1, x_j + c - 1$ with colour c for all $c \in \{2, \dots, \lfloor n/2 \rfloor\}$.
4. If n is odd, label the edge $x_{\lfloor n/2 \rfloor} - k, x_{-1} + k$ with colour $\lceil n/2 \rceil$ for all $k \in \{0, \dots, \lfloor n/2 \rfloor - 1\}$.

Figure 4 depicts the edges coloured 1 after steps 1 and 2 are done. Two cases are given, K_8 and K_9. Figure 5 illustrates the edges coloured 2 after execution of step 3. They correspond to a rotation of the edges coloured 1. For the colour $\lceil n/2 \rceil$, it is concerned by step 3 when n is even and step 4 when n is odd. In the latter case, only the edges of step 1 are rotated, as depicted by Figure 6.

First note that the algorithm above labels exactly $n(n-1)/2$ edges, *i.e.* the number of edges of K_n. Indeed, the first three steps label $(\lfloor n/2 \rfloor + \lceil n/2 \rceil - 1) \times (\lfloor n/2 \rfloor - 1)$ edges, which equals $n(n-1)/2$ if n is even and $(n-1)^2/2$ if n is odd. In the latter case, the missing edges are given by step 4 that labels $(n-1)/2$ edges.

We now prove that no edge is labeled twice. For that purpose, imagine the vertices of K_n as the vertices of a regular n-gon and edges are straight segments, as depicted by the figures above. Thus, all the edges coloured 1 by step 1 are parallel to $x_{-\lceil n/2 \rceil}, x_1$. There is no edge coloured twice according to the restriction given on k. For the same reason, all the edges coloured 1 by step 2 are parallel to $x_{-\lceil n/2 \rceil}, x_2$ and pairwise distinct. Since the two directions $x_{-\lceil n/2 \rceil}, x_1$ and $x_{-\lceil n/2 \rceil}, x_2$ are not parallel,

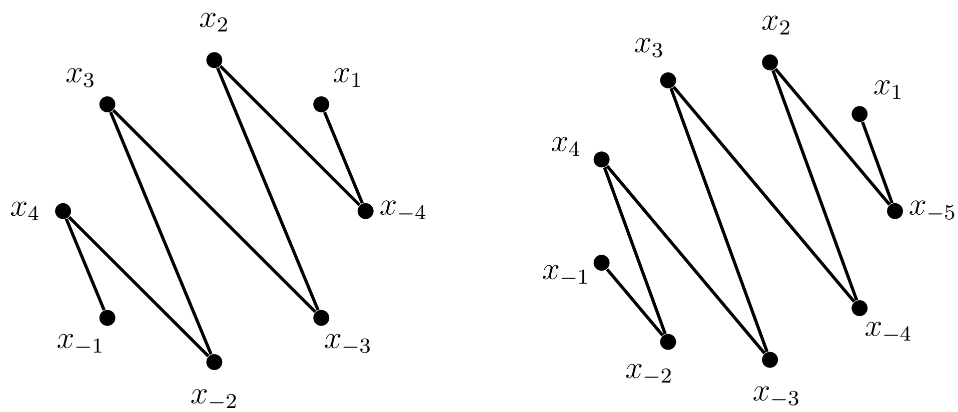


Figure 4: Edges of K_8 and K_9 labeled 1 after steps 1 and 2.

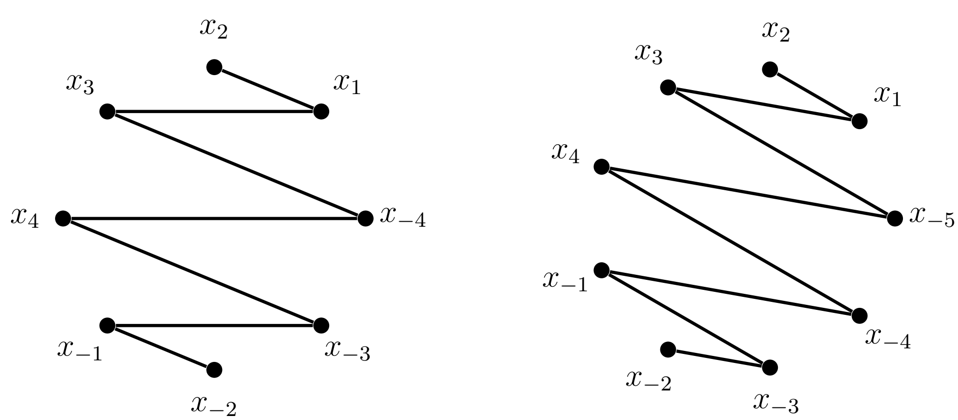


Figure 5: Edges of K_8 and K_9 labeled 2 after step 3.

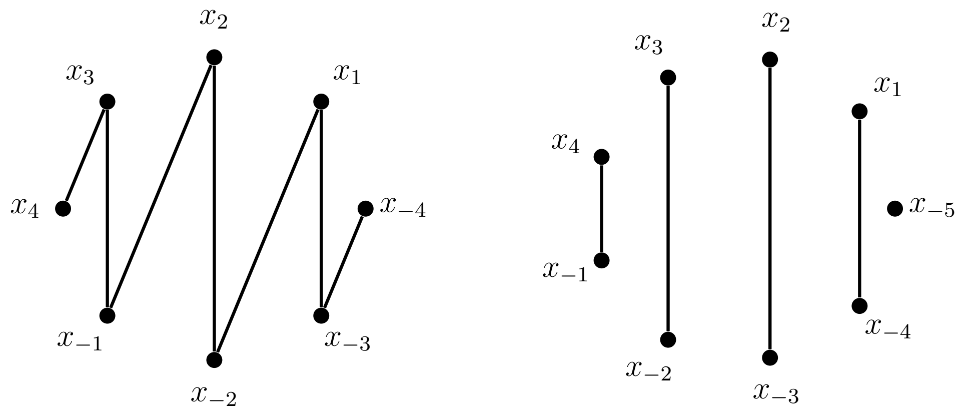


Figure 6: Edges of K_8 and K_9 labeled $\lceil n/2 \rceil$ after step 3 or step 4.

all the edges coloured by steps 1 and 2 are distinct. Figure 4 illustrates this property. For steps 3 and 4, each edge coloured with $c > 1$ is obtained by rotating the edges of colour 1 (see Figure 5 and 6). Consequently, all the edges of a given colour c and chosen by the algorithm are pairwise distinct. In addition, edges of colour c are parallel to $x_c - 1, x_c$ and $x_c - 1, x_c + 1$. Since $c \leq \lfloor n/2 \rfloor$, one can remark that on the n-gon, all the directions $x_c - 1, x_c$ and $x_c - 1, x_c + 1$ are pairwise distinct. Thus, two edges of different colours necessarily have different directions. This ensures that the above algorithm labels all the edges exactly once.

Finally, each vertex is incident to at most twice the same colour. This is true for colour 1 as each vertex is concerned at most once by step 1 and at most once by step 2. Consequently, this is also true for the other colours, as step 3 (and step 4 when n is odd) consist in a rotation of the edges labeled 1.

If the above algorithm provides a 2-relaxed edge colouring, it is not distinguishing as the same sum may appear on two vertices. More precisely, we can compute the sum on each vertex. We define Σ_2 as the quantity

$$\Sigma_2 = 2 \times \sum_{k=1}^{\lceil n/2 \rceil} k$$

One can remark that after steps 1 and 2, each vertex x_i with $|i| \neq 1$ is incident to two edges with label 1, and vertices x_1 and x_{-1} are incident to exactly one edge with label 1 (see Figure 4). Since step 3 consists in "rotating" the edges labeled with colour

1, we have that each vertex x_i is incident twice to every colour $c \in \{1, \ldots, \lfloor n/2 \rfloor\}$ for $c \neq |i|$ and once to colour $|i|$. When n is odd, step 4 ensures that each vertex is incident to exactly one edge with colour $\lceil n/2 \rceil$, except $x_{-\lceil n/2 \rceil}$ that is not incident to this colour (see Figure 6). Consequently, the above algorithm yields for each vertex x_i of K_n:

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| & \text{for each vertex } x_i \text{ of } K_n \text{ when } n \text{ is even} \\ \Sigma_2 - |i| - \lceil n/2 \rceil & \text{for each vertex } x_i \text{ of } K_n \text{ when } n \text{ is odd} \end{cases}$$

In other words, the edge colouring is not distinguishing because vertices x_i and x_{-i} have the same sum. We will now break the ties by replacing some edges of colour 1 with an additional colour. Four cases are considered:

Case $\mathbf{n = 0}$ mod 4 : recolour the edges $x_{-n/2} - k, x_1 + k$ with colour $n/2 + 1$ for all $k \in \{0, \ldots, n/4 - 1\}$. Hence, these edges form a perfect matching of $K_{n/2}$. Let V_{change} be the set of vertices of K_n incident to this new colour. We have $V_{change} = \{x_{-n/4-1}, \ldots, x_{-n/2}, x_1, \ldots, x_{n/4}\}$. Since each vertex of V_{change} is incident to exactly one recoloured edge and the colour $n/2 + 1$ has not been used yet, it remains a 2-relaxed colouring. In addition, the new values of σ are the following:

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| + n/2 & \text{if } x_i \text{ is in } V_{change} \\ \Sigma_2 - |i| & \text{otherwise} \end{cases}$$

Since for all i in $\{1, \ldots, n/2\}$, exactly one vertex among $\{x_i, x_{-i}\}$ is in V_{change}, all the sums are now different and cover all the values of the interval $[\Sigma_2 - n/2, \Sigma_2 - 1 + n/2]$.

Case $\mathbf{n = 2}$ mod 4 : recolour the edges $x_{-n/2} - k, x_2 + k$ with colour $n/2 + 1$ for all $k \in \{0, \ldots, \lfloor n/4 \rfloor - 1\}$. Recolour also the edge $x_{-n/2}, x_1$ with the same colour. Hence the two edges of colour 1 incident to $x_{-n/2}$ have been recoloured, so that $\sigma(x_{-n/2}) = \Sigma_2 - n/2 - 2 + 2(n/2 + 1) = \Sigma_2 + n/2$. For the rest, the set V_{change} can be defined as in the previous case, and it has the same properties, *i.e.* half the vertices have their sum changed. Consequently:

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| + n/2 & \text{if } x_i \text{ is in } V_{change} \text{ and } x_i \neq x_{-n/2} \\ \Sigma_2 + n/2 & \text{if } x_i = x_{-n/2} \\ \Sigma_2 - |i| & \text{otherwise} \end{cases}$$

and the values $\sigma(x_i)$ cover $[\Sigma_2 - n/2, \Sigma_2 - 1] \cup [\Sigma_2 + 1, \Sigma_2 + n/2]$, *i.e.* the labeling is distinguishing.

Case $\mathbf{n = 1}$ mod 4 : recolour the edges $x_{-1} - k, x_{-2} + k$ (initially coloured 1) with colour $\lfloor n/2 \rfloor + 1$ for all $k \in \{0, \ldots, \lfloor n/4 \rfloor - 1\}$. Since n is odd, the colour $\lfloor n/2 \rfloor + 1$

used to replace the colour 1 is already used in step 4 of the algorithm. However, this colouring remains 2-relaxed as in step 4, each vertex is incident to at most one vertex of this colour. After this recolouring, by defining V_{change} as previously:

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| - 1 & \text{if } x_i \text{ is in } V_{change} \\ \Sigma_2 - |i| - \lceil n/2 \rceil & \text{otherwise} \end{cases}$$

One can easily check that it covers all the n values in $[\Sigma_2 - n - 1, \Sigma_2 - 2]$.

Case $\mathbf{n = 3}$ mod $\mathbf{4}$: recolour the edges $x_{\lfloor n/2 \rfloor} - k, x_{-2} + k$ (initially coloured 1) with colour $\lfloor n/2 \rfloor + 2$ for all $k \in \{0, \ldots, \lfloor n/4 \rfloor - 1\}$. Recolour also the edge x_{-2}, x_{-1} with the same colour. Unlike the previous case, we here add a new colour instead of using the colour of step 4. Consequently, the colouring remains 2-relaxed. One can also easily check that it is distinguishing since half the vertices have their sum changed (and exactly one in each pair (x_i, x_{-i})).

To sum up, the above algorithm uses $\lfloor n/2 \rfloor + 1$ colours when n is even, $\lceil n/2 \rceil$ colours when $n = 1 \bmod 4$, and $\lceil n/2 \rceil + 1$ colours when $n = 3 \bmod 4$. This corresponds to the values given in the statement of the theorem.

We now prove that these values are lower bounds. When n is even, imagine there exists a 2-relaxed distinguishing colouring with $\lfloor n/2 \rfloor$ colours. In that case, since the degree of each vertex is $(n-1)$, there is only one colour in $\{1, \ldots, \lfloor n/2 \rfloor\}$ missing in the incident edges of any vertex. Thus, there will be two vertices not distinguished. When $n = 1 \bmod 4$, this argument remains true as there is no 2-relaxed colouring having at most $\lfloor n/2 \rfloor$ colours. When $n = 3 \bmod 4$, it is less obvious to show that $\lceil n/2 \rceil$ colours are not sufficient. By way of contradiction, imagine that there exists a 2-relaxed distinguishing colouring with $\lceil n/2 \rceil$ colours. Since the degree of each vertex is $(n-1)$, there are exactly two colours in $\{1, \ldots, \lceil n/2 \rceil\}$ missing in the incident edges of any vertex (possibly twice the same colour). Then the value $\sigma(x_i)$ of any vertex x_i ranges in $[\Sigma_2 - 2\lceil n/2 \rceil, \Sigma_2 - 2] = [\Sigma_2 - (n+1), \Sigma_2 - 2]$. As this interval is of size n, for each value s in $[\Sigma_2 - (n+1), \Sigma_2 - 2]$, there exists x_i such that $\sigma(x_i) = s$. Now consider the quantity

$$Q = \sum_{x_i \in V} \sigma(x_i)$$

The above remark says it satisfies:

$$Q = \sum_{i=2}^{n+1} (\Sigma_2 - i) = n\Sigma_2 - \frac{(n+1)(n+2)}{2} - 1$$

Since Σ_2 is even and $n = 3 \bmod 4$, it ensures that Q is odd.

Now consider the values $\sigma(x_i)$ and for all i, let c_i and c'_i be the two values in $\{1, \ldots, \lceil n/2 \rceil\}$ such that $\sigma(x_i) = \Sigma_2 - c_i - c'_i$. For each value s in $\{1, \ldots, \lceil n/2 \rceil\}$, let G_s be the subgraph of K_n induced by the edges of colour s. Since the colouring is 2-relaxed, G_s has maximum degree 2. Hence there is an even number of vertices with degree 1. By translating this property in our original context, a vertex x_i of degree 2 in G_s satisfies $c_i \neq s$ and $c'_i \neq s$, a vertex of degree 0 in G_s satisfies $c_i = c'_i = s$, and a vertex of degree 1 has either c_i or c'_i equal to s (but not both). Consequently, there is an even total number of c_i and c'_i (added together) of value s. We denote this number $val(s)$. We now rewrite Q as follows:

$$Q = \sum_{x_i \in V} \sigma(x_i) = \sum_{x_i \in V} (\Sigma_2 - c_i - c'_i) = n\Sigma_2 - \sum_{x_i \in V} (c_i + c'_i) = n\Sigma_2 - \sum_{s=1}^{\lceil n/2 \rceil} val(s)$$

Since Σ_2 is even, the above remark ensures that Q is even, a contradiction.

□

Table 1 summarizes all the known results about the value of $\chi'^{d}_{\sum}(K_n)$.

Value of d	Value of $\chi'^{d}_{\sum}(K_n)$
1	n if n is odd $n+1$ if n is even
2	$\lceil (n-1)/2 \rceil + 1$ if $n \neq 3 \bmod 4$ $\lceil (n-1)/2 \rceil + 2$ if $n = 3 \bmod 4$
$2 < d < \lceil (n-1)/2 \rceil$	open
$\lceil (n-1)/2 \rceil < d \leq n-1$	3 if $n \in \{3, ..., 7\}$ ≤ 4 if $n > 7$

Table 1: Values of $\chi'^{d}_{\sum}(K_n)$

4 Subcubic graphs

Karoński et al. in [8] proved that every subcubic graph with at least three vertices admits a neighbour sum distinguishing 3-relaxed 3-edge colouring. On the other hand the best known upper bound for the neighbour sum distinguishing index of subcubic graphs was proved by Huo et al. [6] and is equal to 6 (*i.e.* every subcubic graph of order at least three has a neighbour sum distinguishing 1-relaxed 6-edge colouring). In this section we prove that every connected subcubic graph with at least three vertices has a neighbour sum distinguishing 2-relaxed 4-edge colouring.

In the proof of this result, we need the following proposition observed by Flandrin et al. in [5].

Proposition 10. [5] *$\chi'_{\sum}(C_5) = 5$, $\chi'_{\sum}(C_m) = 3$ if $m = 0 \mod 3$ and $\chi'_{\sum}(C_m) = 4$, otherwise.*

Some proofs in this section are based on the following theorem of Alon [1].

Theorem 11 (Combinatorial Nullstellensatz [1])**.** *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose the degree $deg(P)$ of P equals $\sum_{i=1}^{n} k_i$, where each k_i is a nonnegative integer, and suppose the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in P is nonzero. Then if $S_1, \dots, S_n$ are subsets of $\mathbb{F}$ with $|Si| > k_i$, there are $s_1 \in S_1, \dots, s_n \in S_n$ so that $P(s_1, \dots, s_n) \neq 0$.*

The main result of this section is that all connected subcubic graphs have a neighbour sum distinguishing 2-relaxed 4-edge colouring (Corollary). To prove this result, we prove by induction a stronger statement that is true for any subcubic graph that is neither K_2 nor C_5.

Theorem 12. *Let G be a connected subcubic graph such that $G \notin \{K_2, C_5\}$. There is a neighbour sum distinguishing 2-relaxed 4-edge colouring of G such that all the vertices of degree 2 have their two adjacent edges of different colours.*

In order to prove this result, we need a first result that allows to simplify graphs having a pending C_5 (i.e. an induced C_5 connected to the rest of the graph by only one vertex).

Lemma 13. *Let G be a subcubic graph. Assume there exists in G an induced C_5, $C = \{u_0, u_1, u_2, u_3, u_4\}$, such that only one vertex, u_0, is connected to the rest of G. Let $G' = G \setminus \{u_1, u_2, u_3, u_4\}$. If G' satisfies Theorem 12 then G also satisfies Theorem 12.*

Proof. Let ω be a neighbour sum distinguishing 2-relaxed 4-edge colouring of G' such that all the vertices of degree 2 have their two adjacent edges of different colours.

Since G is subcubic, u_0 has degree 1 in G'. Let v be its neighbour in G' and c_0 the colour of the edge u_0v. Let $c_1 \in \{1, 2, 3, 4\}$ such that $c_1 \neq c_0$ and $c_1 + 2c_0 \neq \sigma_\omega(v)$. Let $c_3 \in \{1, 2, 3, 4\}$ such that $c_3 \neq c_0, c_1$.

Then we can extend the colouring ω by giving colour c_0 to edges u_4u_0 and u_1u_2, colour c_1 to edges u_0u_1 and u_3u_4 and colour c_2 to edge u_2u_3. Then all the adjacent vertices of C are distinguished as well as u_0 and v. Moreover, vertices u_1 to u_4 are adjacent to edges of different colours, hence this colouring satisfies the conditions of Theorem 12.

□

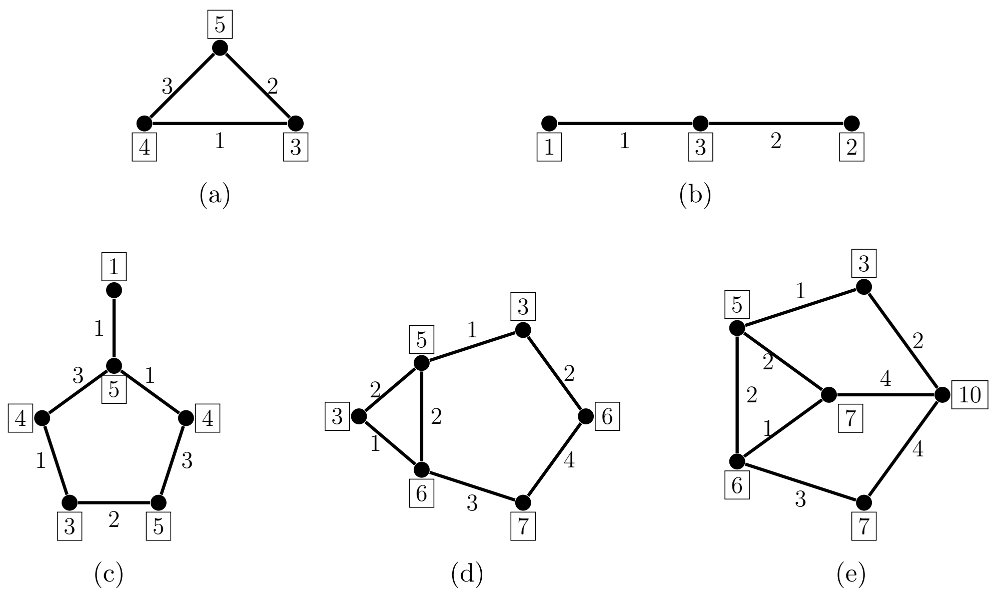


Figure 7: A neighbour sum distinguishing 2-relaxed 4-edge colouring for small graphs satisfying Theorem 12.

We are now ready to present the proof of Theorem 12.

Proof of Theorem 12. We prove Theorem 12 by induction on the number of edges of G. Figure 7a and Figure 7b show that the result is true for all connected graphs having three vertices. Now consider a connected subcubic graph G with at least four vertices and that is not C_5. If G is the graph of Figure 7c, then the result is true. Otherwise, from Lemma 13, one can assume that G has no pending C_5. The proof is now organized with four cases, according to the girth and the minimum degree of G.

Case 1. $\delta(G) = 1$

Let u be a vertex of degree 1. Since G is connected, has at least four vertices and no pending C_5, $G-u$ satisfies the assumptions of the theorem. Hence the graph $G-u$ has a colouring ω that satisfies the theorem. We can extend ω to G as follows. Let v be the neighbour of u. If v has degree 2 in G, then we put to the edge uv a colour different from the colour already adjacent to v and such that v is distinguished from its other neighbour. If v has degree 3 in G, let v_1 and v_2 be the neighbours of v in $G-u$. Then there are at most two forbidden colours for $\omega(uv)$ to distinguish v from v_1 and v_2. Thus there are two remaining possible colours for uv (since v has already two different colours in its neighbourhood). In both cases, we can extend the colouring ω.

Case 2. G has a triangle.

Subcase 2.1. G has a vertex of degree 2 in a triangle

Let u be such a vertex and v, w be its (adjacent) neighbours. Let $G' = G - u$. Figure 7d shows that the result is true if G' is a C_5. Since G has at least four vertices, G' is not K_2. By induction G' satisfies the theorem so has a 2-relaxed 4-edge colouring ω that satisfies the theorem.

Since G has at least four vertices, we can assume that w has degree 3 and let w_1 be its other neighbour. If v has degree 2 in G, then the colour of uv must be different from the colour of vw and u must be distinguished from w, so there are at most two forbidden colours for uv. If v has degree 3 in G, then v must be distinguished from its neighbour in G' and u must be distinguished from w, so again there are at most two forbidden colours for uv. Let S_1 be the set of colours that are not forbidden for uv. Similarly, there are at most two forbidden colours for uw. Let S_2 be the set of colours that are not forbidden for uw. Thus, if we colour uv and uw with colours from S_1 and S_2 then we obtain a colouring that distinguishes vertices of G', distinguishes u from w, u from v, and guarantees that v is adjacent to edges coloured differently. Let x_1, x_2 be colours attributed to edges uv, uw, respectively. To satisfy all conditions of the theorem for colours x_1, x_2 it must verify:

- $x_1 \neq x_2$, since the vertex u must be adjacent to edges coloured differently;
- $x_1 + \alpha \neq x_2 + \omega(ww_1)$ (where α is a colour of the edge vv_1, $v_1 \in N(v) \setminus \{u, w\}$ if v_1 exists and $\alpha = 0$ otherwise), since v and w must be distinguished.

We construct a polynomial

$$P(x_1, x_2) = (x_1 - x_2)(x_1 + \alpha - x_2 - \omega(ww_1)).$$

The coefficient of the monomial x_1x_2 is equal to -2, so is non-zero. Hence, by Theorem 11, there are $x_1 \in S_1, x_2 \in S_2$ such that $P(x_1, x_2) \neq 0$, since $|S_1| > 1, |S_2| > 1$.

We put $\omega(uv) = x_1, \omega(uw) = x_2$, and the resulting colouring satisfies all the assumptions of the theorem.

Subcase 2.2. All the vertices that are in a triangle have degree 3

In this case, we can assume that there exists a vertex u such that there is exactly one edge in the graph induced by its neighbourhood. Indeed, take a vertex u in a triangle. Figure 1 shows that the result is true if G is K_4. Thus we can assume that G is not K_4, which means that there are at most two edges in $G[N(u)]$. Suppose there are exactly two edges, say vw and wy. Vertices v and y are in triangles so they must have degree 3. Consider now v instead of u. The third neighbour of v, says v_1 is not y and thus is not adjacent to u and also not to w whose neighbours are v, u and y.

Let u be such a vertex, v_1, v_2 and v_3 its neighbours with v_2 and v_3 adjacent. Let w_2 and w_3 be the other neighbours of v_2 and v_3 (we potentially have $w_2 = w_3$).

Consider $G' = G - u$. If G' is connected, it cannot be K_2. If it is C_5 then G is isomorphic to the graph of Figure 7e and the theorem is satisfied. If G' is not connected, it must have two components, say G_1, containing v_1 and G_2, containing v_2 and v_3. We can assume that the component G_1 is not isomorophic to K_2 (there would be a vertex of degree 1) nor C_5 (it would be a pendant C_5). Vertices v_2 and v_3 have degree 3 in G so G_2 is not isomorphic to K_2. If G_2 is isomorphic to C_5 then one can replace u by u_2 and then $G - u$ is connected.

Thus, we can assume that all the connected components of G' are distinct from K_2 and C_5 and, by induction, there exists a colouring ω of G' that satisfies the theorem (we coulour independently the components if G' is not connected). We now extend the colouring ω by colouring the three edges adjacent to u.

There are (at least) two possible colours for uv_1 to distinguish v_1 from its two other neighbours, if v_1 has degree 3, or, if v_1 has degree 2, to distinguish v_1 from its neighbour and ensure that v_1 is adjacent to two distinct colours.

Furthermore, v_2 must be distinguished from w_2 and v_3 must be distinguished from w_3, so there at least three possible colours for uv_2 and three possible colours for uv_3 to ensure that. Let S_1, S_2, S_3 be sets of possible colours for uv_1, uv_2, uv_3, respectively, so $|S_2| = |S_3| = 3$ and $|S_1| = 2$. If we colour uv_1, uv_2, uv_3 with colours from S_1, S_2, S_3, then all the vertices of $N(u)$ have at least two distinct colours and are distinguished with their neighbours outside $N[u]$.

Let denote x_1, x_2, x_3 be colours attributed to edges uv_1, uv_3, uv_3, respectively. To distinguish u from its neighbours and make sure that u is adjacent to edges coloured differently for colours x_1, x_2, x_3, it must hold:

- $x_2 \neq x_3$ or $x_1 \neq x_2$ or $x_1 \neq x_3$, since the colouring must be 2-relaxed;

- $x_2 + \omega(v_2w_2) \neq x_3 + \omega(v_3w_3)$, since v_2 and v_3 must be distinguished;
- $x_1 + x_3 \neq \omega(v_2w_2) + \omega(v_2v_3), x_1 + x_2 \neq \omega(v_3w_3) + \omega(v_2v_3), x_2 + x_3 \neq \sigma_\omega(v_1)$, since u must be distinguished from its neighbours.

We construct a polynomial

$$\begin{aligned} P(x_1, x_2, x_3) = \ & (x_2 - x_3)(x_2 - x_3 + \omega(v_2w_2) - \omega(v_3w_3))(x_1 + x_3 - \omega(v_2w_2) - \omega(v_2v_3)) \\ & (x_1 + x_2 - \omega(v_3w_3) - \omega(v_2v_3))(x_2 + x_3 - \sigma_\omega(v_1)). \end{aligned}$$

Consider the coefficient of the monomial $x_1x_2^2x_3^2$, observe that this coefficient in P is the same as in the following polynomial:

$$P(x_1, x_2, x_3) = (x_2 - x_3)(x_2 - x_3)(x_1 + x_3)(x_1 + x_2)(x_2 + x_3).$$

The coefficient of the monomial $x_1x_2^2x_3^2$ is non-zero (is equal to -2). Since $|S_1| > 1, |S_2| > 2, |S_3| > 2$, Theorem 11 implies that there are $x_1 \in S_1, x_2 \in S_2, x_3 \in S_3$ such that $P(x_1, x_2, x_3, x_4) \neq 0$. Thus we can extend ω to G by assigning $\omega(uv_i) = x_i$ for $i = 1, 2, 3$, which proves the theorem.

Case 3. G has a C_4

Let u_1, u_2, u_3, u_4 be the vertices of a 4-cycle C and let G' be obtained from G by removing the four edges of C (but not the vertices).

We can assume that G' has no component isomorphic to K_2 (otherwise there would be a vertex of degree 1 or a triangle in G). Furthermore, the only vertices that have changed their neighbourhood from G to G' are the vertices u_i, that have either degree 1 or degree 0 in G'. Since G has no component isomorphic to C_5, it is also the case for G' and we can apply induction. Thus, there is a colouring ω of G' that satisfies the two conditions of the theorem. We now extend ω by colouring the four edges of the cycle to obtain a neighbour sum distinguishing 2-relaxed edge colouring of G with the vertices of degree 2 adjacent to different colours.

For $i \in \{1, 2, 3, 4\}$, let x_i be the colour that will be assigned to u_iu_{i+1} (indices are taken modulo 4). If u_i has a neighbour, let v_i be this neighbour, $c_i = \omega(u_iv_i)$ and $\alpha_i = \sigma_\omega(v_i) - c_i$. Otherwise, let $c_i = 0 = \alpha_i = 0$.

If for each $i \in \{1, 2, 3, 4\}$ the following conditions are satisfied, then putting the colour x_i to u_iu_{i+1} will extend ω to a colouring satifying the theorem (indices are again taken modulo 4):

- $x_{i-1} + c_i \neq x_{i+1} + c_{i+1}$, to distinguish u_i and u_{i+1};

- $x_i + x_{i-1} \neq \alpha_i$, to distinguish u_i and v_i (if v_i exists, otherwise we keep the condition for simplicity reasons);
- $x_i \neq x_{i-1}$, to have at least two colours adjacent to u_i.

We construct a polynomial

$$P(x_1, x_2, x_3, x_4) = \Pi_{i=1}^{4} \left((x_{i-1} - x_{i+1} + c_i - c_{i+1})(x_i + x_{i-1} - \alpha_i)(x_i - x_{i-1})\right).$$

Consider the coefficient of the monomial $x_1^3x_2^3x_3^3x_4^3$, observe that this coeeficient in P is the same as in the following polynomial:

$$P'(x_1, x_2, x_3, x_4) = \Pi_{i=1}^{4} \left((x_{i-1} - x_{i+1})(x_i^2 - x_{i-1}^2)\right).$$

The coefficient of the monomial $x_1^3x_2^3x_3^3x_4^3$ is non-zero (is equal to 8). Since there are initially four possible values for each x_i, Theorem 11 implies that there are $x_i \in \{1, 2, 3, 4\}$ for $i = 1, 2, 3, 4$ such that $P(x_1, x_2, x_3, x_4) \neq 0$. Thus we can extend ω to G by assigning $\omega(u_iu_{i+1}) = x_i$ for $i \in \{1, ..., 4\}$, which proves the theorem.

Case 4. G has girth at least 5. We consider two subcases according to whether the graph is cubic or not.

Subcase 4.1. $\delta(G) = 2$

By Proposition 10, the theorem is true for all cycles except C_5. Thus we may assume that G has a vertex of degree 3. Therefore, since G is connected, one can find a vertex of degree 2, say u, which has a neighbour of degree 3. Let v, w be the neighbours of u such that $deg(w) = 3$. Let w_1 and w_2 be the neighbours of w different from u.

Since G has girth at least 5, the vertices v and w are not adjacent. This is also the case for the vertices $\{v, w_1, w_2\}$ that form an independent set. Now observe that each component of $G' = G \setminus \{u, w\}$ admits, by induction hypothesis, a colouring that satisfy the theorem. Indeed, since $\delta(G) > 1$ and there is no pending C_5, there is no component isomorphic to K_2 or C_5 in G', except if the graph is of the form of Figure 8. In that case, it suffices to consider y as the new vertex u of degree 2, and v its neighbour of degree 3.

Let ω be such a colouring of G'.

The vertex v must be distinguished from its neighbours in G'. If v has two neighbours, then there are potentially two forbidden colours for vu. Thus there are two remaining possible colours for uv, since v has already two different colours in its

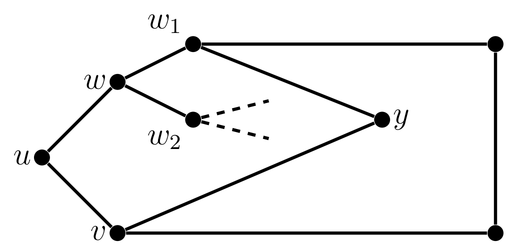


Figure 8: Subcase 4.1: if G is of this form, then consider G' as $G \setminus \{y, v\}$.

neighbourhood. If v has exactly one neighbour in G', then v must be distinguished from its neighbour and additionally the colour of uv must be different from the colour of the other edge incident to v. Thus again we have at most two forbidden colours. Let S_1 be a set colours that are not forbidden for uv. Vertices w_1 and w_2 must be distinguished from their neighbours in G' and if they have degree two in G, then they must be adjacent to edges coloured differently. Thus again there are potentially two forbidden colours for ww_1 and ww_2. Let S_2 and S_3 be the sets of colours that are not forbidden for w_1w and w_2w, respectively. The vertex u must be distinguished from v, so there is at most one forbidden colour for wu. Let S_4 be a set colours that are not forbidden for uw. Thus $|S_1| = |S_2| = |S_3| = 2$ and $|S_4| = 3$. Summarize our reasoning, if for each edge uv, w_1w, w_2w, uw we choose colours from S_1, S_2, S_3, S_4, respectively, then we obtain a colouring that distinguishes all vertices of G', distinguishes u from v and furthermore guarantees that w_1 and w_2 are adjacent to edges coloured differently. To obtain a colouring that satisfies all the conditions of the theorem we need to add some additional restrictions on colours that we choose for uv, w_1w, w_2w and uw. Let x_1, x_2, x_3, x_4 be the colours attributed to edges uv, w_1w, w_2w, uw, respectively. Thus for colours x_1, x_2, x_3, x_4 it must hold:

- $x_2 + x_3 \neq x_1$, since u and w must be distinguished;
- $x_3 + x_4 \neq \sigma_1$ (where σ_1 is the colour of w_1 in G'), since w and w_1 must be distinguished;
- $x_2 + x_4 \neq \sigma_2$ (where σ_2 is the colour of w_2 in G'), since w and w_2 must be distinguished;
- $x_1 \neq x_4$, since u must be adjacent to edges coloured differently;
- $x_3 \neq x_4$ or $x_3 \neq x_2$ or $x_2 \neq x_4$, since w must be adjacent to edges coloured differently.

We construct a polynomial

$$P(x_1, x_2, x_3, x_4) = (x_2 + x_3 - x_1)(x_3 + x_4 - \sigma_1)(x_2 + x_4 - \sigma_2)(x_1 - x_4)(x_3 - x_4).$$

Observe that if there are $x_i \in S_i$ ($i \in \{1, 2, 3, 4\}$) such that $P(x_1, x_2, x_3, x_4) \neq 0$, then we put $\omega(uv) = x_1, \omega(w_1w) = x_2, \omega(w_2w) = x_3, \omega(uw) = x_4$, and the resulting colouring satisfies all the conditions of the theorem. To prove that there are $x_i \in S_i$ ($i \in \{1, 2, 3, 4\}$) such that $P(x_1, x_2, x_3, x_4) \neq 0$, we use the Combinatorial Nullstellensatz. The coefficient of the monomial $x_1x_2x_3x_4^2$ is equal to -1, so is non-zero. Since $|S_1| > 1, |S_2| > 1, |S_3| > 1, |S_4| > 2$, Theorem 11 implies that there are $x_i \in S_i$ ($i \in \{1, 2, 3, 4\}$) such that $P(x_1, x_2, x_3, x_4) \neq 0$. Thus there is a colouring that satisfy all conditions of the theorem.

Subcase 4.2. $\delta(G) = 3$

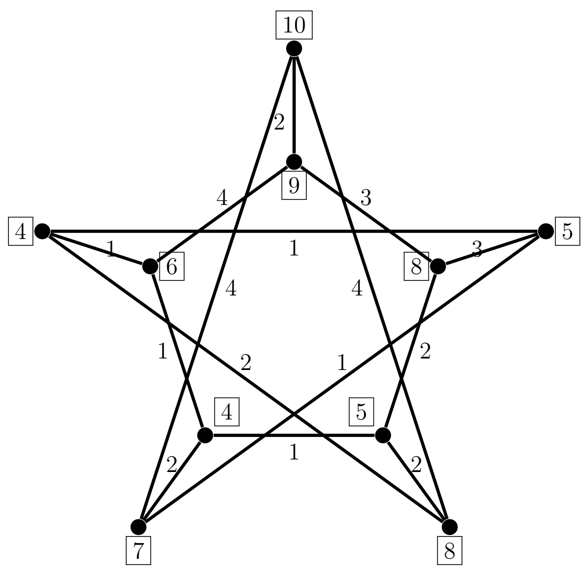


Figure 9: A neighbour sum distinguishing 2-relaxed 4-edge colouring for Petersen graph, appearing in Case 3 of Theorem 12.

Let C be a cycle of smallest size and $u_0, \ldots, u_{\ell-1}$ its vertices. For each vertex u_i of the cycle, u_i has a neighbour v_i outside C (since C is minimal).

We remove all the vertices u_i from G to obtain a graph G'.

If $G' = C_5$, then G is isomorphic to the Petersen graph and satisfies the theorem according to Figure 9. Thus we may assume that G is not isomorphic to the graph in Figure 9 and then G' has no component isomorphic to C_5. Since C is of smallest size and G is cubic, there is also no isolated K_2 in G'. Thus by induction, there is a colouring ω of G' that is distinguishing, 2-relaxed, and assigns two distinct colours to the edges incident with vertices of degree 2.

Since C has size at least 5 and by minimality of C, all the vertices v_i are distinct (but might be adjacent). For each vertex v_i, we denote by σ_i the sum of the colours on the two edges adjacent to v_i that are already coloured (there are exactly two such edges). Note that to distinguish u_i and v_i, the colours of $u_{i-1}u_i$ and u_iu_{i+1} must not sum to σ_i.

In order to keep v_i distinguished from its neighbours outside of C, at most two colours are forbidden for the edge u_iv_i, and thus at least two colours remain possible for the edge u_iv_i. We denote by L_i the list of possible colours for the edge u_iv_i.

Note that if v_i is adjacent to another vertex v_j, then, since there is no C_4, we have $|j-i|>1$ (subscripts are taken modulo ℓ). Furthermore, until either the edge v_ju_j or v_iu_i is coloured, the lists L_i and L_j have size at least three and at least two if one of the edges is coloured.

Subcase 4.2.1 C has length 5 and there is a pair of adjacent vertices in $\{v_0,\dots,v_4\}$.

Subcase 4.2.1.1. There is $\sigma_i \neq 5$ for $i \in \{0,\dots,4\}$

First we colour edges of the cycle C in such a way that every vertex u_i is adjacent to two differently coloured edges and the pairs (u_i,v_j) are distinguished for $i \in \{0,\dots,4\}$. We claim using Theorem 11 that such a colouring exists. Without loss of generality we assume that $\sigma_1 \neq 5$. Observe that assumption on σ_1 implies that there is a colour $c_0 \in \{1,2,3,4\}$ such that $\sigma_1 - c_0 \notin \{1,2,3,4\}$. We colour u_0u_1 with c_0. Thus if we colour u_1u_2 with any colour from $\{1,2,3,4\}$, then the pair (u_1,v_1) is distinguished. Since u_1 must be adjacent to edges coloured differently, we assume that $S_1=\{1,2,3,4\}\setminus\{c_0\}$ is the set of possible colours for u_1u_2. Let x_1,x_2,x_3,x_4 be the colours attributed to edges $u_1u_2,u_2u_3,u_3u_4,u_4u_0$, respectively. Let S_i be the set of colours that are possible for x_i, so $|S_1|=3$ and $|S_i|=4$ for $i\in\{2,3,4\}$. To obtain the above described colouring we need that the colours additionally satisfy:

- $x_i + x_{i+1} \neq \sigma_{i+1}$ for $i \in \{1,2,3\}$, since (u_{i+1},v_{i+1}) must be distinguished;
- $x_i \neq x_{i+1}$ for $i \in \{1,2,3\}$, since u_{i+1} must be adjacent to edges with different colours;
- $x_4 + c_0 \neq \sigma_0$, since (u_0,v_0) must be distinguished;
- $x_4 \neq c_0$, since u_0 must be adjacent to edges with different colours.

We construct a polynomial

$$\begin{aligned} P(x_1,x_2,x_3,x_4) = \quad & (x_1+x_2-\sigma_2)(x_1-x_2)(x_2+x_3-\sigma_3)(x_2-x_3) \\ & (x_3+x_4-\sigma_4)(x_3-x_4)(x_4+c_0-\sigma_0)(x_4-c_0). \end{aligned}$$

Consider the coefficient of the monomial $x_1^2x_2^2x_3^2x_4^2$, observe that this coefficient in P is the same as in the following polynomial

$$P_1(x_1,x_2,x_3,x_4) = (x_1^2 - x_2^2)(x_2^2 - x_3^3)(x_3^2 - x_4^2)x_4^2.$$

The coefficient of the monomial $x_1^2x_2^2x_3^2x_4^2$ is 1, so since $|S_1| > 2, |S_i| > 3$ for $i \in \{2,3,4\}$, Theorem 11 implies that there are $x_i \in S_i$ $(i \in \{1,2,3,4\})$ such that $P(x_1,x_2,x_3,x_4) \neq 0$ and equivalently there is a desired colouring of the edges of C. Let $c_i \in S_i$ $(i \in \{1,2,3,4\})$ be colours such that $P(c_1,c_2,c_3,c_4) \neq 0$, we put $\omega(u_1u_2) = c_1, \omega(u_2u_3) = c_2, \omega(u_3u_4) = c_3, \omega(u_4u_0) = c_4$.

Now we colour the edges u_iv_i for $i \in \{0,\dots,4\}$. The colours that we choose for these edges must distinguish adjacent vertices of C and adjacent vertices of $\{v_0,\dots,v_4\}$. Let $t_i = |N(v_i) \cap \{v_0,v_1,v_2,v_3,v_4\}|$ for $i \in \{0,\dots,4\}$.

Claim 14. *There are two consecutive vertices v_i, v_{i+1} such that $t_i > t_{i+1}$.*

Proof. Suppose that there is i such that $t_i = 2$. Since G is not isomorphic to the graph of Figure 9, there is $t_j < 2$. Thus we find two consecutive vertices v_i, v_{i+1} such that $t_i > t_{i+1}$. If $t_i < 2$ for $i \in \{0,\dots,4\}$, then there are at most two pairs of adjacent vertices in $\{v_0,\dots,v_4\}$ and consequently there is $t_i = 0$. Since in $\{v_0,\dots,v_4\}$ there are two adjacent vertices, there is $t_j = 1$ and this implies that there are two vertices v_i, v_{i+1} such that $t_i > t_{i+1}$. □

Renaming vertices, if it is necessary, assume that $t_0 > t_1$. Note that this operation is possible, even though it has been assumed previously that $\sigma_1 \neq 5$. Indeed, we will not use anymore this property in the rest of the proof.

Observe that $|L_i| \geq 2 + t_i$ for $i \in \{0,\dots,4\}$. We choose the colour $b_0 \in L_0$ for u_0v_0 such that $|L_1 \setminus \{b_0 + c_4 - c_1\}| \geq 2 + t_1$. Because $t_0 > t_1$, we can find such a colour. We colour u_0v_0 with b_0. Then we modify the list L_1, i.e. we delete the colour $b_0 + c_4 - c_1$ from the list whenever such a colour is in L_1. Now each colour from L_1 will distinguish u_0 and u_1 and still $|L_1| \geq 2 + t_1$. Moreover for every neighbour v_i of v_0 in $\{v_1,v_2,v_3,v_4\}$ we delete the colour $\sigma_0 + b_0 - \sigma_i$ from the list L_i. After such a list modification, every colour in L_i will distinguish v_0 and v_i.

Next we colour u_4v_4, we choose the colour from L_4 that distinguishes u_0 and u_4. Because $|L_4| \geq 2$ we can find the proper colour. Let b_4 be such a colour, which we put on u_4v_4. We modify the lists of neighbours of v_4 in $\{v_0,v_1,v_2,v_3\}$: if v_i is the neighbour of v_4, then we delete the colour $\sigma_4 + b_4 - \sigma_i$ from L_i. Then we repeat procedures of colouring and list modifications for the edges u_3v_3, u_2v_2 and u_1v_1. Observe that when we colour u_1v_1 we do not need to care about the vertex

u_0, because the colour b_0 on u_0v_0 guarantees that every colour in L_1 distinguishes u_0 and u_1. Each time when we choose colour for the edge u_iv_i the list L_i contains at least two colours, because initially every list L_i had at least $2 + t_i$ colours. Thus eventually we obtain the neighbour sum distinguishing 2-relaxed edge colouring of G.

Subcase 4.2.1.2 $\sigma_i = 5$ **for** $i \in \{0, \ldots, 4\}$

By our assumption there is a pair of adjacent vertices in $\{v_0, \ldots, v_4\}$. Without loss of generality, we assume that $v_0v_2 \in E(G)$. Thus $|L_0| \geq 3$. Recall that $|L_i| \geq 2 + t_i$, where $t_i = |N(v_i) \cap \{v_0, v_1, v_2, v_3, v_4\}|$.

First we colour the edges of C in the following way $\omega(u_0u_1) = 1, \omega(u_1u_2) = 3, \omega(u_2u_3) = 4, \omega(u_3u_4) = 3, \omega(u_4u_0) = 1$. Such a colouring distinguishes the pairs (u_i, v_i) for $i \in \{0, \ldots, 4\}$, furthermore, all vertices of C, except u_0, are adjacent to edges with different colours.

Now we colour edges u_iv_i ($i \in \{1, \ldots, 4\}$) in such a way that we distinguish all adjacent pairs of C and $\{v_0, \ldots, v_4\}$. The vertex u_0 must be adjacent to edges with different colours, so from L_0 we delete the colour 1, whenever it is on the list L_0. Thus the edge u_0v_0 has at least two possible colours.

Consider the pair (u_1, u_2). If we give u_1v_1 a colour $b_1 \leq 3$, then whatever the colour will be on u_2v_2, the pair (u_1, u_2) will be distinguished. Since u_1v_1 has at least two possible colours, there is a colour $b_1 \leq 3$, which we can put on u_1v_1. After colouring u_1v_1 with b_1, for every neighbour v_i of v_1 in $\{v_0, \ldots, v_4\}$ we delete the colour $\sigma_1 + b_1 - \sigma_i$ from L_i. After such a modification of L_i every colour in L_i will distinguish v_1 and v_i.

Next, as in the case 3.1.1, we colour the edges $u_0v_0, u_4v_4, u_3v_3, u_2v_2$, one by one. We start with the edge u_0v_0, we choose the colour from L_0 that distinguishes u_0 and u_1, because $|L_0| \geq 2$ we can choose the proper colour. After colouring u_0v_0 we modify the lists of neighbours of v_0 in $\{v_0, \ldots, v_4\}$ in such a way that every colour on the list of the neighbour will distinguishes v_0 from its neighbour, i.e. for every neighbour v_i of v_0 we delete the colour $\sigma_0 + b_0 - \sigma_i$ (where b_0 is the colour of u_0v_0) from L_i. Then we colour u_4v_4 in such a way that the vertices u_0 and u_4 are distinguished, so if b_4 is a colour of u_4v_4, then $b_4 \in L_4$ and $b_4 + 3 \neq b_0 + 1$. We delete the colour $\sigma_4 + b_4 - \sigma_i$ from L_i for every neighbour v_i of v_4. We do the same procedure for u_3v_3 and u_2v_2. The colour b_1, which w chose for u_1v_1 provide that every colour on L_2 distinguishes vertices u_1 and u_2, so if we colour u_2v_2 we do not take care about the vertex v_1. Since initially every list L_i had at least $2 + t_i$ colours, when we colour the edge u_iv_i the actual list has at least two colour and so we can find a proper colour. Eventually we obtain the neighbour sum distinguishing 2-relaxed edge colouring of G.

Subcase 4.2.2 C **has length 5 and the set** $\{v_0, \dots, v_4\}$ **is independent or** C **has length at least 6.**

We first consider that we are not in the case where all the lists L_i have size 2, are the same and all the σ_i sums to 5. In this case, we first colour some edges with particular conditions (given in Claim 15) and then extend this partial colouring to a complete colouring.

We say that a partial edge colouring is *good* if a vertex that has all its edges coloured has at least two colours in its neighbourhood and if two adjacent vertices x and y of degree 3 have their four edges distinct from xy coloured, then those vertices are distinguished (i.e. the sum in x and y are distinct). Observe that the partial edge colouring ω of G is good.

Claim 15. *There is a partial good colouring that satisfies the following conditions :*

- *the edges of* C *that are coloured are* u_1u_2 *and* u_2u_3*;*
- *the edges* u_iv_i *are coloured for* $0 \leq i \leq 2$*;*
- *the pair* (u_2, v_2) *is distinguished;*
- *the pairs* (u_0, u_1) *and* (u_1, u_2) *are necessarily distinguished whatever will be the colours on* $u_{\ell-1}u_0$ *and* u_0u_1*;*
- *the vertices* u_1 *and* u_2 *are adjacent to edges of different colours.*

Proof of Claim 15. Observe that our assumptions that either $\{v_0, \dots, v_4\}$ is independent when C has length 5 or C has length at least 6 imply that if v_i is adjacent to another vertex v_j, then we have $|j - i| > 2$. Thus if we colour u_iv_i, then L_j does not change for $|i - j| \leq 2$. We first prove the claim is true in the following cases:

(i) There are three consecutive lists L_i with 1 or 2 in the first one, 3 in the second and 1 in the last list.

(ii) There are three consecutive lists L_i with 1 or 2 in the first one, 4 in the second and 1 in the last list.

(iii) There are two consecutive lists L_i with 2 in the first one and 4 in the second.

(iv) There are two consecutive lists L_i with 1 in the first one and 3 in the second.

(i-ii) *There are three consecutive lists* L_i *with* 1 *or* 2 *in the first one,* 3 *(resp.* 4*) in the second and* 1 *in the last list.*

Without loss of generality, we assume that $1 \in L_0$ (or $2 \in L_0$), $3 \in L_1$ and $1 \in L_2$. We assign colour 1 (or 2) to the edge u_0v_0, colour 3 to the edge u_1v_1 and 1 to the edge u_2v_2. Observe that by our assumption that C has length 5 and the set $\{v_0, \dots, v_4\}$ is independent or C has length at least 6, the vertices v_0, v_1, v_2 are independent, so after colouring u_0v_0, u_1v_1 and u_2v_2 we still have a good colouring. Now we choose for the edges u_1u_2, u_2u_3 colours $4, 1$ if $\sigma_2 \neq 5$ and $4, 2$, otherwise (see Figure 10a and 10b). In this way we make sure that pairs of vertices (u_0, u_1), (u_1, u_2) and (u_2, v_2) will be distinguished. Indeed, for the last pair, we have $\omega(u_1u_2) + \omega(u_2u_3) \neq \sigma_2$. For the pair (u_1, u_2), the sum in u_1 will be at least 8 whereas the sum in u_2 is 6 or 7. For the pair (u_0, u_1) the sum in u_1 is $\omega(u_0u_1) + 7$ whereas the sum in u_0 will be at most $\omega(u_0u_1) + 2 + \omega(u_{\ell-1}u_0)$ which is smaller, since $7 > 2 + \omega(u_{\ell-1}u_0)$. We also have that the vertices u_1 and u_2 are adjacent to edges of different colours.

Similarly we can show that if there exist three consecutive lists L_i with 1 or 2 in the first one, 4 in the second and 1 in the last list, then the claim holds (see Figure 10c and 10d).

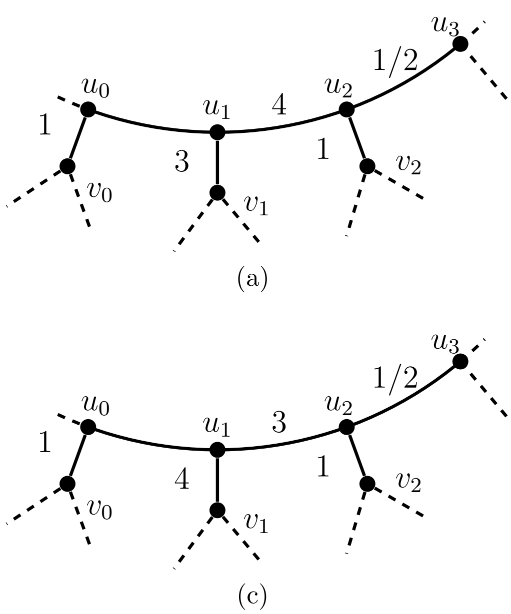


Figure 10: Pre-colouring of Claim 15 (i-ii)

(iii) *There are two consecutive lists L_i with 2 in the first one and 4 in the second.*

Without loss of generality, we assume that $2 \in L_1$ and $4 \in L_2$ and we assign colours 2 to the edge u_1v_1 and 4 to the edge u_2v_2.

We now consider the list L_0 and choose $c_0 \in L_0$ such that $c_0 \neq 2$ and assign this colour to u_0v_0. If possible, we choose $c_0 > 2$.

We first assume that we are not in the case where $c_0 = 1$ and $\sigma_2 = 7$. Depending on c_0 and the value of σ_2, we attribute colours to u_1u_2 and u_2u_3 in the following way:

	σ_2	$\omega(u_1u_2)$	$\omega(u_2u_3)$
$c_0 > 2$	$\neq 5$	1	4
$c_0 > 2$	5	1	3
$c_0 = 1$	$\neq 7$	4	3

We can see that we satisfy all the conditions of Claim 15.

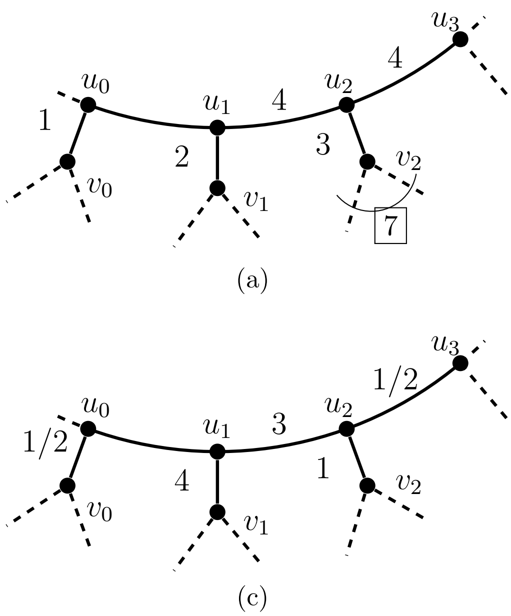


Figure 11: Pre-colouring of Claim 15 (iii)

Assume now that we are in the case $c_0 = 1$ and $\sigma_2 = 7$. Then necessarily $L_0 = \{1, 2\}$. If $3 \in L_2$, we choose for the edges u_0v_0, u_1v_1, u_2v_2 the colours $1, 2, 3$ and for u_1u_2 and u_2u_3 the colour 4 (see Figure 11a). Then as before we satisfy all the conditions of Claim 15. If $1 \in L_2$, we choose for the edges u_0v_0, u_1v_1, u_2v_2 the colours $1, 2, 1$ and for u_1u_2 and u_2u_3 the colour 4 and 1 (see Figure 11b). Then as before we satisfy all the conditions of Claim 15. Hence we can assume that $L_2 = \{2, 4\}$. If $3 \in L_1$ or $4 \in L_1$ then we have three consecutive lists with 1 in the first one, 3 (or 4) in the second and 2 in the last list. Then as we show before Claim 15 holds. Hence we can assume that $L_1 = \{1, 2\}$ and $L_2 = \{2, 4\}$.

Now, we reverse the role of u_1 and u_2, assuming that $L_1 = \{2, 4\}$ and $L_2 = \{1, 2\}$, and also the roles of u_0 and u_3. We choose for u_1v_1 colour 4 and for u_2v_2 colour 1 and we put to u_1u_2 colour 3 and to u_2u_3 colour 1 or 2 to have a sum different to the new σ_2. Then if 1 or 2 belongs to L_0 we will satisfy Claim 15 (see Figure 11c). Thus we can assume that $L_0 = \{3, 4\}$. But then we are back to the first case by exchanging the role of 0 and 2. We have now $L_1 = \{2, 4\}$ and $L_2 = \{3, 4\}$ and thus can find colours to satisfy Claim 15.

Therefore, we have proved that we can satisfy Claim 15 whenever there are two consecutive lists with a 2 and a 4.

(iv) *There are two consecutive lists L_i with 1 in the first one and 3 in the second.*

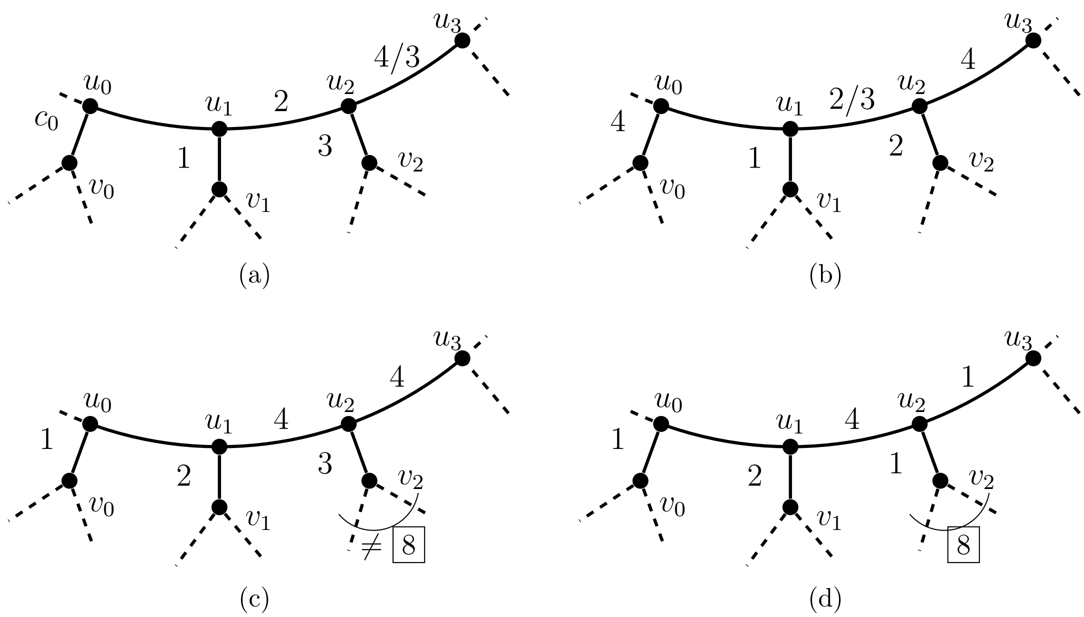


Figure 12: Pre-colouring of Claim 15 (iv)

Assume that $1 \in L_1$ and $3 \in L_2$ and we affect colour 1 to the edge u_1v_1 and 3 to the edge u_2v_2.

Consider the list L_0 and choose $c_0 \in L_0$. If $c_0 > 2$, then we assign colour 2 to the edge u_1u_2 and we assign colour 4 to the edge u_2u_3 whenever $\sigma_2 \neq 6$ and colour 3 otherwise (see Figure 12a). Hence, we satisfy all the conditions of Claim 15.

Thus we may assume that $L_0 = \{1, 2\}$.

Since Claim 15 is true when there are two consecutive lists the first with 2 and the second with 4, we may assume that $4 \notin L_1$. If $3 \in L_1$, then $1 \notin L_2$ and $2 \notin L_2$, since by our previous observation are no three consecutive lists with colours $1, 3, 1$ or $1, 3, 2$, respectively. Thus the argument $3 \in L_1$ implies that $L_2 = \{3, 4\}$. In this case, we reverse the role of u_0 and u_2 assuming that $L_0 = \{3, 4\}$ and $L_2 = \{1, 2\}$. Then we may assign colours $4, 1, 2$ to edges u_0v_0, u_1v_1, u_2v_2 and assign to colours $2, 4$ (if $\sigma_0 \neq 6$) or $3, 4$ (otherwise) u_1u_2, u_2u_3 (see Figure 12b). Thus again Claim 15 holds.

Now assume that $L_1 = \{1, 2\}$. Observe that $4 \notin L_2$ because, otherwise, there would be two consecutive lists the first with 2 and the second with 4 (which is case (iii)). Assume that $1 \in L_2$. If $\sigma_2 \neq 8$, then we assign colours $1, 2, 3$ to edges u_0v_0, u_1v_1, u_2v_2 and colours $4, 4$ to edges u_1u_2, u_2u_3 (see Figure 12c). If $\sigma_2 = 8$, then we assign colours $1, 2, 1$ to edges u_0v_0, u_1v_1, u_2v_2 and colours $4, 1$ to edges u_1u_2, u_2u_3 (see Figure 12d). Hence, we satisfy all the conditions of Claim 15.

We may assume that $1 \notin L_2$ and $4 \notin L_2$, so $L_2 = \{2, 3\}$. Then we consider L_3. If $4 \in L_3$ then we have two two consecutive lists the first one with 2 and the second with 4, so Claim 15 is true. Since $|L_3| \geq 2$ we have that $1 \in L_3$ or $2 \in L_3$. Thus we have three consecutive lists with colours $1, 3, 1$ or $1, 3, 2$ so by our previous observation Claim 15 is true.

Thus we may assume that there are no two consecutive lists L_i with 2 in the first one and 4 in the second and there are no two consecutive lists L_i with 1 in the first one and 3 in the second. If none of these two cases appear, but there are at least two different lists, it means that the lists are necessarily alternating $\{1, 3\}$ with $\{2, 4\}$. But then three consecutive lists L_i with 1 in the first one, 3 in the second and 1 in the last list appear, so we are done.

Thus we can now assume that all the lists have size 2 and are the same, say $L_i = \{a, b\}$ with $a < b$. By hypothesis, there exists i with $\sigma_i \neq 5$ and without loss of generality, we can assume that $i = 2$. Assume first that $a \neq 1$. Then we assign the following colours: u_0v_0 and u_2v_2 get b whereas u_1v_1 gets a, u_1u_2 gets 1 and u_2u_3 gets 4 (see Figure 13a). If $b \neq 4$, we do the reverse: u_0v_0 and u_2v_2 get a whereas u_1v_1 gets b, u_1u_2 gets 4 and u_2u_3 gets 1. Finally, if $\{a, b\} = \{1, 4\}$, then u_0v_0 and u_2v_2 get 1 whereas u_1v_1 gets 4, u_1u_2 gets 3 and u_2u_3 gets 2. In all these cases, Claim 15 is satisfied.

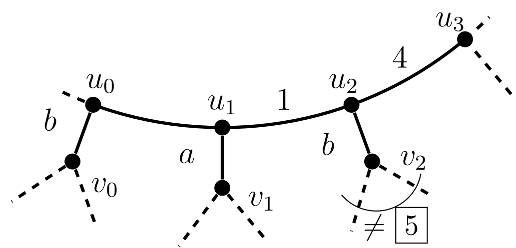
□

We now extend the partial colouring of Claim 15 to a neighbour sum distinguishing 2-relaxed colouring in the following way.

We first colour the edge u_3v_3 with a colour in L_3 that is not the colour of $\omega(u_2u_3)$ and we colour all the other edges u_iv_i with any colour in L_i.

Now we colour edge by edge the edges of C from u_3u_4 to u_0u_1. To colour u_iu_{i+1} we ask for:

- u_i and v_i to be distinguished, thus $\omega(u_iu_{i+1}) + \omega(u_iu_{i-1}) \neq \sigma_i$



(a)

Figure 13: Pre-colouring of Claim 15

- u_{i+1} to have their two adjacent edges of different colours: $\omega(u_iu_{i+1})$ must be different from $\omega(u_{i+1}v_{i+1})$
- u_i and u_{i-1} to be distinguished.

To colour the last edge u_0u_1, we replace the second condition (since u_1 already has two colours) by a condition to distinguish u_1 from v_1. There are three conditions giving three forbidden colours, thus at least one colour is always available.

This way, we colour all the edges of the cycle. At the end, all the adjacent vertices (u_i, v_i) are distinguished and all the adjacent vertices of the cycle are also distinguished (since the pairs u_0, u_1 and u_1, u_2 are sure to be distinguished by Claim 15). Moreover, there are at most twice the same colour on a vertex. Thus we obtain a neighbour sum distinguishing 2-relaxed colouring.

We now consider the final case where all the lists L_i are the same list $\{a, b\}$ and all the σ_i are equal to 5. Then we colour all the edges u_iv_i with the colour a.

Let $r = \ell \bmod 4$. We colour the edges of the cycle following the pattern 1342, except for the last $4 + r$ edges. For these ones, we use the same pattern but we double r colours that are not equal to a. For example, if $r = 2$ and $a = 3$, we finish by 113442. This way, all the vertices of the cycle have at least two distinct colours. All the couples u_i, v_i are distinguished since there are no consecutive edges of the cycle that sum to 5. And finally all the pairs u_i, u_{i+1} are also distinguished since all the edges at distance 2 on the cycle are different, which is enough to distinguish the edge between them.

□

Since $\chi'^{2}_{\sum}(C_5) = 3$, Theorem 12 implies the following corollary.

Corollary 16. *If G is a connected subcubic graph with at least three vertices, then $\chi'^{2}_{\sum}(G) \leq 4$.*

5 Concluding remarks

In the paper we propose a new version of the distinguishing edge colouring of a graph, in which the monochromatic set of edges induces a subgraph with bounded maximum degree. If the maximum degree of the monochromatic subgraph is bounded either by maximum degree of the graph or by 1, then we obtain two well-know versions of distinguishing edge colourings related with Conjecture 1 or 2, respectively. According to these equivalences, the following conjecture can be considered as a generalization of both Conjectures 1 and 2.

Conjecture 17. *If G is a connected graph on at least 3 vertices and $G \neq C_5$, then $\chi'^{d}_{\sum}(G) \leq \left\lceil \frac{\Delta(G)}{d} \right\rceil + 2$.*

As a support for Conjecture 17 we proved the validity of it for three families of graphs. In Section 3 we proved that for every value of d, it holds for trees. In Section 4 we proved that Conjecture 17 is true for complete graphs when $d = 2$ or $d \geq \lceil (V(G) - 1)/2 \rceil$. In Section 4 we proved that Conjecture 17 is also true for subcubic graphs.

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